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Final exam
Mathematical Methods for Economists
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Instructions:

- You will be provided a calculator.
- Start every problem on a new page, and write at the top of the page which problem it belongs to.
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

There are six problems, some with multiple parts. The problems are not ordered according to difficulty.

Problem 1. (5p) Compute the degree 2 Taylor polynomial of $f(x) = x \cdot e^{1/x}$ around the point $x_0 = -1$. The answer should be expressed on the form $ax^2 + bx + c$.

Solution. We need the derivatives. First, product rule and chain rule gives

$$f'(x) = e^{1/x} + xe^{1/x}(-x^{-2}) = e^{1/x}\left(1 - x^{-1}\right).$$

Taking the derivative again gives

$$f''(x) = e^{1/x}x^{-2} + \left(1 - x^{-1}\right)e^{1/x}(-x^{-2}) = x^{-3}e^{1/x}.$$

Now, $f(-1) = -e^{-1}$, $f'(-1) = 2e^{-1}$, $f''(-1) = -e^{-1}$. The Taylor polynomial of degree 2 at $x_0 = -1$ is

$$\begin{aligned} f(-1) + f'(-1)(x+1) + f''(-1)(x+1)^2/2 &= -e^{-1} + 2e^{-1}(x+1) - e^{-1}(x+1)^2/2 \\ &= e^{-1}(-1 + 2x + 2 - x^2/2 - x - 1/2) \\ &= \frac{1}{2e} + \frac{1}{e}x - \frac{1}{2e}x^2. \end{aligned}$$

Problem 2. Consider the function

$$f(x) = (2x-7)(x+4)\sum_{k=0}^{\infty}\left(\frac{x}{4}\right)^k.$$

(a) (1p) Show that $f(x)$ is defined on the interval $(-4, 4)$ but nowhere else.

(b) (2p) Find all critical points of $f(x)$.

(c) (2p) Find the minimum value of $f(x)$ on its domain, and sketch the graph. *Pay extra attention to the endpoints of the domain.*

Solution. (a) The sum is a geometric series for $r = x/4$. The sum is therefore $\frac{1}{1-\frac{x}{4}} = \frac{4}{4-x}$, but only if $-1 < x/4 < 1$. Hence, the function is only defined on the interval $-4 < x < 4$. On this interval,

$$f(x) = \frac{4(2x-7)(x+4)}{4-x} = 4\frac{2x^2+x-28}{4-x}. \tag{1}$$

(b) We need to solve $f'(x) = 0$. The quotient rule tells us

$$f'(x) = 4\frac{(4x+1)(4-x)-(2x^2+x-28)(-1)}{(4-x)^2} = 4\frac{-2x^2+16x-24}{(4-x)^2} = -8\frac{(x-6)(x-2)}{(4-x)^2}.$$

Thus, the critical points of f are $x = 2$ and $x = 6$, but only $x = 2$ lies in the interval where f is defined.
(c) By examining the derivative further, we see that $f'(x)$ is negative on the interval $(-4, 2)$ and positive on $(2, 4)$. Hence, $x = 2$ is a local minimum, and since f is continuous on $(-4, 4)$, this is a global minimum also. The minimum value is therefore $f(2) = 4(2\cdot4+2-28)/2 = -36$. By (1), we see that $\lim_{x\to4^-} f(x) = \infty$, since the numerator is positive at $x = 4$, and the denominator approaches 0 from above. We can also see that $\lim_{x\to-4} f(x) = 0$ due to the $(x + 4)$ factor.

Problem 3. Solve the following problems:

(a) (3p) Compute the integral $\displaystyle\int \frac{\ln(\sqrt{x}+1)}{\sqrt{x}}\,dx$.

(b) (2p) Compute the limit $\displaystyle\lim_{t\to\infty}\int_t^{2t} \frac{1+x}{x^2}\,dx$.

Solution. (a) We use the substitution $u = \sqrt{x} + 1$, $du = \frac{1}{2\sqrt{x}}dx$. This gives

$$\int 2\ln(u)\,du = \{\text{partial integration}\} = 2u\ln(u) - \int 2u\cdot\frac{1}{u}\,du = 2u\ln(u) - 2u + C.$$

Factoring out $2u$ and substituting back, we find that the answer is $2(\sqrt{x}+1)(\ln(\sqrt{x}+1)-1)+C$.
(b) We have that

$$\int_t^{2t} \frac{1+x}{x^2}\,dx = \int_t^{2t} x^{-1} + x^{-2}\,dx = \ln|x| - x^{-1}\Big|_t^{2t}.$$

This equals

$$\ln|2t| - 1/(2t) - (\ln|t| - 1/t) = \ln\left(\frac{2t}{t}\right) - \frac{1}{2t} + \frac{1}{t}.$$

As $t \to \infty$, we see that only $\ln(2)$ remains.

Problem 4. For every $C \in \mathbb{R}$, the equation $\sqrt{x}y^2 + x\sqrt{y} + \frac{4}{\sqrt{xy}} = C$ defines a curve in the plane.

(a) (1p) Determine C, such that the point $(4, 1)$ lies on the curve.

(b) (3p) For this value of C, find the slope of the tangent line at $(4, 1)$.

(c) (1p) Find the equation of the tangent line at $(4, 1)$.

Solution. (a) Plugging in $x = 4$, $y = 1$, we find that $C = 8$.
(b) We rewrite the square roots as rational powers, and differentiate both sides with respect to x (remembering that y is a function of x).

$$D[x^{\frac{1}{2}}y^2 + xy^{\frac{1}{2}} + 4x^{-\frac{1}{2}}y^{-\frac{1}{2}}] = 0$$

$$\left(\frac{1}{2}x^{-\frac{1}{2}}y^2 + x^{\frac{1}{2}}2yy'\right) + \left(y^{\frac{1}{2}} + x\frac{1}{2}y^{-\frac{1}{2}}y'\right) + \left(-2x^{-\frac{3}{2}}y^{-\frac{1}{2}} - 2x^{-\frac{1}{2}}y^{-\frac{3}{2}}y'\right) = 0$$

We now set $x = 4$, $y = 1$, and get

$$\left(\frac{1}{4} + 4y'(4)\right) + (1 + 2y'(4)) + \left(-\frac{1}{4} - y'(4)\right) = 0.$$

Solving for $y'(4)$ gives $y'(4) = -\frac{1}{5}$. This is the slope of the tangent line.
(c) The tangent line must go through $(4, 1)$, so its equation is $y = -\frac{1}{5}(x - 4) + 1$.

Problem 5. Consider the function $f(x, y) = 2x^2 + 2y^2 + 3x + 4y$, and let D be the domain determined by the inequalities $x \geq 0$ and $x^2 + y^2 \leq 16$.

(a) (1p) Draw the domain D.

(b) (1p) Find all critical points of f and determine their type.

(c) (3p) Find the maximum of f on D.

Solution. (a) The domain is a disk with radius 4, centered at the origin, but only the points with non-negative x-coordinate, i.e., the right hand half-disk.
(b) We compute,

$$f'_x = 4x + 3, \quad f'_y = 4y + 4, \quad f''_{xx} = 4, \quad f''_{yy} = 4, \quad f''_{xy} = 0.$$

so there is one critical point, namely $(-3/4, -1)$. By the criteria below, we see that this must be a local minimum.
(c) We note that the critical point we found in (b) is not inside the region D. The maximum must therefore be on the boundary. There are two parts to examine; the line $x = 0$, (with $-4 \leq y \leq 4$) and the part where $x^2 + y^2 = 16$, $x \geq 0$.

On the line $x = 0$, the function is $f(0, y) = 2y^2 + 4y$. If $g(y) = 2y^2 + 4y$, then $g'(y) = 4y - 4$, and we see $y = 1$ is a minimum. Hence, the corners, $(0, 4)$ and $(0, -4)$, are potential locations for the maximum.
To treat the circle sector, we introduce the Lagrangian,

$$L(x, y, \lambda) = 2x^2 + 2y^2 + 3x + 4y + \lambda(x^2 + y^2 - 16).$$

We have

$$\frac{\partial L}{\partial x} = 4x + 3 + 2\lambda x, \quad \frac{\partial L}{\partial y} = 4y + 4 + 2\lambda y, \quad \frac{\partial L}{\partial \lambda} = x^2 + y^2 - 16.$$

We want to find where all these vanish, so we want to simultaneously solve

$$4x + 3 + 2\lambda x = 0, \qquad 4y + 4 + 2\lambda y = 0, \qquad x^2 + y^2 = 16.$$

From the first two equations, we find $x = -\frac{3}{2(\lambda+2)}$, $y = -\frac{2}{\lambda+2}$. This is substituted into the last gives

$$\begin{aligned}
\left(-\frac{3}{2(\lambda+2)}\right)^2 + \left(-\frac{2}{(\lambda+2)}\right)^2 &= 16 \\
\left(-\frac{3}{2}\right)^2 + (-2)^2 &= 16(\lambda+2)^2 \\
\frac{9}{4} + \frac{16}{4} &= 16(\lambda+2)^2 \\
\frac{25}{4 \cdot 16} &= (\lambda+2)^2 \\
-2 \pm \frac{5}{8} &= \lambda.
\end{aligned}$$

Only $\lambda = -2 - \frac{5}{8}$ results in a positive x (remember, in the region, $x \geq 0$), so we get $x = -\frac{3}{2(-2-5/8+2)} = \frac{3 \cdot 8}{2 \cdot 5} = \frac{12}{5}$, and $y = -2/(-2 - 5/8 + 2) = \frac{16}{5}$.
In conclusion, the point $(\frac{12}{5}, \frac{16}{5})$ is an extremal point on the boundary of D. Finally, we just need to compare

$$f(0, 4) = 48, \qquad f(0, -4) = 16, \qquad f\left(\frac{12}{5}, \frac{16}{5}\right) = 52.$$

Thus, the maximal value of f on D is 52.

Problem 6. Consider the matrices

$$A = \begin{pmatrix} 2 & k \\ 1 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}, \quad X = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

(a) (1p) Compute $|A|$ as a function of k.

(b) (1p) For what values of k is $|A|$ non-zero?

(c) (3p) For the values of k you got in (b), solve the matrix equation $AX = B$. You should find x, y, z and w; some of these might depend on k.

Solution. (a) The determinant of A is $2 \cdot 1 - 1 \cdot k = 2 - k$.
(b) The determinant is non-zero whenever $k \neq 2$.
(b) We must solve

$$\begin{pmatrix} 2 & k \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Matrix multiplication in the left-hand side gives

$$\begin{pmatrix} 2x + kz & 2y + kw \\ x + z & y + w \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}.$$

Rewriting this as a system of equations, we get

$$\begin{cases} 2x + kz & = 1 \\ x + z & = 2 \\ 2y + kw & = 2 \\ y + w & = 1. \end{cases} \sim \left(\begin{array}{cccc|c} 2 & 0 & k & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & k & 2 \\ 0 & 1 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 2 & 0 & k & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 2 & 0 & k & 2 \end{array}\right)$$

We now perform Gaussian elimination. Using row 1 and 3 to eliminate in row 2 and 4, we get

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 0 & k-2 & 0 & -3 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & k-2 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & -\frac{3}{k-2} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 + \frac{3}{k-2} \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -\frac{3}{k-2} \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

where we in the first step, rearrange the row order and divide by $k - 2$ (this is allowed as we are working under the assumption $k \neq 2$). In the last step, we eliminate the remaining off-diagonal entries. This now tells us that

$$x = 2 + \frac{3}{k-2}, \quad y = 1, \quad z = -\frac{3}{k-2}, \quad w = 0 \implies X = \begin{pmatrix} 2 + \frac{3}{k-2} & 1 \\ -\frac{3}{k-2} & 0 \end{pmatrix}.$$

Formula for geometric series

We have $1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$, and $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ whenever $-1 < r < 1$. The infinite series does not converge if $r \geq 1$ or $r \leq -1$.

Formula for Taylor polynomials

Taylor polynomial of degree k for $f(x)$ at $x = a$, is

$$f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(k)}(a)\frac{(x-a)^k}{k!}.$$

Characterization of critical points

Let $f(x, y)$ be differentiable, and let $H = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{vmatrix}$. If, at critical point (x, y) we have

- $H > 0$ and $f''_{xx} > 0$, $f''_{yy} > 0$ then f has a local minimum at this critical point.
- $H > 0$ and $f''_{xx} < 0$, $f''_{yy} < 0$ then f has a local maximum at this critical point.
- $H < 0$ then f has neither a local maximum or minimum at this critical point.