Instructions: Work alone. You are not allowed to use the textbook and the class notes. You can quote results that you learned in the class. Be sure to state clearly what results you are using.

Justify all your answers with a proof or a counterexample. A simple Yes or No answer, even if correct, may get partial or no credit.

Problems have multiple parts. In some cases, later parts depend on earlier ones. Even if you could not do the earlier parts, you **are allowed** to use the results of the earlier parts in the later parts.

1. Let τ be the following set of subsets of the real line $\mathbb{R}$: A set $U \subset \mathbb{R}$ is in τ if and only if one of the following conditions holds

   1. U is empty,
   2. $3 \in U$.

   You may take for granted that τ defines a topology on $\mathbb{R}$. Throughout this problem, let $\mathbb{R}$ denote the real line with the standard topology, and $\mathbb{R}_\tau$ denote the real line with the topology τ. Remember to justify your answers.

   (a) [1 pt] Is $\mathbb{R}_\tau$ compact?

   Solution: No. For all $3 \neq x \in \mathbb{R}$, let $U_x = \{3, x\}$. Then $\{U_x \mid x \in \mathbb{R} \setminus \{3\}\}$ is an open cover of $\mathbb{R}_\tau$, that has no finite subcover. Indeed, it has no proper subcover.

   (b) [2 pts] Is $\mathbb{R}_\tau$ path-connected?

   Solution: Yes. It is enough to show that for every $x \neq 3$, there is a path from 3 to x. By composition of paths, it will follow that every two points can be connected by a path.
   Let us define $\alpha\colon [0,1] \to \mathbb{R}_\tau$ as follows

   $$\alpha(t) = \begin{cases} 3 & t \in [0,1) \\ x & t = 1 \end{cases}$$

   We need to check that α is continuous. Let U be an open set of $\mathbb{R}_\tau$. We need to prove that $\alpha^{-1}(U)$ is open in $[0,1]$. If U is empty then it is clear. Suppose U is non-empty. Then $3 \in U$. If $x \notin U$ then $\alpha^{-1}(U) = [0,1)$. If $x \in U$ then $\alpha^{-1}(U) = [0,1]$. In all cases, $\alpha^{-1}(U)$ is open.

   (c) [1 pt] Is $\mathbb{R}_\tau$ first-countable?

   Solution: Yes. Every point $x \in \mathbb{R}_\tau$ has a local basis consisting of a single open set: $\{3\} \cup \{x\}$. In particular, it has a countable local basis.

   (d) [2 pts] Is $\mathbb{R}_\tau$ second-countable?

   Solution: No. I claim that every basis of $\mathbb{R}_\tau$ must contain the set $\{3\} \cup \{x\}$, for every $x \in \mathbb{R}_\tau$. In particular every basis is uncountable. Indeed for every x, there has to be a basis element containing x and contained in $\{3\} \cup \{x\}$. But the only open set satisfying these two conditions is $\{3\} \cup \{x\}$.

(e) [2 pts] Describe all the continuous maps $\mathbb{R}_\tau \to \mathbb{R}$.

Solution: I claim that a function $f\colon \mathbb{R}_\tau \to \mathbb{R}$. is continuous if and only if it is constant. Proof: only the "only if" part need proof. Suppose $f\colon \mathbb{R}_\tau \to \mathbb{R}$ is a non-constant map. Let $x = f(3)$. Since f is not constant, $f^{-1}(\mathbb{R} \setminus \{x\})$ is a non-empty subset of $\mathbb{R}_\tau \setminus \{3\}$. By the definition of τ, such a set can not be open. But this means that f is not continuous.

2. Let $f\colon X \to Y$ be a map (it goes without saying that a "map" is continuous). Recall that a retraction of f is a map $r\colon Y \to X$ such that $r \circ f = 1_X$.

(a) [3 pts] Suppose f has a retraction, and Y is Hausdorff. Prove that that $f(X)$ is a closed subset of Y.

Solution: Define the map $\nabla\colon Y \to Y \times Y$ by the formula $\nabla(y) = (y, f(r(y)))$. Let $Y_\Delta \subset Y \times Y$ be the diagonal. This means that $Y_\Delta = \{(y, y) \in Y \times Y \mid y \in Y\}$. I claim that $f(X) = \nabla^{-1}(Y_\Delta)$. Since Y is Hausdorff, Y_Δ is a closed subset of $Y \times Y$, and the desired result follows. It remains to prove the claim.

Suppose first that $y \in f(X)$. This means that $y = f(x)$ for some $x \in X$. Then

$$\nabla(y) = (y, f(r(y))) = (f(x), f(r(f(x)))) = (f(x), f(x)) \in Y_\Delta,$$

and thus $y \in \nabla^{-1}(Y_\Delta)$. We have proved that $f(X) \subset \nabla^{-1}(Y_\Delta)$.

Now suppose that $y \in \nabla^{-1}(Y_\Delta)$. This means that $y = f(r(y))$, so $y \in f(X)$. We proved that $\nabla^{-1}(Y_\Delta) \subset f(X)$.

(b) [2 pts] Under the same conditions as in part (a), prove that f is a closed map. Remember: you are allowed to use the conclusion of part (a) even if you did not do part (a).

Solution: Let $C \subset X$ be a closed subset. We need to prove that $f(C)$ is a closed subset of Y. I claim that $f(C) = r^{-1}(C) \cap f(X)$. Since $r^{-1}(C)$ is closed by continuity of r and $f(X)$ is closed by part (a), it follows that $f(X)$ is closed. It remains to prove the claim.

Let us prove first that $f(C) \subset r^{-1}(C) \cap f(X)$. We need to show that $f(C) \subset r^{-1}(C)$ and $f(C) \subset f(X)$. The first inclusion follows from the relation $r(f(x)) = x$ and the second inclusion follows from the fact that $C \subset X$. Now let us prove that $r^{-1}(C) \cap f(X) \subset f(C)$. Let $y \in r^{-1}(C) \cap f(X)$. Then $y = f(x)$ for some $x \in X$, and $r(y) \in C$, so $r(f(x)) \in C$. But $r(f(x)) = x$, so $x \in C$ and $y = f(x)$ is in $f(C)$.

(c) [1 pt] Show that if Y is not Hausdorff, then the conclusion of part (a) is not necessarily true.

Solution: Let Y be any set with at least two points, endowed with the trivial topology. Let $X \subset Y$ be a non-empty proper subspace. Then the inclusion has a retraction, but X is not a closed subset of Y.

3. [4 pts] Let X be a space and $A \subset X$ a subspace. Suppose $C \subset X$ is a connected subspace, such that $C \cap A$ and $C \cap (X \setminus A)$ are both non-empty. Prove that $C \cap \partial A$ is non-empty. Here $\partial A = \bar{A} \cap \overline{X \setminus A}$ is the boundary of A.

Solution: I claim that C is not contained in $\text{int}(A) \cup \text{int}(X \setminus A)$. Indeed, if $C \subset \text{int}(A)$, this contradicts the assumption that $C \cap X \setminus A$ is non-empty. Similarly, if $C \subset \text{int}(X \setminus A)$, this contradicts that $X \cap A$ is non-empty. Finally, if $C \subset \mathrm{int}(A) \cup \mathrm{int}(X \setminus A)$, but is not contained in either of the two parts of the union, then $C \cap \mathrm{int}(A)$ and $C \cap \mathrm{int}(X \setminus A)$ would give a separation of C, contradicting the assumption that C is connected.

We have shown that $C \cap (X \setminus (\mathrm{int}(A) \cup \mathrm{int}(X \setminus A)))$ is non-empty. But $X \setminus (\mathrm{int}(A) \cup \mathrm{int}(X \setminus A))$ is precisely ∂A. Indeed

$$X \setminus (\mathrm{int}(A) \cup \mathrm{int}(X \setminus A)) = (X \setminus \mathrm{int}(A)) \cap (X \setminus \mathrm{int}(X \setminus A)) = \overline{X \setminus A} \cap \bar{A}.$$

4. Let X be a locally compact Hausdorff space.

   (a) [2 pts] Prove that a closed subset of X is locally compact.

   Solution: Let C be a closed subset of X, and $x \in C$. Since X is locally compact, there exists an open neighborhood U of x in X, such that $\bar{U}$ is compact. But then $U \cap C$ is an open neighborhood of x in C, and the closure of $U \cap C$ in C is $\bar{U} \cap C$, which is a closed subset of $\bar{U}$, so is compact. We have shown that C is locally compact.

   (b) [3 pts] Prove that an open subset of X is locally compact.

   Solution: Let W be an open subset of X, and $x \in W$. We want to prove that there exists an open neighborhood V of x whose closure (taken in X) is contained in W, and is compact. As before, we can find a neighborhood $x \in U \subset X$, such that $\bar{U}$ is compact. We use the notation $\bar{U}$ to denote closure in X. The problem is that we do not know that $\bar{U}$, or even just U, is contained in W.

   Let $K = \bar{U} \setminus W$. Then K is a closed subset of $\bar{U}$, and therefore is compact. Furtheremore, $x \notin K$, since $x \in W$. It follows that one can find disjoint open subsets P and Q of X, such that $K \subset P$ and $x \in Q$. Let $V = U \cap Q$. We claim that V is the desired neighborhood. Indeed, U and Q both contain x, so V contains x. U and Q are open, so V is open. The closure of V is contained in the closure of U, and therefore is compact. Finally, the closure of V is disjoint from K, because $K \subset P$ and P is disjoint from Q, so K is disjoint from the closure of Q. It follows that $\bar{V} \subset \bar{U} \setminus K = \bar{U} \cap W$, so $\bar{V} \subset W$, and we are done.

   (c) [1 pt] Show an example of a subspace of a locally compact Hausdorff space that is not locally compact.

   Solution: The line $\mathbb{R}$ is locally compact, but the subspace $\mathbb{Q}$ consisting of the rational numbers is not locally compact. Indeed, it is not hard to show that no subset of $\mathbb{Q}$ that contains a non-empty open interval is compact, and therefore no non-empty open subset of $\mathbb{Q}$ has a compact closure in $\mathbb{Q}$.

5. (a) [2 pts] Prove that every map from $\mathbb{R}P^2$ to S^1 is homotopic to a constant map.

   Solution: The fundamental group of $\mathbb{R}P^2$ is $\mathbb{Z}/2$ and the fundamental group of S^1 is $\mathbb{Z}$. It follows that the only homomorphism from $\pi_1(\mathbb{R}P^2)$ to $\pi_1(S^1)$ is the zero homomorphism. It follows that any map $f\colon \mathbb{R}P^2 \to S^1$ factors through the universal cover of S^1. But the universal cover of S^1 is $\mathbb{R}$, which is contractible. It follows that every map $\mathbb{R}P^2 \to S^1$ is null-homotopic.

(b) [2 pts] Give an example of two pointed spaces X and Y, and two pointed maps $f, g\colon X \to Y$, such that f and g are homotopic, but not pointed homotopic.

Solution: The fundamental group of $S^1 \vee S^1$ is the free group on two generators. Let's call them x and y. Let $f, g\colon S^1 \to S^1 \vee S^1$ be pointed maps representing the elements x and $y^{-1}xy$ in $\pi_1(S^1 \vee S^1)$. Then f and g are not homotopic as pointed maps, because they represent different elements of the fundamental group. But they represent conjugate elements of the fundamental group, so they are "freely" homotopic.

6. [3 pts] Let $X \subset \mathbb{R}^3$ be the space consisting of the four points $(0,0,0), (0,0,1), (0,1,0), (1,0,0)$, and the six line segments connecting them. In other words, X consists of the edges of a tetrahedron. What is the fundamental group of X?

Solution: X is a graph. The complement of a spanning tree consists of three edges. In other words, X is obtained by attaching three one-dimensional cells to a contractible space. It follows that $\pi_1(X)$ is the free group on three generators. You can also prove it using the van Kampen theorem (you may need to apply it twice).