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Written Exam
Logic II
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The maximum score on this written exam is 40 points (p). Grading (after inclusion of bonus points): A requires at least 32 p, B at least 28 p, C at least 22 p, D at least 18 p and E at least 16 p. Maximum score for each problem is indicated below.

The allowed time for the exam is five hours. No aids are permitted except paper and pen. Write clearly and motivate your answers carefully.

1. Define primitive recursive functions $q, r\colon \mathbb{N}^2 \to \mathbb{N}$ that computes quotient and remainder of two numbers, i.e. such that $x = q(x,y)y + r(x,y)$ with $r(x,y) < y$ for all $y > 0$, and $q(x,0) = 0$ and $r(x,0) = x$. (5p)

2. *(Un)decidable properties of partial recursive functions.* Determine (with proof) whether each of the following sets is recursive or not.

   (a) $\{x \in \mathbb{N} : (\exists y)\phi^1_x(y)\downarrow\}$

   (b) $\{x \in \mathbb{N} : W_x \text{ is infinite}\}$

   (c) $\{(x,y,n) \in \mathbb{N}^3 :$ Turing machine with index x stops on input y at the nth step $\}$

   Hint: Rice's theorem may be useful. (5p)

3. Show that if a theory T has only finite models, then there is $n \in \omega$ such that every model of T has cardinality less than n. (5 p)

4. State the three main equivalent forms of the axiom choice (AC, Zorn's lemma and the well-ordering principle). Prove that AC is equivalent to: for every non-empty set A and map $f\colon A \to B$, there is $g\colon B \to A$ such that $fgf = f$. (5 p)

5. Calculate the cardinalities (you may assume the axiom of choice) of the following sets and order them according to size :

   (a) $S_1 = \mathbb{R}^{\mathbb{R}}$

   (b) $S_2 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is continuous}\}$ (Hint: such continuous functions are determined by their values for rational numbers.)

   (c) $S_3 = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a bijection}\}$.

(d) Use the Generalized Continuum Hypothesis to state the above cardinalities on the form $\aleph_\alpha$. (5 p)

6. Define a partial recursive function $f\colon \mathbb{N} \to \mathbb{N}$ that cannot be extended to a total recursive one, i.e. such that for every total recursive g there is $x \in \mathrm{dom}(f)$ such that $g(x) \neq f(x)$. (Hint: use the enumeration theorem and a diagonal argument.) (5p)

7. Give the definition of complete theory, then show that a theory T is complete if and only if

   (a) there is no formula F such that $T \vdash F$ and $T \vdash \neg F$, and

   (b) for every closed formula F, $T \vdash F$ or $T \vdash \neg F$.

   (5p)

8. Let T be a finite set of closed formulas in the language of arithmetic. Prove that the theory

$$\mathrm{PA} \cup T$$

   is incomplete or inconsistent. Show that this can be false if T is infinite. (5p)
