Solutions to Linear analysis, February 01, 2017

1. The cosine series coefficients are

$$a_0 = \frac{2}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{2}{\pi}\int_{\pi-a}^{\pi} dx = \frac{2a}{\pi}$$

and

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_{\pi-a}^{\pi}\cos nx\,dx = \frac{2}{\pi}\left[\frac{\sin nx}{n}\right]_{\pi-a}^{\pi} =$$
$$= \frac{2}{\pi n}\big(\sin n\pi - \sin(n\pi - na)\big) = \frac{2}{\pi}(-1)^n\frac{\sin na}{n},\quad n \geq 1.$$

Hence

$$f(x) \sim \frac{a}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}(-1)^n\frac{\sin na}{n}\cos nx \tag{1}$$

is the cosine series expansion of $f(x)$ on $0 \leq x \leq \pi$. Let $g(x)$ be the 2π-periodic function on $\mathbf{R}$ for which $g(x) = 0$ for $0 \leq x \leq \pi - a$, $g(x) = 1$ for $\pi - a < x < \pi - a$ and $g(x) = 0$ for $\pi + a \leq x < 2\pi$. Then $g(x) = f(x)$ for $0 \leq x \leq \pi$ and $g(x)$ is an even function on $\mathbf{R}$. It follows that the Fourier series expansion of $g(x)$ om $\mathbf{R}$ is $g(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx$, $x \in \mathbf{R}$, with the same coefficients a_n, $n = 0, 1, \ldots$ as above. The function $g(x)$ is piecewise C^1 on $\mathbf{R}$. The Dirichlet convergence theorem for Fourier series therefore shows that the Fourier series of $g(x)$ converges for all $x \in \mathbf{R}$ and that its sum is equal to $\frac{1}{2}\big(g(x+) + g(x-)\big)$ for all $x \in \mathbf{R}$. That is $\sum_{n=1}^{\infty} a_n\cos nx = \frac{1}{2}\big(g(x+) + g(x-)\big) - \frac{1}{2}a_0$ for all $x \in \mathbf{R}$. In particular

$$\sum_{n=1}^{\infty} a_n\cos nx = \begin{cases} -\frac{a}{\pi} & \text{if } 0 \leq x < \pi - a \\ \frac{1}{2} - \frac{a}{\pi} & \text{if } x = \pi - a \\ 1 - \frac{a}{\pi} & \text{if } \pi - a < x < \pi + a \\ \frac{1}{2} - \frac{a}{\pi} & \text{if } x = \pi + a \\ -\frac{a}{\pi} & \text{if } \pi + a < x < 2\pi. \end{cases} \tag{2}$$

We now make repeated use of a theorem of integration of Fourier series expansion. The expansion $g(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n\cos nx$, $x \in \mathbf{R}$, implies

$$\int_0^x g(t)\,dt = \int_0^x \frac{1}{2}a_0\,dt + \sum_{n=1}^{\infty} a_n\int_0^x \cos nt\,dt \quad \text{for all } x \in \mathbf{R} \quad \Longleftrightarrow$$

$$\int_0^x g(t)\,dt = \frac{a}{\pi}x + \sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx \quad \text{for all } x \in \mathbf{R}.$$

Set $h(x) = \int_0^x g(t)\,dt$, $x \in \mathbf{R}$. Then $h(x) - \frac{a}{\pi}x$ is 2π-periodic since $h(x) - \frac{a}{\pi}x = \sum_{n=1}^{\infty}\frac{a_n}{n}\sin nx$ for all $x \in \mathbf{R}$. Calculation gives

$$h(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq \pi - a \\ x - (\pi - a) & \text{if } \pi - a < x < \pi + a \\ 2a & \text{if } \pi + a \leq x < 2\pi. \end{cases}$$

The expansion $h(x) - \frac{a}{\pi}x = \sum_{n=1}^{\infty} \frac{a_n}{n} \sin nx$, $x \in \mathbf{R}$, implies

$$\int_0^x h(t)\,dt - \int_0^x \frac{a}{\pi} t\,dt = \sum_{n=1}^{\infty} \frac{a_n}{n} \int_0^x \sin nt\,dt \quad \text{for all } x \in \mathbf{R} \quad \Longleftrightarrow$$

$$\int_0^x h(t)\,dt - \frac{a}{2\pi} x^2 = b - \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nx \quad \text{for all } x \in \mathbf{R}$$

where $b = \sum_{n=1}^{\infty} \frac{a_n}{n^2}$. Set $k(x) = \int_0^x h(t)\,dt$, $x \in \mathbf{R}$. Then $k(x) - \frac{a}{2\pi}x^2$ is 2π-periodic since $k(x) - \frac{a}{2\pi}x^2 = b - \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nx$ for all $x \in \mathbf{R}$. Calculation gives

$$(3) \qquad k(x) = \begin{cases} 0 & \text{if } 0 \le x \le \pi - a \\ \frac{1}{2}(x - (\pi - a))^2 & \text{if } \pi - a < x < \pi + a \\ 2a(x - \pi) & \text{if } \pi + a \le x < 2\pi. \end{cases}$$

Finally integrating the expansion $k(x) - \frac{a}{2\pi}x^2 = b - \sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nx$, $x \in \mathbf{R}$, over the interval $0 \le x \le 2\pi$ we get

$$\int_0^{2\pi} k(x)\,dx - \int_0^{2\pi} \frac{a}{2\pi} t^2\,dt = \int_0^{2\pi} b\,dx - \sum_{n=1}^{\infty} \frac{a_n}{n^2} \int_0^{2\pi} \cos nx\,dx \Longleftrightarrow b = \frac{1}{2\pi} \int_0^{2\pi} k(x)\,dx - \frac{2\pi a}{3}.$$

Calculation gives $b = -\frac{1}{6\pi} a(\pi^2 - a^2)$. Consequently

$$\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nx = \frac{a}{2\pi} x^2 - \frac{1}{6\pi} a(\pi^2 - a^2) - k(x), \quad x \in \mathbf{R}.$$

Together with (3) this gives the sum of $\sum_{n=1}^{\infty} \frac{a_n}{n^2} \cos nx$ for $0 \le x < 2\pi$ (and therefore for all $x \in \mathbf{R}$ since the sum is 2π-periodic).

Using (2) for $x = \pi$ gives $\sum_{n=1}^{\infty} (-)^n a_n = 1 - \frac{a}{\pi} \Longleftrightarrow \sum_{n=1}^{\infty} \frac{\sin na}{n} = \frac{1}{2}(\pi - a)$. Here $0 < a < \pi$. Clearly $\sum_{n=1}^{\infty} \frac{\sin nu}{n} = 0$ for $u = 0$. We also note that $\sum_{n=1}^{\infty} \frac{\sin nu}{n}$ is an odd 2π-periodic function. Let $\alpha(u)$ be the 2π-periodic function for which $\alpha(0) = 0$ and $\alpha(u) = \frac{1}{2}(\pi - u)$ for $0 < u < 2\pi$. Then $\alpha(u)$ is an odd 2π-periodic function which is equal to $\sum_{n=1}^{\infty} \frac{\sin nu}{n}$ for $0 \le u < \pi$. It follows that $\sum_{n=1}^{\infty} \frac{\sin nu}{n} = \alpha(u)$ for all $u \in \mathbf{R}$. In particular $\sum_{n=1}^{\infty} \frac{\sin nu}{n} = 0$ for $u = 0$ and $\sum_{n=1}^{\infty} \frac{\sin nu}{n} = \frac{1}{2}(\pi - u)$ for $0 < u < 2\pi$.

Applying the Parseval identity to the expansion (1) gives

$$\frac{1}{2} \left(\frac{2a}{\pi} \right)^2 + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\sin^2 na}{n^2} = \frac{2}{\pi} \int_0^{\pi} |f(x)|^2\,dx.$$

But $\int_0^{\pi} |f(x)|^2 = a$. Thus $\sum_{n=1}^{\infty} \frac{\sin^2 na}{n^2} = \frac{1}{2} a(\pi - a)$. Here $0 < a < \pi$. Clearly $\sum_{n=1}^{\infty} \frac{\sin^2 nu}{n^2} = 0$ for $u = 0$. We also note that $\sum_{n=1}^{\infty} \frac{\sin^2 nu}{n^2}$ is a π-periodic function since $\sin^2 n(\pi + u) = \sin^2 nu$ for all $u \in \mathbf{R}$. Let $\beta(u)$ be the π-periodic function for which $\beta(u) = \frac{1}{2}u(\pi - u)$ for $0 \le u < \pi$. It follows then that $\sum_{n=1}^{\infty} \frac{\sin^2 nu}{n^2} = \beta(u)$ for all $u \in \mathbf{R}$. In particular $\sum_{n=1}^{\infty} \frac{\sin^2 nu}{n^2} = \frac{1}{2}u(\pi - u)$ for $0 \le u < \pi$ and $\sum_{n=1}^{\infty} \frac{\sin^2 nu}{n^2} = \frac{1}{2}(u - \pi)(2\pi - u)$ for $\pi \le u < 2\pi$.

2. The Fourier transform of $g(x)$ is

$$\hat{g}(u) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x)e^{-iux}\,dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-|x|}e^{-iux}\,dx =$$

$$= \frac{1}{2\pi}\int_0^{\infty} e^{-x}\left(e^{-iux} + e^{iux}\right)dx = \frac{1}{2\pi}\int_0^{\infty}\left(e^{-(1+iu)x} + e^{-(1-iu)x}\right)dx =$$

$$= \frac{1}{2\pi}\left[\frac{e^{-(1+iu)x}}{-(1+iu)} + \frac{e^{-(1-iu)x}}{-(1-iu)}\right]_0^{\infty} = \frac{1}{2\pi}\left(\frac{1}{1+iu} + \frac{1}{1-iu}\right) = \frac{1}{\pi(1+u^2)}.$$

The function $g(x)$ is continuous on $\mathbf{R}$ and has a piecewise continuous derivative on $\mathbf{R}$. Also $\int_{-\infty}^{\infty} |\hat{g}(u)|\,du < \infty$. Thus

$$\int_{-\infty}^{\infty} \frac{1}{\pi}\frac{1}{1+u^2}e^{ixu}\,du = e^{-|x|} \quad \text{for all } x \in \mathbf{R}$$

by the Fourier inversion formula. Equivalently

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{1+u^2}e^{-ixu}\,du = \frac{1}{2}e^{-|x|} \quad \text{for all } x \in \mathbf{R}.$$

Consequently $f(x)$ has the Fourier transform $\hat{f}(u) = \frac{1}{2}e^{-|u|}$. The function sequence $f_n(x)$ $n = 1, 2, \ldots$ on $\mathbf{R}$ is defined by: $f_1(x) = f(x)$ and $f_k(x) = (f * f_{k-1})(x)$ for $k = 2, 3, \ldots$. Applying the Fourier transform to the formula $f_k(x) = (f * f_{k-1})(x)$, $k \geq 2$, gives $\hat{f}_k(u) = 2\pi\hat{f}(u)\hat{f}_{k-1}(u)$, $k \geq 2$. Now let the integer $n \geq 1$ be arbitrary. Repeated application of the formula $\hat{f}_k(u) = 2\pi\hat{f}(u)\hat{f}_{k-1}(u)$, $k \geq 2$, gives

$$\hat{f}_n(u) = 2\pi\hat{f}(u)\hat{f}_{n-1}(u) = 2\pi\hat{f}(u)\; 2\pi\hat{f}(u)\hat{f}_{n-2}(u) = (2\pi)^2(\hat{f}(u))^2\hat{f}_{n-2}(u) = \ldots$$

$$= (2\pi)^{n-1}(\hat{f}(u))^{n-1}\hat{f}_1(u) = (2\pi)^{n-1}(\hat{f}(u))^{n-1}\hat{f}(u) = (2\pi)^{n-1}(\hat{f}(u))^n.$$

Hence

$$\hat{f}_n(u) = (2\pi)^{n-1}(\hat{f}(u))^n = (2\pi)^{n-1}\left(\frac{1}{2}e^{-|u|}\right)^n = \pi^{n-1}\frac{1}{2}e^{-|nu|} = \pi^{n-1}\hat{f}(nu)$$

and therefore

$$f_n(x) = \pi^{n-1}\frac{1}{n}f\left(\frac{x}{n}\right) = \pi^{n-1}\frac{1}{n}\frac{1}{1+\dfrac{x^2}{n^2}} = \pi^{n-1}\frac{n}{n^2+x^2}.$$

3. a) By definition

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx,\ n \geq 0, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx,\ n \geq 1$$

and

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx,\ n \in \mathbf{Z}.$$

Hence

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,du = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\cos nx - i\sin nx)\,dx = \frac{1}{2}\big(a_n - ib_n\big),\ n \geq 1$$

and

$$c_{-n} = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{inx}\,du = \frac{1}{2\pi}\int_0^{2\pi} f(x)(\cos nx + i\sin nx)\,dx = \frac{1}{2}\big(a_n + ib_n\big),\ n \geq 1.$$

Also $c_0 = \frac{1}{2}a_0$. Consequently

$$\sum_{n=-N}^{N} c_n e^{inx} = c_0 + \sum_{n=1}^{N}\big(c_n e^{inx} + c_{-n}e^{-inx}\big) =$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{N}\left(\frac{1}{2}\big(a_n - ib_n\big)\big(\cos nx + i\sin nx\big) + \frac{1}{2}\big(a_n + ib_n\big)\big(\cos nx - i\sin nx\big)\right) =$$

$$= \frac{1}{2}a_0 + \sum_{n=1}^{N}\big(a_n\cos nx + b_n\sin nx\big) \quad \text{for all integers } N \geq 1 \text{ and all } x \in \mathbf{R}.$$

b) The complex Fourier series coefficients of $f(x)$ are

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-iax}e^{-inx}\,dx = \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n+a)x}\,dx =$$

$$= \frac{1}{2\pi}\left[\frac{e^{-i(n+a)x}}{-i(n+a)}\right]_0^{2\pi} = \frac{1}{2\pi i}\left(1 - e^{-i\,2\pi a}\right)\frac{1}{n+a}, \quad n \in \mathbf{Z}.$$

Hence

$$f(x) \sim \sum_{n=-\infty}^{\infty}\frac{1}{2\pi i}\left(1 - e^{-i\,2\pi a}\right)\frac{1}{n+a}e^{inx}$$

is the complex Fourier series expansion of $f(x)$. Let a_n, $n \geq 0$, and b_n, $n \geq 1$, be the Fourier series coefficients of $f(x)$. The function $f(x)$ is piecewise C^1 on $\mathbf{R}$. The Dirichlet convergence theorem for Fourier series therefore shows that the Fourier series of $f(x)$ converges for all $x \in \mathbf{R}$ and that its sum is equal to $\frac{1}{2}\big(f(x+) + f(x-)\big)$ for all $x \in \mathbf{R}$. That is $\frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{1}{2}\big(f(x+) + f(x-)\big)$ for all $x \in \mathbf{R}$, or equivalently

$$\frac{1}{2}a_0 + \sum_{n=1}^{N}(a_n\cos nx + b_n\sin nx) \to \frac{1}{2}\big(f(x+) + f(x-)\big) \quad \text{as } N \to \infty \text{ for all } x \in \mathbf{R}.$$

Using a) we get

$$(4) \qquad \sum_{n=-N}^{N}\frac{1}{2\pi i}\left(1 - e^{-i\,2\pi a}\right)\frac{1}{n+a}e^{inx} \to \frac{1}{2}\big(f(x+) + f(x-)\big) \quad \text{as } N \to \infty \text{ for all } x \in \mathbf{R}.$$

Applying (4) with $x = 0$ we get

$$\sum_{n=-N}^{N}\frac{1}{2\pi i}\left(1 - e^{-i\,2\pi a}\right)\frac{1}{n+a} \to \frac{1}{2}\left(1 + e^{-i\,2\pi a}\right) \quad \text{as } N \to \infty.$$

But

$$\pi i\,\frac{1+e^{-i\,2\pi a}}{1-e^{-i\,2\pi a}} = \pi i\,\frac{e^{i\,\pi a}+e^{-i\,\pi a}}{e^{i\,\pi a}-e^{-i\,\pi a}} = \pi i\,\frac{2\cos\pi a}{2i\,\sin\pi a} = \frac{\pi}{\cot\pi a}.$$

Hence

$$\sum_{n=-N}^{N}\frac{1}{n+a} \to \frac{\pi}{\cot\pi a} \quad \text{as } N\to\infty.$$

4. The corresponding Sturm-Liouville eigenvalue problem is $X''(x)+\lambda X(x)=0$, $0<x<1$, with boundary values $X(0)=0$ and $X'(1)=0$. The theory of Sturm-Liouville eigenvalue problems shows that any eigenvalue to this eigenvalue problem must be ≥ 0. The value $\lambda=0$ gives $X(x)=0$ so $\lambda=0$ is not an eigenvalue. Now suppose $\lambda>0$. Then $\lambda=\alpha^2$ where $\alpha>0$, and $X''(x)+\lambda X(x)=0$ gives $X(x)=A\sin\alpha x+B\cos\alpha x$. The boundary value $X(0)=0$ implies $B=0$. The boundary value $X'(1)=0$ then shows that there is a non-zero solution $X(x)$ if and only if $\cos\alpha=0$. Since $\alpha>0$ this happens if and only if $\alpha=\left(n+\frac{1}{2}\right)\pi$ and $n=0,1,\ldots$. The Sturm-Liouville problem here therefore has the eigenvalues $\lambda_n=\left(n+\frac{1}{2}\right)^2\pi^2$, $n=0,1,\ldots$ and the corresponding eigenfunctions $X_n(x)=\sin\left(n+\frac{1}{2}\right)\pi x$, $n=0,1,\ldots$. The eigenfunctions constitute an orthogonal system with respect to the inner product $\langle f,g\rangle=\int_0^1 f(x)\overline{g(x)}\,dx$. The corresponding norm is $\|f\|=\sqrt{\langle f,f\rangle}$. Since $\|X_n\|^2=\int_0^1\sin^2\left(n+\frac{1}{2}\right)\pi x\,dx=\frac{1}{2}$ for $n=0,1,\ldots$ the Fourier series expansion of a function $f(x)$ on $0\leq x\leq 1$ with respect to the orthogonal system $X_n(x)$ $n=0,1,2,\ldots$ is

$$f(x)\sim\sum_{n=0}^{\infty}\alpha_n X_n(x) \quad \text{where} \quad \alpha_n = 2\int_0^1 f(x)X_n(x)\,dx \;\; n=0,1,2,\ldots\,.$$

We now expand the sought function $u(x,y)$ in a Fourier series with respect to the orthogonal system $X_n(x)$ $n=0,1,2,\ldots$. From the general Sturm-Liouville theory we know that $u(x,y)$ then is equal to its Fourier series, that is

$$(5) \qquad u(x,y)=\sum_{n=0}^{\infty}Y_n(y)X_n(x) \text{ where } Y_n(y)=2\int_0^1 u(x,y)X_n(x)\,dx \;\; n=0,1,2,\ldots\,.$$

We also know that

$$u_{xx}(x,y)\sim-\sum_{n=0}^{\infty}\pi^2\left(n+\frac{1}{2}\right)^2 Y_n(y)X_n(x) \quad \text{and} \quad u_{yy}(x,y)\sim\sum_{n=0}^{\infty}Y_n''(y)X_n(x)$$

so

$$u_{xx}(x,y)+u_{yy}(x,y)\sim\sum_{n=0}^{\infty}\left(Y_n''(y)-\pi^2\left(n+\frac{1}{2}\right)^2 Y_n(y)\right)X_n(x).$$

But $u_{xx}(x,y)+u_{yy}(x,y)=0$. Hence

$$(6) \qquad Y_n''(y)-\pi^2\left(n+\frac{1}{2}\right)^2 Y_n(y)=0, \quad n=0,1,\ldots\,.$$

From (5) we further get

$$(7) \qquad u(x,0)=\sum_{n=0}^{\infty}Y_n(0)X_n(x) \quad \text{and} \quad u(x,1)=\sum_{n=0}^{\infty}Y_n(1)X_n(x).$$

We know that

$$u(x,0) = u(x,1) = x^2 - 2x. \tag{8}$$

In order to be able to exploit that we expand $f(x) = x^2 - 2x$ in a Fourier series with respect to the orthogonal system $X_n(x)$, $n = 0, 1, \ldots$. The Fourier series coefficients in that expansion are

$$2\int_0^1 f(x)X_n(x)\,dx = 2\int_0^1 (x^2-2x)\sin\left(n+\frac{1}{2}\right)\pi x\,dx =$$

$$= 2\left(\left[(x^2-2x)\left(-\frac{\cos\left(n+\frac{1}{2}\right)\pi x}{\left(n+\frac{1}{2}\right)\pi}\right)\right]_0^1 + \frac{2}{\left(n+\frac{1}{2}\right)\pi}\int_0^1 (x-1)\cos\left(n+\frac{1}{2}\right)\pi x\,dx\right) =$$

$$= \frac{4}{\left(n+\frac{1}{2}\right)\pi}\left(\left[(x-1)\frac{\sin\left(n+\frac{1}{2}\right)\pi x}{\left(n+\frac{1}{2}\right)\pi}\right]_0^1 - \frac{1}{\left(n+\frac{1}{2}\right)\pi}\int_0^1 \sin\left(n+\frac{1}{2}\right)\pi x\,dx\right) =$$

$$= -\frac{4}{\left(n+\frac{1}{2}\right)^2\pi^2}\left[-\frac{\cos\left(n+\frac{1}{2}\right)\pi x}{\left(n+\frac{1}{2}\right)\pi}\right]_0^1 = \frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3}, \quad n = 0, 1, \ldots.$$

Thus

$$x^2 - 2x \sim \sum_{n=0}^{\infty} \frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3} X_n(x). \tag{9}$$

Comparison of (7), (8) and (9) gives

$$Y_n(0) = Y_n(1) = \frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3}, \quad n = 0, 1, \ldots. \tag{10}$$

From (6) we get

$$Y_n(y) = A_n e^{\left(n+\frac{1}{2}\right)\pi y} + B_n e^{-\left(n+\frac{1}{2}\right)\pi y} \quad n = 0, 1, \ldots.$$

Using (10) we then get

$$A_n + B_n = \frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3} \quad \text{and} \quad A_n e^{\left(n+\frac{1}{2}\right)\pi} + B_n e^{-\left(n+\frac{1}{2}\right)\pi} = \frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3} \quad n = 0, 1, \ldots$$

which gives

$$A_n = \frac{1}{1+e^{\left(n+\frac{1}{2}\right)\pi}}\frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3} \quad \text{and} \quad B_n = \frac{1}{1+e^{-\left(n+\frac{1}{2}\right)\pi}}\frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3} \quad n = 0, 1, \ldots.$$

The sought solution $u(x,y) = \sum_{n=0}^{\infty} Y_n(y)X_n(x)$ therefore is

$$u(x,y) = \sum_{n=0}^{\infty} \frac{4}{\left(n+\frac{1}{2}\right)^3\pi^3}\left(\frac{e^{\left(n+\frac{1}{2}\right)\pi y}}{1+e^{\left(n+\frac{1}{2}\right)\pi}} + \frac{e^{-\left(n+\frac{1}{2}\right)\pi y}}{1+e^{-\left(n+\frac{1}{2}\right)\pi}}\right)\sin\left(n+\frac{1}{2}\right)\pi x.$$

5. For $u \neq 0$ we get

$$\frac{3}{4}\int_0^1 \cos ux\,dx - \frac{1}{4}\int_0^3 \cos ux\,dx = \frac{3}{4}\left[\frac{\sin ux}{u}\right]_{x=0}^{x=1} - \frac{1}{4}\left[\frac{\sin ux}{u}\right]_{x=0}^{x=3} =$$

$$= \frac{1}{u}\left(\frac{3}{4}\sin u - \frac{1}{4}\sin 3u\right).$$

But

$$\sin^3 u = \left(\frac{e^{iu} - e^{-iu}}{2i}\right)^3 = -\frac{1}{8i}\left(e^{i\,3u} - 3e^{iu} + 3e^{-iu} - e^{-i\,3u}\right) =$$

$$= \frac{3}{4}\,\frac{e^{iu} - e^{-iu}}{2i} - \frac{1}{4}\,\frac{e^{i\,3u} - e^{-i\,3u}}{2i} = \frac{3}{4}\sin u - \frac{1}{4}\sin 3u \quad \text{for all } u \in \mathbf{R}.$$

Hence

$$\frac{3}{4}\int_0^1 \cos ux\,dx - \frac{1}{4}\int_0^3 \cos ux\,dx = \frac{\sin^3 u}{u} \quad \text{for } u \neq 0.$$

For $u = 0$ we get

$$\frac{3}{4}\int_0^1 \cos ux\,dx - \frac{1}{4}\int_0^3 \cos ux\,dx = \frac{3}{4}\int_0^1 dx - \frac{1}{4}\int_0^3 dx = 0.$$

Set

$$f(x) = \begin{cases} 0, & x < -3 \\ -\frac{1}{4}, & -3 \leq x < -1 \\ \frac{1}{2}, & -1 \leq x \leq 1 \\ -\frac{1}{4}, & 1 < x \leq 3 \\ 0, & x > 3. \end{cases}$$

Then $f(x)$ is an an even function and

$$\frac{3}{4}\int_0^1 \cos ux\,dx - \frac{1}{4}\int_0^3 \cos ux\,dx = \frac{1}{2}\int_0^1 \cos ux\,dx - \frac{1}{4}\int_1^3 \cos ux\,dx = \int_0^3 f(x)\cos ux\,dx =$$

$$= \frac{1}{2}\int_{-3}^3 f(x)\cos ux\,dx = \frac{1}{2}\int_{-3}^3 f(x)\big(\cos ux - i\sin ux\big)\,dx = \frac{1}{2}\int_{-3}^3 f(x)e^{-iux}\,dx =$$

$$= \pi\,\frac{1}{2\pi}\int_{-3}^3 f(x)e^{-iux}\,dx = \pi\,\frac{1}{2\pi}\int_{-\infty}^\infty f(x)e^{-iux}\,dx = \pi\,\hat{f}(u).$$

That is

$$\frac{\sin^3 u}{u} = \pi\,\hat{f}(u) \quad \text{if } u \neq 0. \tag{11}$$

For an arbitrary number $T > 0$, an arbitrary $x \in \mathbf{R}$ and with the use of (11) we get

$$\int_0^T \frac{\sin^3 u}{u}\cos ux\,dx = \frac{1}{2}\int_{-T}^T \frac{\sin^3 u}{u}\cos ux\,dx = \frac{1}{2}\int_{-T}^T \frac{\sin^3 u}{u}\big(\cos ux + i\sin ux\big)\,dx = \tag{12}$$

$$= \frac{1}{2}\int_{-T}^{T} \frac{\sin^3 u}{u} e^{iux}\, dx = \frac{\pi}{2}\int_{-T}^{T} \hat{f}(u) e^{iux}\, dx.$$

But $f(x)$ is piecewise C^1 on $\mathbf{R}$ and therefore

$$\int_{-T}^{T} \hat{f}(u) e^{iux}\, dx \to \frac{1}{2}\big(f(x+) + f(x-)\big) \quad \text{as } T \to \infty \text{ for each } x \in \mathbf{R} \tag{13}$$

by the inversion formula for the Fourier transform. Combining (12) and (13) it follows that

$$\int_{0}^{T} \frac{\sin^3 u}{u} \cos ux\, dx \to \frac{\pi}{4}\big(f(x+) + f(x-)\big) \quad \text{as } T \to \infty \text{ for each } x \in \mathbf{R}.$$

That is the generalized integral

$$\int_{0}^{\infty} \frac{\sin^3 u}{u} \cos ux\, dx$$

is convergent for each $x \in \mathbf{R}$ and

$$\int_{0}^{\infty} \frac{\sin^3 u}{u} \cos ux\, dx = \frac{\pi}{4}\big(f(x+) + f(x-)\big) = \begin{cases} 0, & x < -3 \\ -\frac{\pi}{16}, & x = -3 \\ -\frac{\pi}{8}, & -3 < x < -1 \\ \frac{\pi}{16}, & x = -1 \\ \frac{\pi}{8}, & -1 < x < 1 \\ \frac{\pi}{16}, & x = 1 \\ -\frac{\pi}{8}, & 1 < x < 3 \\ -\frac{\pi}{16}, & x = 3 \\ 0, & x > 3. \end{cases}$$

Finally using (11) and the Plancheral identity we get

$$\int_{0}^{\infty} \frac{\sin^6 u}{u^2}\, du = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^6 u}{u^2}\, du = \frac{1}{2}\int_{-\infty}^{\infty} \left|\frac{\sin^3 u}{u}\right|^2 du = \frac{\pi^2}{2}\int_{-\infty}^{\infty} \left|\hat{f}(u)\right|^2 du =$$

$$= \frac{\pi}{4}\int_{-\infty}^{\infty} |f(x)|^2\, dx = \frac{\pi}{4}\int_{-3}^{3} |f(x)|^2\, dx = \frac{\pi}{2}\int_{0}^{3} |f(x)|^2\, dx =$$

$$= \frac{\pi}{2}\left(\frac{1}{4} + 2\cdot\frac{1}{16}\right) = \frac{3\pi}{16}.$$

6. a) Let $f(x) \in A$ and let $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the complex Fourier series expansion of $f(x)$. Then $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ by the definition of A. But $\left|c_n e^{inx}\right| = |c_n|$ for all $n \in \mathbf{Z}$ and all $x \in \mathbf{R}$. Hence $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ converges absolutely and uniformly on $\mathbf{R}$. Set $g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for $x \in \mathbf{R}$. Each function $c_n e^{inx}$ is continuous on $\mathbf{R}$. An infinite sum of continuous functions on $\mathbf{R}$ converging uniformly on $\mathbf{R}$ is continuous on $\mathbf{R}$. Hence $g(x)$ is continuos on $\mathbf{R}$. The function $g(x)$ is also 2π-periodic on $\mathbf{R}$ since each function $c_n e^{inx}$ is 2π-periodic on $\mathbf{R}$. The m-th complex Fourier series coefficient of $g(x)$ is

$$\frac{1}{2\pi}\int_0^{2\pi} g(x)e^{-imx}\,dx = \frac{1}{2\pi}\int_0^{2\pi}\left(\sum_{n=-\infty}^{\infty} c_n e^{inx}\right)e^{-imx}\,dx =$$

$$= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}\,dx = c_m.$$

The interchange of integration and summation above is justified by the uniform convergence of the sum on $\mathbf{R}$. Above we have also used that

$$\frac{1}{2\pi}\int_0^{2\pi} e^{i(n-m)x}\,dx = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m. \end{cases}$$

The functions $f(x)$ and $g(x)$ thus have the same complex Fourier series coefficients. Consequently all complex Fourier series coefficients of $f(x) - g(x)$ are zero. Hence

$$\int_0^{2\pi} |f(x) - g(x)|^2\,dx = 0$$

by the Parseval theorem. Since $f(x) - g(x)$ is continuous on $\mathbf{R}$ ($f(x)$ is continuous on $\mathbf{R}$ since $f(x) \in A$) it follows that $f(x) = g(x)$ on $0 \le x \le 2\pi$ and consequently on $\mathbf{R}$ because of the 2π-periodicity of both functions. Hence $f(x) = g(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$ for all $x \in \mathbf{R}$.

b) Let $f(x)$ be a complex-valued 2π-periodic continuous function $f(x)$ on $\mathbf{R}$ and assume that $f(x)$ has a piecewise continuous derivative on $\mathbf{R}$ Let $f(x) \sim \sum_{n=-\infty}^{\infty} c_n e^{inx}$ be the complex Fourier series expansion of $f(x)$. The n-th complex Fourier series coefficient of $f'(x)$ then is

$$\frac{1}{2\pi}\int_0^{2\pi} f'(x)e^{-inx}\,dx = \frac{1}{2\pi}\left(\left[f(x)e^{-inx}\right]_0^{2\pi} - \int_0^{2\pi} f(x)\left(-ine^{-inx}\right)\,dx\right) =$$

$$= in\,\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}\,dx = in\,c_n.$$

Hence $f'(x) \sim \sum_{n=-\infty}^{\infty} in\,c_n e^{inx}$ is the complex Fourier series expansion of $f'(x)$. Using the Cauchy-Schwarz inequality and Bessel's inequality for $f'(x)$ we get

$$\sum_{\substack{-N\le n\le N\\ n\neq 0}} |c_n| = \sum_{\substack{-N\le n\le N\\ n\neq 0}} \frac{1}{|n|}\,|nc_n| \le \left(\sum_{\substack{-N\le n\le N\\ n\neq 0}} \frac{1}{n^2}\right)^{1/2}\left(\sum_{\substack{-N\le n\le N\\ n\neq 0}} n^2\,|c_n|^2\right)^{1/2} \le$$

$$\le \left(2\sum_{n=1}^{\infty}\frac{1}{n^2}\right)^{1/2}\left(\frac{1}{2\pi}\int_0^{2\pi}|f'(x)|^2\,dx\right)^{1/2} \quad \text{for all integers } N \ge 1.$$

Thus $\sum_{n=-\infty}^{\infty} |c_n| < \infty$ since $\sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$. Consequentlly $f(x) \in A$.

c) We first note that

$$h(x+2\pi) = \int_{-\pi}^{\pi} f(x+2\pi-y)g(y)\,dy = \int_{-\pi}^{\pi} f(x-y)g(y)\,dy = h(x) \quad \text{for all } x \in \mathbf{R}$$

since $f(x)$ is 2π-periodic. Hence $h(x)$ is 2π-periodic. We now show that $h(x)$ is continuous on $\mathbf{R}$. This can be proved in different ways. Here we give a Fourier analysis proof. Fix an arbitrary $x \in \mathbf{R}$ and let $\delta \in \mathbf{R}$. Then

$$\text{(14)} \qquad |h(x+\delta)-h(x)| = \left| \int_{-\pi}^{\pi} f(x+\delta-y)g(y)\,dy - \int_{-\pi}^{\pi} f(x-y)g(y)\,dy \right| =$$

$$= \left| \int_{-\pi}^{\pi} \big(f(x-y+\delta)-f(x-y)\big)g(y)\,dy \right| \leq \int_{-\pi}^{\pi} \big|f(x-y+\delta)-f(x-y)\big|\,\big|g(y)\big|\,dy \leq$$

$$\leq \left(\int_{-\pi}^{\pi} \big|f(x-y+\delta)-f(x-y)\big|^2\,dy \right)^{1/2} \left(\int_{-\pi}^{\pi} \big|g(y)\big|^2\,dy \right)^{1/2}$$

using the triangle inequality and the Cauchy-Schwarz inequality for integrals. Also

$$\text{(15)} \quad \int_{-\pi}^{\pi} \big|f(x-y+\delta)-f(x-y)\big|^2\,dy = \int_{x-\pi}^{x+\pi} \big|f(u+\delta)-f(u)\big|^2\,du = \int_{-\pi}^{\pi} \big|f(u+\delta)-f(u)\big|^2\,du$$

since $f(x)$ is 2π-periodic. Let $f(u) \sim \sum_{n=-\infty}^{\infty} a_n e^{inu}$ be the complex Fourier series expansion of $f(u)$. Then $f(u+\delta)-f(u) \sim \sum_{n=-\infty}^{\infty} a_n \left(e^{in\delta}-1\right) e^{inu}$ is the complex Fourier series expansion of $f(u+\delta)-f(u)$. Applying the Parseval identity to $f(u)$ and $f(u+\delta)-f(u)$ we get

$$\text{(16)} \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(u)|^2\,du = \sum_{n=-\infty}^{\infty} |a_n|^2$$

and

$$\text{(17)} \qquad \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(u+\delta)-f(u)|^2\,du = \sum_{n=-\infty}^{\infty} |a_n|^2 \left| e^{in\delta}-1 \right|.$$

Since $\left|e^{in\delta}-1\right| \leq 2$ for all $n \in \mathbf{Z}$ and all $\delta \in \mathbf{R}$ it follows from (16) that $\sum_{n=-\infty}^{\infty} |a_n|^2 \left|e^{in\delta}-1\right|$ converges uniformly on $\mathbf{R}$. Since each function $|a_n|^2 \left|e^{in\delta}-1\right|$ is a continuous function of δ on $\mathbf{R}$ it follows that $\sum_{n=-\infty}^{\infty} |a_n|^2 \left|e^{in\delta}-1\right|$ is a continuous function of δ on $\mathbf{R}$. Hence

$$\text{(18)} \qquad \sum_{n=-\infty}^{\infty} |a_n|^2 \left|e^{in\delta}-1\right| \to \sum_{n=-\infty}^{\infty} |a_n|^2 \left|e^{0}-1\right| = 0 \quad \text{as } \delta \to 0.$$

From (14), (15), (17) and (18) we see that $h(x+\delta) \to h(x)$ as $\delta \to 0$ so $h(x)$ is continuous on $\mathbf{R}$.

We now complete the proof that $h(x) \in A$ by proving the absolute convergence of the complex Fourier series coefficients of $h(x)$. Let c_n, $n \in \mathbf{Z}$, be the complex Fourier series coefficients of

$h(x)$, let b_n, $n \in \mathbf{Z}$, be the complex Fourier series coefficients of $g(x)$, and as above let a_n, $n \in \mathbf{Z}$, be the complex Fourier series coefficients of $f(x)$. Then

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} h(x)e^{-inx}\,dx = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(x-y)g(y)\,dy\right)e^{-inx}\,dx =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(x-y)e^{-in(x-y)}\,dx\right)g(y)e^{-iny}\,dy =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi+y}^{\pi+y} f(x)e^{-in(x)}\,dx\right)g(y)e^{-iny}\,dy =$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\int_{-\pi}^{\pi} f(x)e^{-in(x)}\,dx\right)g(y)e^{-iny}\,dy =$$

$$= 2\pi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-in(x)}\,dx\right)\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} g(y)e^{-iny}\,dy\right) = 2\pi\, a_n b_n \text{ for all } n \in \mathbf{Z}.$$

Using this, the Cauchy-Schwarz inequality and Bessel's inequality for $f(x)$ and $g(x)$ we get

$$\sum_{n=-N}^{N} |c_n| = 2\pi \sum_{n=-N}^{N} |a_n|\,|b_n| \leq 2\pi\left(\sum_{n=-N}^{N} |a_n|^2\right)^{1/2}\left(\sum_{n=-N}^{N} |b_n|^2\right)^{1/2} \leq$$

$$\leq \left(\int_{-\pi}^{\pi} |f(x)|^2\,dx\right)^{1/2}\left(\int_{-\pi}^{\pi} |g(x)|^2\,dx\right)^{1/2} \quad \text{for all integers } N \geq 1.$$

Hence $\sum_{n=-\infty}^{\infty} |c_n| < \infty$. Thus $h(x) \in A$.