Solutions to Linear analysis, January 05, 2017

1. The Fourier coefficients are

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi} x\,dx + \frac{1}{\pi}\int_\pi^{2\pi} (\pi - x)\,dx =$$

$$= \frac{1}{\pi}\left[\frac{1}{2}x^2\right]_0^\pi + \frac{1}{\pi}\left[-\frac{1}{2}(\pi-x)^2\right]_\pi^{2\pi} = 0,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\cos nx\,dx + \frac{1}{\pi}\int_\pi^{2\pi} (\pi - x)\cos nx\,dx =$$

$$= \frac{1}{\pi}\left(\left[x\frac{\sin nx}{n}\right]_0^\pi - \frac{1}{n}\int_0^\pi \sin nx\,dx\right) + \frac{1}{\pi}\left(\left[(\pi-x)\frac{\sin nx}{n}\right]_\pi^{2\pi} + \frac{1}{n}\int_\pi^{2\pi} \sin nx\,dx\right) =$$

$$= \frac{1}{\pi n}\left(-\left[-\frac{\cos nx}{n}\right]_0^\pi + \left[-\frac{\cos nx}{n}\right]_\pi^{2\pi}\right) = -\frac{2}{\pi n^2}\big(1-(-1)^n\big),\quad n \geq 1$$

and

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{\pi} x\sin nx\,dx + \frac{1}{\pi}\int_\pi^{2\pi} (\pi - x)\sin nx\,dx =$$

$$= \frac{1}{\pi}\left(\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^\pi + \frac{1}{n}\int_0^\pi \cos nx\,dx\right) + \frac{1}{\pi}\left(\left[(\pi-x)\left(-\frac{\cos nx}{n}\right)\right]_\pi^{2\pi} - \frac{1}{n}\int_\pi^{2\pi} \cos nx\,dx\right) =$$

$$= \frac{1}{n}\big(1-(-1)^n\big),\quad n \geq 1$$

Hence

$$f(x) \sim \sum_{n=1}^\infty \left(-\frac{4}{\pi(2n-1)^2}\cos(2n-1)x + \frac{2}{2n-1}\sin(2n-1)x\right)$$

is the Fourier series expansion of the function $f(x)$. The function $f(x)$ is piecewise C^1 and therefore it follows from the Dirichlet convergence theorem for Fourier series that the Fourier series of $f(x)$ converges for all $x \in \mathbf{R}$ and that its sum is equal to $\frac{1}{2}\big(f(x+) + f(x-)\big)$ for all $x \in \mathbf{R}$. In particular

$$\sum_{n=1}^\infty \left(-\frac{4}{\pi(2n-1)^2}\cos(2n-1)x + \frac{2}{2n-1}\sin(2n-1)x\right) = \begin{cases} -\frac{\pi}{2} & \text{if } \ x = 0 \\ x & \text{if } \ 0 < x < \pi \\ \frac{\pi}{2} & \text{if } \ x = \pi \\ \pi - x & \text{if } \ \pi < x < 2\pi. \end{cases}$$

For $x = 0$ this gives $\displaystyle\sum_{n=1}^\infty \frac{1}{(2n-1)^2} = \frac{\pi^2}{8}$, and for $x = \frac{\pi}{2}$ this gives $\displaystyle\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1} = \frac{\pi}{4}$.

The given integral is minimized when $a = b_1 = 2$, $b = b_2 = 0$ and $c = b_3 = \frac{2}{3}$. The minimum value is

$$\int_0^{2\pi} |f(x)|^2\,dx - \pi(|b_1|^2 + |b_2|^2 + |b_3|^2) =$$

$$= \int_0^{\pi} x^2\,dx + \int_{\pi}^{2\pi} (\pi - x)^2\,dx - \pi\left(4 + 0 + \frac{4}{9}\right) = \frac{2\pi^3}{3} - \frac{40\pi}{9}.$$

2. Set $g(x) = e^{-|x|}$, $x \in \mathbf{R}$. The given equation can then be written $f(x) + (f * g)(x) = g(x)$. Applying the Fourier transform to this equation we get $\hat{f}(u) + 2\pi\hat{f}(u)\hat{g}(u) = \hat{g}(u)$ for all $u \in \mathbf{R}$. Thus

$$\hat{f}(u) = \frac{\hat{g}(u)}{1 + 2\pi\hat{g}(u)} \quad \text{for all } u \in \mathbf{R}.$$

But

$$\hat{g}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-iux}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-iux}\,dx =$$

$$= \frac{1}{2\pi}\int_0^{\infty} e^{-x}\left(e^{-iux} + e^{iux}\right)\,dx = \frac{1}{2\pi}\int_0^{\infty}\left(e^{-(1+iu)x} + e^{-(1-iu)x}\right)\,dx =$$

$$= \frac{1}{2\pi}\left[\frac{e^{-(1+iu)x}}{-(1+iu)} + \frac{e^{-(1-iu)x}}{-(1-iu)}\right]_0^{\infty} = \frac{1}{2\pi}\left(\frac{1}{1+iu} + \frac{1}{1-iu}\right) = \frac{1}{\pi(1+u^2)}.$$

Hence

$$\hat{f}(u) = \frac{\dfrac{1}{\pi(1+u^2)}}{1 + 2\pi\dfrac{1}{\pi(1+u^2)}} = \frac{1}{\pi(3+u^2)} = \frac{1}{3\pi\left(1 + \left(\dfrac{u}{\sqrt{3}}\right)^2\right)} = \frac{1}{3}\,\hat{g}\left(\frac{u}{\sqrt{3}}\right)$$

and therefore

$$f(x) = \frac{1}{\sqrt{3}}\,g(\sqrt{3}x) = \frac{1}{\sqrt{3}}\,e^{-\sqrt{3}\,|x|}, \text{ for all } x \in \mathbf{R}.$$

3. We solve the heat equation $u_t - u_{xx} = cx$, $0 < x < 1$, $t > 0$, with initial value $u(x,0) = x$, $0 < x < 1$, and boundary values $u_x(0,t) = 0$ and $u_x(1,t) = 0$, $t > 0$, where c is a constant.

Setting $c = 0$ in the solution to this problem we get the solution to problem a), and setting $c = 1$ in the solution to this problem we get the solution to problem b).

The Sturm-Liouville problem corresponding to our heat equation is the eigenvalue problem $X''(x) + \lambda X(x) = 0$, $0 < x < 1$, with boundary values $X'(0) = 0$ and $X'(1) = 0$. It has eigenvalues $\lambda_n = \pi^2 n^2$ $n = 0, 1, 2, \ldots$ and corresponding eigenfunctions $X_n(x) = \cos \pi n x$ $n = 0, 1, 2, \ldots$. The eigenfunctions constitute an orthogonal system with respect to the inner product $\langle f, g\rangle = \int_0^1 f(x)\overline{g(x)}\,dx$. The corresponding norm is $\|f\| = \sqrt{\langle f, f\rangle}$. Since $\|X_0\|^2 = \int_0^1 dx = 1$ and $\|X_n\|^2 = \int_0^1 \cos^2 \pi n x\,dx = \frac{1}{2}$ for $n = 1, 2, \ldots$ the Fourier series expansion of a function $f(x)$ on $0 \le x \le 1$ with respect to the orthogonal system $X_n(x)$ $n = 0, 1, 2, \ldots$ is

$$f(x) \sim \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n X_n(x) \quad \text{where} \quad \alpha_n = 2\int_0^1 f(x)X_n(x)\,dx \;\; n = 0, 1, 2, \ldots .$$

We now expand the sought function $u(x,t)$ in a Fourier series with respect to the orthogonal system $X_n(x)$ $n = 0, 1, 2, \ldots$. From the general Sturm-Liouville theory we know that $u(x,t)$ then is equal to its Fourier series, that is

$$(1)\quad u(x,t) = \frac{1}{2}Y_0(t) + \sum_{n=1}^{\infty} Y_n(t)X_n(x) \text{ where } Y_n(t) = 2\int_0^1 u(x,t)X_n(x)\,dx \quad n = 0,1,2,\ldots \;.$$

We also know that

$$u_t(x,t) \sim \frac{1}{2}Y_0'(t) + \sum_{n=1}^{\infty} Y_n'(t)X_n(x) \quad \text{and} \quad u_{xx}(x,t) \sim -\sum_{n=1}^{\infty} \pi^2 n^2\, Y_n(t)X_n(x)$$

so

$$(2)\qquad u_t(x,t) - u_{xx}(x,t) \sim \frac{1}{2}Y_0'(t) + \sum_{n=1}^{\infty} \big(Y_n'(t) + \pi^2 n^2\, Y_n(t)\big) X_n(x).$$

But $u_t(x,t) - u_{xx}(x,t) = cx$. We now rewrite this equality in the form (2) by expanding $f(x) = x$ with respect to the orthogonal system $X_n(x)$ $n = 0, 1, 2, \ldots$. The Fourier series coefficients in that expansion are

$$2\int_0^1 f(x)X_0(x)\,dx = 2\int_0^1 x\,dx = 1$$

and

$$2\int_0^1 f(x)X_n(x)\,dx = 2\int_0^1 x\cos\pi nx\,dx = 2\left(\left[x\frac{\sin\pi nx}{\pi n}\right]_0^1 - \frac{1}{\pi n}\int_0^1 \sin\pi nx\,dx\right) =$$

$$= -\frac{2}{\pi n}\left[-\frac{\cos\pi nx}{\pi n}\right]_0^1 = -\frac{2}{\pi^2 n^2}\big(1 - (-1)^n\big) \quad \text{for } n = 1, 2, \ldots \;.$$

Thus

$$(3)\qquad x \sim \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi^2(2n-1)^2} X_{2n-1}(x)$$

and therefore $u_t(x,t) - u_{xx}(x,t) = cx$ is equivalent with

$$(4)\qquad u_t(x,t) - u_{xx}(x,t) \sim \frac{1}{2}c - \sum_{n=1}^{\infty} \frac{4c}{\pi^2(2n-1)^2} X_{2n-1}(x).$$

Comparison of (2) and (4) gives

$$(5)\qquad \begin{cases} Y_0'(t) = c \\ Y_{2n}'(t) + \pi^2(2n)^2\, Y_{2n}(t) = 0, & n = 1, 2, \ldots \\ Y_{2n-1}'(t) + \pi^2(2n-1)^2\, Y_{2n-1}(t) = -\dfrac{4c}{\pi^2(2n-1)^2}, & n = 1, 2, \ldots \;. \end{cases}$$

From (1) we further get

$$u(x,0) = \frac{1}{2}Y_0(0) + \sum_{n=1}^{\infty} Y_n(0)X_n(x). \tag{6}$$

But $u(x,0) = x$, and using the expansion (3) we get

$$u(x,0) \sim \frac{1}{2} - \sum_{n=1}^{\infty} \frac{4}{\pi^2(2n-1)^2} X_{2n-1}(x). \tag{7}$$

Comparison of (6) and (7) gives

$$\begin{cases} Y_0(0) = 1 & \\ Y_{2n}(0) = 0, & n = 1, 2, \ldots \\ Y_{2n-1}(0) = -\dfrac{4}{\pi^2(2n-1)^2}, & n = 1, 2, \ldots \ . \end{cases} \tag{8}$$

The differential equations in (5) are all of the type $Y'(t) + \alpha Y(t) = \beta$. For $\alpha \neq 0$ we get

$$Y'(t) + \alpha Y(t) = \beta \quad \Longleftrightarrow \quad \frac{d}{dt}\left(e^{\alpha t}\, Y(t)\right) = \beta\, e^{\alpha t} \quad \Longleftrightarrow$$

$$e^{\alpha t}\, Y(t) = \frac{\beta}{\alpha} e^{\alpha t} + Y(0) - \frac{\beta}{\alpha} \quad \Longleftrightarrow \quad Y(t) = Y(0)e^{-\alpha t} + \frac{\beta}{\alpha}\left(1 - e^{-\alpha t}\right).$$

For $\alpha = 0$ we get $Y(t) = \beta\, t + Y(0)$. The differential equations in (5) together with the initial values in (8) therefore imply that

$$\begin{cases} Y_0(t) = c\,t + 1 & \\ Y_{2n}(t) = 0, & n \geq 1 \\ Y_{2n-1}(t) = -\dfrac{4}{\pi^2(2n-1)^2} e^{-\pi^2(2n-1)^2\, t} - \dfrac{4c}{\pi^4(2n-1)^4}\left(1 - e^{-\pi^2(2n-1)^2\, t}\right), & n \geq 1 \end{cases} \tag{9}$$

Inserting (9) into (1) we finally get

$$u(x,t) = \frac{1}{2}c\,t + \frac{1}{2} - \sum_{n=1}^{\infty}\left(\frac{4}{\pi^2(2n-1)^2} e^{-\pi^2(2n-1)^2\, t} + \frac{4c}{\pi^4(2n-1)^4}\left(1 - e^{-\pi^2(2n-1)^2\, t}\right)\right) \cos \pi(2n-1)x.$$

4. It is given that

$$f(x) \sim \sum_{n=1}^{\infty} a_n \sin nx, \quad 0 \leq x \leq \pi, \tag{10}$$

is the sine series expansion of $f(x)$ on $0 \leq x \leq \pi$. Hence $a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$, $n = 1, 2, \ldots$. Let

$$f'(x) \sim \frac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n \cos nx, \quad 0 \leq x \leq \pi,$$

be the cosine series expansion of $f'(x)$ on $0 \le x \le \pi$, and let

$$f''(x) \sim \sum_{n=1}^{\infty} c_n \cos nx, \quad 0 \le x \le \pi,$$

be the sine series expansion of $f''(x)$ on $0 \le x \le \pi$. Then

$$b_0 = \frac{2}{\pi}\int_0^\pi f'(x)\,dx = \frac{2}{\pi}\big(f(\pi) - f(0)\big) = 0,$$

$$b_n = \frac{2}{\pi}\int_0^\pi f'(x)\cos nx\,dx = \frac{2}{\pi}\left([f(x)\cos nx]_0^\pi - \int_0^\pi f(x)(-n\sin nx)\,dx\right) =$$

$$= n\,\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx = na_n, \quad n = 1, 2, \ldots$$

and

$$c_n = \frac{2}{\pi}\int_0^\pi f''(x)\sin nx\,dx = \frac{2}{\pi}\left(\big[f'(x)\sin nx\big]_0^\pi - \int_0^\pi f'(x) n\cos nx)\,dx\right) =$$

$$= -n\,\frac{2}{\pi}\int_0^\pi f'(x)\cos nx\,dx = -nb_n = -n^2 a_n \quad n = 1, 2, \ldots\ .$$

This proves that

$$f'(x) \sim \sum_{n=1}^{\infty} na_n \cos nx, \quad 0 \le x \le \pi, \tag{11}$$

is the cosine series expansion of $f'(x)$ on $0 \le x \le \pi$, and that

$$f''(x) \sim \sum_{n=1}^{\infty} -n^2 a_n \sin nx, \quad 0 \le x \le \pi, \tag{12}$$

is the sine series expansion of $f''(x)$ on $0 \le x \le \pi$. Applying the Parseval identity to the expansions (10), (11) and (12) we get

$$\int_0^\pi |f(x)|^2\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} |a_n|^2, \tag{13}$$

$$\int_0^\pi |f'(x)|^2\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} n^2|a_n|^2 \tag{14}$$

and

$$\int_0^\pi |f''(x)|^2\,dx = \frac{\pi}{2}\sum_{n=1}^{\infty} n^4|a_n|^2 \tag{15}$$

From (13) and (14) it is obvious that

$$\int_0^\pi |f(x)|^2\,dx \le \int_0^\pi |f'(x)|^2\,dx$$

with equality if and only if $a_n = 0$ for all integers $n \geq 2$, that is if and only if $f(x) = a\sin x$ for some constant a. The equalities (10), (11) and (12) together with the Cauchy-Schwarz inequality furthermore show that

$$\int_0^\pi |f'(x)|^2\,dx = \frac{\pi}{2}\sum_{n=1}^\infty n^2|a_n|^2 = \frac{\pi}{2}\sum_{n=1}^\infty |a_n|\cdot n^2|a_n| \leq$$

$$\leq \frac{\pi}{2}\left(\sum_{n=1}^\infty \big(|a_n|\big)^2\right)^{1/2}\left(\sum_{n=1}^\infty \big(n^2|a_n|\big)^2\right)^{1/2} =$$

$$= \left(\frac{\pi}{2}\sum_{n=1}^\infty |a_n|^2\right)^{1/2}\left(\frac{\pi}{2}\sum_{n=1}^\infty n^4|a_n|^2\right)^{1/2} = \left(\int_0^\pi |f(x)|^2\,dx\right)^{1/2}\left(\int_0^\pi |f''(x)|^2\,dx\right)^{1/2}$$

with equality if and only if the sequences $(|a_1|, |a_2|, |a_3|, \ldots)$ and $(|a_1|, 2^2|a_2|, 3^2|a_3|, \ldots)$ are linearly dependent, that is if and only if $a_n \neq 0$ for at most one integer $n \geq 1$, that is if and only if $f(x) = a\sin nx$ for some constant a and some integer $n \geq 1$.

5. Using the trigonometric identities $\cos 2\alpha = 1 - 2\sin^2\alpha$ and $\sin 2\alpha = 2\sin\alpha\cos\alpha$ we get

$$\frac{1}{2}\int_0^2 \sin ux\,dx - \frac{1}{8}\int_0^4 \sin ux\,dx = \frac{1}{2}\left[-\frac{\cos ux}{u}\right]_{x=0}^{x=2} - \frac{1}{8}\left[-\frac{\cos ux}{u}\right]_{x=0}^{x=4} =$$

$$= \frac{1}{2u}(1-\cos 2u) - \frac{1}{8u}(1-\cos 4u) = \frac{1}{u}\sin^2 u - \frac{1}{4u}\sin^2 2u =$$

$$= \frac{1}{u}(\sin^2 u - \sin^2 u\cos^2 u) = \frac{1}{u}\big(\sin^2 u - \sin^2 u(1-\sin^2 u)\big) = \frac{\sin^4 u}{u} \quad \text{if } u \neq 0.$$

If $u = 0$ the result is obviously 0. Set

$$f(x) = \begin{cases} 0, & x < -4 \\ \frac{1}{8}, & -4 \leq x < -2 \\ -\frac{3}{8}, & -2 \leq x < 0 \\ 0, & x = 0 \\ \frac{3}{8}, & 0 < x \leq 2 \\ -\frac{1}{8}, & 2 < x \leq 4 \\ 0, & x > 4. \end{cases}$$

Then $f(x)$ is an an odd function and

$$\frac{1}{2}\int_0^2 \sin ux\,dx - \frac{1}{8}\int_0^4 \sin ux\,dx = \frac{3}{8}\int_0^2 \sin ux\,dx - \frac{1}{8}\int_2^4 \sin ux\,dx = \int_0^4 f(x)\sin ux\,dx =$$

$$= \frac{1}{2}\int_{-4}^4 f(x)\sin ux\,dx = -\frac{1}{2i}\int_{-4}^4 f(x)\big(\cos ux - i\sin ux\big)\,dx = -\frac{1}{2i}\int_{-4}^4 f(x)e^{-iux}\,dx =$$

$$= \pi i \,\frac{1}{2\pi}\int_{-4}^{4} f(x)e^{-iux}\,dx = \pi i\,\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-iux}\,dx = \pi i\,\hat{f}(u).$$

That is

$$\frac{\sin^4 u}{u} = \pi i\,\hat{f}(u) \quad \text{if } u \neq 0. \tag{16}$$

For an arbitrary number $T > 0$, an arbitrary $x \in \mathbf{R}$ and with the use of (16) we get

$$\int_0^T \frac{\sin^4 u}{u}\sin ux\,dx = \frac{1}{2}\int_{-T}^{T}\frac{\sin^4 u}{u}\sin ux\,dx = \frac{1}{2i}\int_{-T}^{T}\frac{\sin^4 u}{u}\big(\cos ux + i\sin ux\big)\,dx = \tag{17}$$

$$= \frac{1}{2i}\int_{-T}^{T}\frac{\sin^4 u}{u}e^{iux}\,dx = \frac{\pi}{2}\int_{-T}^{T}\hat{f}(u)e^{iux}\,dx.$$

But $f(x)$ is piecewise C^1 on $\mathbf{R}$ and therefore

$$\int_{-T}^{T}\hat{f}(u)e^{iux}\,dx \to \frac{1}{2}\big(f(x+)+f(x-)\big) \quad \text{as } T\to\infty \text{ for each } x \in \mathbf{R} \tag{18}$$

by the inversion formula for the Fourier transform. Combining (17) and (18) it follows that

$$\int_0^T \frac{\sin^4 u}{u}\sin ux\,dx \to \frac{\pi}{4}\big(f(x+)+f(x-)\big) \quad \text{as } T\to\infty \text{ for each } x\in\mathbf{R}.$$

That is the generalized integral

$$\int_0^\infty \frac{\sin^4 u}{u}\sin ux\,dx$$

is convergent for each $x \in \mathbf{R}$ and

$$\int_0^\infty \frac{\sin^4 u}{u}\sin ux\,dx = \frac{\pi}{4}\big(f(x+)+f(x-)\big) = \begin{cases} 0, & x < -4 \\ \frac{\pi}{32}, & x = -4 \\ \frac{\pi}{16}, & -4 < x < -2 \\ -\frac{\pi}{16}, & x = -2 \\ -\frac{3\pi}{16}, & -2 < x < 0 \\ 0, & x = 0 \\ \frac{3\pi}{16}, & 0 < x < 2 \\ \frac{\pi}{16}, & x = 2 \\ -\frac{\pi}{16}, & 2 < x < 4 \\ -\frac{\pi}{32}, & x = 4 \\ 0, & x > 4. \end{cases}$$

Finally using (16) and the Plancheral identity we get

$$\int_0^\infty \frac{\sin^8 u}{u^2}\,du = \frac{1}{2}\int_{-\infty}^\infty \frac{\sin^8 u}{u^2}\,du = \frac{1}{2}\int_{-\infty}^\infty \left|\frac{\sin^4 u}{u}\right|^2 du = \frac{\pi^2}{2}\int_{-\infty}^\infty \left|\hat{f}(u)\right|^2 du =$$

$$= \frac{\pi}{4}\int_{-\infty}^\infty |f(x)|^2\,dx = \frac{\pi}{4}\int_{-4}^4 |f(x)|^2\,dx = \frac{\pi}{2}\int_0^4 |f(x)|^2\,dx =$$

$$= \frac{\pi}{2}\left(2\cdot\frac{9}{64} + 2\cdot\frac{1}{64}\right) = \frac{5\pi}{32}.$$

6. Let λ be an arbitrary real number ≥ 1 and let N be an arbitrary integer ≥ 1.

The triangle inequality and the Cauchy-Schwarz inequality for integrals show that

(19)
$$\left|\int_0^{2\pi} f(x)\big(g(\lambda x) - S_N(\lambda x)\big)\,dx\right| \leq \int_0^{2\pi} |f(x)|\,\big|g(\lambda x) - S_N(\lambda x)\big|\,dx \leq$$

$$\leq \left(\int_0^{2\pi} |f(x)|^2\,dx\right)^{1/2} \left(\int_0^{2\pi} \big|g(\lambda x) - S_N(\lambda x)\big|^2\,dx\right)^{1/2}.$$

But

(20)
$$\int_0^{2\pi} \big|g(\lambda x) - S_N(\lambda x)\big|^2\,dx = \frac{1}{\lambda}\int_0^{2\pi\lambda} \big|g(x) - S_N(x)\big|^2\,dx \leq$$

$$\leq \frac{1}{\lambda}\int_0^{2\pi([\lambda]+1)} \big|g(x) - S_N(x)\big|^2\,dx = \frac{[\lambda]+1}{\lambda}\int_0^{2\pi} \big|g(x) - S_N(x)\big|^2\,dx \leq$$

$$\leq 2\int_0^{2\pi} \big|g(x) - S_N(x)\big|^2\,dx.$$

Here $[\lambda]$ denotes the largest integer less than or equal to λ, and we have used that $\frac{[\lambda]+1}{\lambda} < 2$ for $\lambda > 1$. The function $g(x) - S_N(x)$ has the Fourier series expansion

$$g(x) - S_N(x) \sim \sum_{n=N+1}^\infty (a_n \cos nx + b_n \sin nx)$$

and therefore

(21)
$$\int_0^{2\pi} \big|g(x) - S_N(x)\big|^2\,dx = \pi \sum_{n=N+1}^\infty \left(|a_n|^2 + |b_n|^2\right).$$

by the Parseval identity. Combining (19), (20) and (21) we get

(22)
$$\left|\int_0^{2\pi} f(x)\big(g(\lambda x) - S_N(\lambda x)\big)\,dx\right| \leq$$

$$\leq \sqrt{2\pi}\left(\int_0^{2\pi} |f(x)|^2\,dx\right)^{1/2}\left(\sum_{n=N+1}^\infty \left(|a_n|^2 + |b_n|^2\right)\right)^{1/2}$$

for all real numbers $\lambda > 1$ and all integers $N \geq 1$.

Next write

$$\int_0^{2\pi} f(x)g(\lambda x)\,dx = \int_0^{2\pi} f(x)S_N(\lambda x)\,dx + \int_0^{2\pi} f(x)\big(g(\lambda x) - S_N(\lambda x)\big)\,dx$$

and note that

$$\int_0^{2\pi} f(x)S_N(\lambda x)\,dx =$$

$$= \left(\int_0^{2\pi} f(x)\,dx\right)\frac{a_0}{2} + \int_0^{2\pi} f(x)\left(\sum_{n=1}^{N}\big(a_n\cos n\lambda x + b_n \sin n\lambda x\big)\right)dx =$$

$$= \frac{1}{2\pi}\left(\int_0^{2\pi} f(x)\,dx\right)\left(\int_0^{2\pi} g(x)\,dx\right) + \sum_{n=1}^{N}\left(a_n\int_0^{2\pi} f(x)\cos n\lambda x\,dx + b_n\int_0^{2\pi} f(x)\sin n\lambda x\,dx\right).$$

Hence

$$\int_0^{2\pi} f(x)g(\lambda x)\,dx \; - \frac{1}{2\pi}\left(\int_0^{2\pi} f(x)\,dx\right)\left(\int_0^{2\pi} g(x)\,dx\right) = \tag{23}$$

$$= \int_0^{2\pi} f(x)\big(g(\lambda x) - S_N(\lambda x)\big)\,dx + \sum_{n=1}^{N}\left(a_n\int_0^{2\pi} f(x)\cos n\lambda x\,dx + b_n\int_0^{2\pi} f(x)\sin n\lambda x\,dx\right)$$

for all real numbers $\lambda > 1$ and all integers $N \geq 1$.

Now choose an arbitrary $\varepsilon > 0$. The sum $\sum_{n=1}^{\infty}\big(|a_n|^2 + |b_n|^2\big)$ converges by the Parseval identity. Thus $\sum_{n=N+1}^{\infty}\big(|a_n|^2 + |b_n|^2\big) \to 0$ as $N \to \infty$, which together with (22) show that there is an integer $N_0 \geq 1$ such that

$$\left|\int_0^{2\pi} f(x)\big(g(\lambda x) - S_{N_0}(\lambda x)\big)\,dx\right| < \varepsilon \quad \text{for all } \lambda > 1. \tag{24}$$

Moreover

$$\int_0^{2\pi} f(x)\cos n\lambda x\,dx \to 0 \quad \text{and} \quad \int_0^{2\pi} f(x)\sin n\lambda x\,dx \to 0 \quad \text{as } \lambda \to \infty$$

for each integer $n \geq 1$ by the Riemann-Lebesgue lemma. Hence there is a real number $\lambda_0 > 1$ such that

$$\left|\sum_{n=1}^{N_0}\left(a_n\int_0^{2\pi} f(x)\cos n\lambda x\,dx + b_n\int_0^{2\pi} f(x)\sin n\lambda x\,dx\right)\right| < \varepsilon \quad \text{for all } \lambda > \lambda_0. \tag{25}$$

Combining (23), (24) and (25) we see that

$$\left|\int_0^{2\pi} f(x)g(\lambda x)\,dx \; - \frac{1}{2\pi}\left(\int_0^{2\pi} f(x)\,dx\right)\left(\int_0^{2\pi} g(x)\,dx\right)\right| < 2\varepsilon \quad \text{for all } \lambda > \lambda_0.$$

Since ε here is an arbitrary number > 0 it follows that

$$\int_0^{2\pi} f(x)g(\lambda x)\,dx \to \frac{1}{2\pi}\left(\int_0^{2\pi} f(x)\,dx\right)\left(\int_0^{2\pi} g(x)\,dx\right) \quad \text{as } \lambda \to \infty.$$