MATEMATISKA INSTITUTIONEN STOCKHOLMS UNIVERSITET Avd. Matematik.
Examinator: Leo Tzou
Tentamensskrivning i Linjär Analys den 22-02-2014.
No books, no notes, no calculators.

1. Consider the function defined by

$$f(x) := \begin{cases} 0 & \text{if } -\pi \leq x \leq 0 \\ cosx & \text{if } 0 < x \leq \pi \end{cases}$$

a) Write out the Fourier Series of f, namely $\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx)$. (10 pt)

b) Draw the graph of the function $g(x) := \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(nx) + b_n sin(nx)$ on the interval $[-2\pi, 2\pi]$. Pay close attention to jump points. (10 pt)

2. Consider the function $F(x) := x/2$ on the interval $(-\pi, \pi)$.
a) Show that

$$F(x) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

b) Use part a) to compute the sum $\sum_{n=1}^{\infty} \frac{1}{n^2}$. (10 pt)

3. Solve the initial value problem in the set $0 \leq x \leq \pi$, $t \geq 0$. (20 pt)

$$u_t - u_{xx} = -1, u(0,t) = 0, u_x(\pi,t) = \pi, u(x,0) = sin\frac{3x}{2} + \frac{x^2}{2}$$

4. Let $h(x) := \sum_{n=1}^{\infty} \frac{sin(nx)}{n^{\frac{3}{2}}}$. Is this function continuous? (10 pt)
5. Prove that for $x > 0$ the function $e^{-x}e^{-ix}$ can be expressed as

$$e^{-x}e^{-ix} = c\int_{-\infty}^{\infty} e^{iwx} \frac{1}{w^2 + 2w + 2} dw$$

for some constant c. (20 pt)
6. Let X be the space of functions defined by

$$X := \{u \in C^1([-\pi, \pi]) \mid \int_{-\pi}^{\pi} u(x)cos(nx)dx = 0 \quad \forall n \in \mathbb{N}\}.$$

a) Given $F \in C^1([-\pi, \pi])$, solve the minimization problem

$$\inf_{u \in X} \int_{-\pi}^{\pi} |u - F|^2 dx.$$

That is, find $\tilde{u} \in X$ such that

$$\int_{-\pi}^{\pi} |\tilde{u} - F|^2 dx \leq \int_{-\pi}^{\pi} |u - F|^2 dx$$

for all $u \in X$. Justify your answer.(hint: think Parseval). (8 pt)

b) Express the solution $\tilde{u}$ of the above minimization problem in a formula containing only the function F. (2 pt)