DEPARTMENT OF MATHEMATICS
STOCKHOLM UNIVERSITY
Section mathematics
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Written examination
Linear analysis
February 1, 2017

No aids are allowed. Answer in English or Swedish.

1. Let a be a real number such that $0 < a < \pi$. Set $f(x) = 0$ for $0 \le x \le \pi - a$ and $f(x) = 1$ for $\pi - a < x \le \pi$. Let $f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ be the cosine series expansion of $f(x)$ on $0 \le x \le \pi$. Determine this cosine series expansion. Determine the sum of each of the series

$$\sum_{n=1}^{\infty} a_n \cos nu, \quad \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nu, \quad \sum_{n=1}^{\infty} \frac{\sin nu}{n} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{\sin^2 nu}{n^2}$$

for $0 \le u < 2\pi$. 10 p

2. Set $f(x) = \frac{1}{1+x^2}$, $x \in \mathbf{R}$, and $g(x) = e^{-|x|}$, $x \in \mathbf{R}$. Determine the Fourier transform of $f(x)$ and of $g(x)$. Define a sequence of functions $f_n(x)$ $n = 1, 2, \ldots$ on $\mathbf{R}$ in the following way: $f_1(x) = f(x)$ and $f_k(x) = \int_{-\infty}^{\infty} f(x-y) f_{k-1}(y)\, dy$ for $k = 2, 3, \ldots$. Determine the function $f_n(x)$ for an arbitrary integer $n \ge 1$. 10 p

3. a) Let $f(x)$ be a complex-valued 2π-periodic piecewise continuous function on $\mathbf{R}$. Let $f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$ be the Fourier series expansion of $f(x)$ and let $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the complex Fourier series expansion of $f(x)$. Show that $\sum_{n=-N}^{N} c_n e^{inx} = \frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n \cos nx + b_n \sin nx)$ for all integers $N \ge 1$ and all $x \in \mathbf{R}$.

   b) Let a be a real number but not an integer. Let $f(x)$ be the 2π-periodic function on $\mathbf{R}$ for which $f(x) = e^{-iax}$ for $0 < x \le 2\pi$. Determine the complex Fourier series expansion of $f(x)$. Show that the limit

$$\lim_{N\to\infty} \sum_{n=-N}^{N} \frac{1}{n+a}$$

   exists and determine the limit. 10 p

4. Find the solution $u(x, y)$ to the partial differential equation $u_{xx} + u_{yy} = 0$, $0 < x < 1$, $0 < y < 1$, with boundary values $u(x, 0) = u(x, 1) = x^2 - 2x$, $0 < x < 1$, and $u(0, y) = u_x(1, y) = 0$, $0 < y < 1$. 10 p

5. For $u \in \mathbf{R}$ show that

$$\frac{3}{4}\int_0^1 \cos ux\, dx - \frac{1}{4}\int_0^3 \cos ux\, dx = \begin{cases} \dfrac{\sin^3 u}{u} & \text{if } u \ne 0 \\ 0 & \text{if } u = 0. \end{cases}$$

   Show that the generalized integral

$$\int_0^{\infty} \frac{\sin^3 u}{u} \cos ux\, du$$

   is convergent for each $x \in \mathbf{R}$ and determine the value of the generalized integral for each $x \in \mathbf{R}$. Determine the value of

$$\int_0^{\infty} \frac{\sin^6 u}{u^2}\, du.$$

   10 p

Continuation on the other side.

6. Let A be the set of every complex-valued 2π-periodic continuous function $f(x)$ on $\mathbf{R}$ having a complex Fourier series expansion $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ such that $\sum_{-\infty}^{\infty} |c_n| < \infty$.

   a) Let $f(x) \in A$ and let $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the complex Fourier series expansion of $f(x)$. Show that $f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for all $x \in \mathbf{R}$.

   b) Let $f(x)$ be a complex-valued 2π-periodic continuous function $f(x)$ on $\mathbf{R}$ and assume that $f(x)$ has a piecewise continuous derivative on $\mathbf{R}$. Show that $f(x) \in A$.

   c) Let $f(x)$ and $g(x)$ be two complex-valued 2π-periodic piecewise continuous functions on $\mathbf{R}$. Set $h(x) = \int_{-\pi}^{\pi} f(x-y)g(y)\,dy$ for all $x \in \mathbf{R}$. Show that $h(x) \in A$. 10 p

Return of examination scripts Thursday February 9 between 11.45 and 12.00 outside room 32 house 5, and thereafter in room 204 house 6.