MATEMATISKA INSTITUTIONEN STOCKHOLMS UNIVERSITET Avd. Matematik.
Examinator: Leo Tzou
Tentamensskrivning i Linjär Analys den 18-01-2014.
No books, no notes, no calculators.

1. Consider the function defined by

$$f(x) := \begin{cases} 1 & \text{if } -\pi \le x \le 0 \\ sinx & \text{if } 0 < x \le \pi \end{cases}$$

a) Write out the Fourier Series of f, namely $\frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx)$. (10 pt)
The Fourier coefficients are defined by

$$a_n = \frac{2}{\pi}(\int_{-\pi}^{0} cos(nx)dx + \int_{0}^{\pi} cos(nx)sin(x)dx)$$
$$b_n = \frac{2}{\pi}(\int_{-\pi}^{0} sin(nx)dx + \int_{0}^{\pi} sin(nx)sin(x)dx)$$

and compute the integrals.

b) Draw the graph of the function $g(x) := \frac{a_0}{2} + \sum\limits_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx)$ on the interval $[-2\pi, 2\pi]$. Pay close attention to jump points. (10 pt)
The graph of $g(x)$ is exactly the same as the 2π periodic extension of $f(x)$ except for where the extension is discontinuous. At the points of discontinuity $g(x)$ is the average value of its left and right limits.

2. Consider the function $F(x) := x^2$ on the interval $(-\pi, \pi)$.
a) Show that

$$F(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx.$$

Remember to argue why $F(x)$ is EQUAL to the infinite sum point-wise rather than just having the sum as a Fourier series representation. (10 pt)
The function is continuous on $(-\pi, \pi)$ so the Fourier series coincides with the function. Since $F(x)$ is even the sine coefficients vanish and therefore we only have the cosine coefficients which are given by

$$a_n = \frac{2}{\pi}\int_{-\pi}^{\pi} x^2 cos(nx)dx.$$

We have that $a_0 = 2\pi^2/3$ and $a_n = 4\frac{(-1)^n}{n^2}$.

b) Use part a) to compute the sum $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^4}$. (10 pt)
The first sum is obtained by plugging in $x = \pi$. The second sum is obtained by using parseval's identity which gives

$$\int_{-\pi}^{\pi} x^4 = \int_{-\pi}^{\pi} F(x)^2 = \frac{\pi^4}{9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

3. Solve the initial value problem in the set $0 \le x \le \pi$, $t \ge 0$. (20 pt)

$$u_t - u_{xx} = 0, u(0,t) = u_x(\pi, t) = 0, u(x,0) = sin\frac{3x}{2}$$

We use ansatz $u(x,t) = \sum_{n=0}^{\infty} a_n(t)cos(\lambda_n x) + b_n(t)sin(\lambda_n x)$. The boundary condition $u(0,t) = 0$ forces $a_n(t) = 0$ for all n. Now plug in $u_x(\pi, t) = 0$ we obtain that $cos(\lambda_n \pi) = 0$ which forces $\lambda_n = \frac{1}{2} + n$.
So our ansatz is $u(x,t) = \sum_{n=1}^{\infty} b_n(t)sin((\frac{1}{2} + n)x)$. Plug this into $u_t = u_{xx}$ we get that for each n

$$b_n'(t) = -(1/2 + n)^2 b_n(t)$$

which gives the solution $b_n(t) = b_n(0)e^{-(1/2+n)^2 t}$. So $u(x,t) = \sum_{n=1}^{\infty} b_n(0)e^{-(1/2+n)^2 t} sin((\frac{1}{2}+n)x)$. We now see that initial condition $u(x,0) = sin(3/2x)$ forces $b_n(0) = 0$ for all n except for $b_n(0) = 1$.

4. Let $h(x) := \sum_{n=1}^{\infty} \frac{cos(nx)}{n^2}$. Is this function continuous? (10 pt)
Yes. Since $|\frac{cos(nx)}{n^2}| \leq \frac{1}{n^2}$ which is a convergent sum, we can use the M-test to show that $\sum_{n=1}^{\infty} \frac{cos(nx)}{n^2}$ converges uniformly. Uniform convergent sum of continuous functions are continuous.

5. Prove that for $x > 0$ the function $e^{-x}e^{ix}$ can be expressed as

$$e^{-x}e^{ix} = c\int_{-\infty}^{\infty} e^{iwx}\frac{1}{w^2 - 2w + 2}dw$$

for some constant c. (20 pt)
We define $f(x) = e^{-|x|}e^{ix}$. This function is continuous and integrable so we can compute its Fourier transform by

$$\mathcal{F}(f)(w) = c\int_{-\infty}^{\infty} e^{-|x|}e^{ix}e^{-ixw}dx = c\frac{1}{w^2 - 2w + 2}$$

for some constant c. The inverse Fourier transform formula then gives the expression for $f(x)$ for $x \geq 0$.

6. Let X be the space of functions defined by

$$X := \{u \in C^1([-\pi,\pi]) \mid \int_{-\pi}^{\pi} u(x)sin(nx)dx = 0 \quad \forall n \in \mathbb{N}\}.$$

a) Given $F \in C^1([-\pi,\pi])$, solve the minimization problem

$$\inf_{u \in X} \int_{-\pi}^{\pi} |u - F|^2 dx.$$

That is, find $\tilde{u} \in X$ such that

$$\int_{-\pi}^{\pi} |\tilde{u} - F|^2 dx \leq \int_{-\pi}^{\pi} |u - F|^2 dx$$

for all $u \in X$. Justify your answer.(hint: think Parseval). (8 pt)
Let u be a function in X. Then by the fact that $\int_{-\pi}^{\pi} u(x)sin(nx) = 0$, we have that $u(x) = a_0/2 + \sum_{n=1}^{\infty} a_n cos(nx)$. Now let $F \in C^1[-\pi,\pi]$ have Fourier series given by $F = \hat{a}_0/2 + \sum_{n=1}^{\infty} \hat{b}_n sin(nx) + \hat{a}_n cos(nx)$. By Parseval's identity we have now for any u in X

$$\int_{-\pi}^{\pi} |u - F|^2 dx = |a_0 - \hat{a}_0|^2/2 + \sum_{n=1}^{\infty} |\hat{b}_n|^2 + |a_n - \hat{a}_n|^2.$$

Therefore the optimal $\tilde{u} \in X$ would have Fourier series $\tilde{a}_n = \hat{a}_n$.

b) Express the solution $\tilde{u}$ of the above minimization problem in a formula containing only the function F. (2 pt)
We showed above that the optimal solution $\tilde{u}$ would have Fourier expansion

$$\tilde{u}(x) = \hat{a}_0/2 + \sum_{n=1}^{\infty} \hat{a}_n cos(nx)$$

where $\hat{a}_n$ are the cosine coefficients of F. This is precisely the even part of F. Namely $\tilde{u}(x) = \frac{F(x)+F(-x)}{2}$.