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Final exam
Mathematical Methods for Economists
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Instructions:

- You will be provided a calculator.
- Start every problem on a new page, and write at the top of the page which problem it belongs to.
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear solutions even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

There are six problems, some with multiple parts. The problems are not ordered according to difficulty.

Problem 1. (5p) Use the degree 3 Taylor polynomial of $f(x) = x \cdot \ln(x)$ around the point $x_0 = 1$, and use the expansion to give an approximation of $f(1.5)$.

Solution. We have that

$$f'(x) = \ln(x) + \frac{x}{x}, \qquad f''(x) = \frac{1}{x}, \qquad f'''(x) = -\frac{1}{x^2}.$$

Hence, the Taylor polynomial of degree 3 at $x_0 = 1$ is

$$P_3(x) = f(1)+f'(1)(x-1)+\frac{f''(1)}{2}(x-1)^2+\frac{f'''(1)}{6}(x-1)^3 = 0+(x-1)+\frac{1}{2}(x-1)^2-\frac{1}{6}(x-1)^3.$$

Plugging in $x = 1.5$ gives $P_3(1.5) = \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{4} - \frac{1}{6} \cdot \frac{1}{8} = \frac{29}{48}$. So, $f(1.5) \approx 0.604$.

Problem 2. Consider the function

$$f(x) = (10 - x)^2 \left(1 - \frac{3}{x}\right) \left(1 + \frac{x}{10} + \frac{x^2}{100} + \frac{x^3}{1000} + \cdots\right),$$

defined on $3 < x < 10$.

(a) (1p) Give an expression for the geometric series $1 + \frac{x}{10} + \frac{x^2}{100} + \frac{x^3}{1000} + \cdots$.

(b) (1p) Compute the limit $\lim_{x\to 10} f(x)$.

(c) (2p) Find all critical points of $f(x)$.

(d) (1p) Find the maximum of $f(x)$ on the interval $3 < x < 10$.

If you cannot solve (a), you may assume $f(x) = \frac{10(10-x)(x-3)}{x}$ *for (b), (c) and (d).*

Solution. (a) The geometric series is (by using the formula for infinite geometric series)

$$1 + \frac{x}{10} + \frac{x^2}{100} + \frac{x^3}{1000} + \cdots = \frac{1}{1 - \frac{x}{10}}$$

and this converges whenever $-10 < x < 10$. Hence,

$$f(x) = (10-x)^2 \left(1 - \frac{3}{x}\right) \frac{1}{1 - \frac{x}{10}} = (10-x)^2 \frac{10}{x} \frac{(x-3)}{(10-x)} = \frac{10(10-x)(x-3)}{x}.$$

(b) We can see that $\lim_{x \to 10} f(x) = 0$ by just plugging in $x = 10$.
(c) We now compute $f'(x)$, by applying the quotient rule:

$$\begin{aligned} f'(x) &= 10 \frac{D[(10-x)(x-3)]x - D[x](10-x)(x-3)}{x^2} \\ &= 10 \frac{D[10x - 30 - x^2 + 3x]x - (10x - 30 - x^2 + 3x)}{x^2} \\ &= 10 \frac{(13-2x)x + (x^2 - 13x + 30)}{x^2} \\ &= 10 \frac{30 - x^2}{x^2}. \end{aligned}$$

Hence, the critical points of f are $x = \pm\sqrt{30}$, but only $\sqrt{30} \approx 5.48$ lies in the interval $(3, 10)$ where f is defined.
(d) The function is continuous, with continuous derivative on $(3, 10)$, and it approaches 0 on the endpoints of this interval. Moreover, $f(\sqrt{30})$ is clearly positive (we see this in the original definition). Hence, $x = \sqrt{30}$ is the global maximum, and the maximum value is therefore

$$f(\sqrt{30}) = \frac{10(10 - \sqrt{30})(\sqrt{30} - 3)}{\sqrt{30}} \approx 20.45.$$

Problem 3. Solve the following problems:

(a) (2p) Compute the integral $\displaystyle\int \frac{(\sqrt{x}+1)^5}{\sqrt{x}}\, dx$.

(b) (2p) Compute the integral $\int_1^5 xe^x \, dx$

(c) (1p) Compute

$$\int_1^3 xe^x + e^{x^2} \, dx \; + \; \int_3^5 xe^x + e^{x^2} \, dx \; - \; \int_1^5 e^{x^2} \, dx.$$

Solution. (a) We use the substitution $u = \sqrt{x} + 1$, and $du = \frac{1}{2\sqrt{x}}$. This gives

$$\int \frac{2(\sqrt{x}+1)^5}{2\sqrt{x}}\, dx = \int 2u^5\, du = \frac{2u^6}{6} + C.$$

Substituting back, we arrive at the final answer $\frac{(\sqrt{x}+1)^6}{3} + C$.

(b) We solve this via partial integration:

$$\int xe^x\, dx = xe^x - \int e^x\, dx = (x-1)e^x + C.$$

Hence, the value of the integral is $[(x-1)e^x]_1^5 = 4e^5 - 0 = 4e^5$.

(c) Properties of the integral states that

$$\int_1^3 xe^x + e^{x^2}\, dx \; + \; \int_3^5 xe^x + e^{x^2}\, dx = \int_1^5 xe^x + e^{x^2}\, dx$$

Now, more properties states that

$$\int_1^5 xe^x + e^{x^2}\, dx - \int_1^5 e^{x^2}\, dx = \int_1^5 xe^x + e^{x^2} - e^{x^2}\, dx$$

and we end up with the same integral (and answer) as in (b).

Problem 4. Let $f(x) = g(h(x))$, where you are given that $g'(2) = 5$, $h(1) = 2$ and $h'(1) = a$, for some unknown value of a.

(a) (3p) Compute $f'(1)$ in terms of a.

(b) (2p) What must the value of a be, if the tangent line of f at $x = 1$ should have slope 20?

Solution. (a) The chain rule states that $f'(x) = g'(h(x))h'(x)$, so we get

$$f'(1) = g'(h(1)) \cdot h'(1) = g'(2)a = 5a.$$

(b) We know from (a) that the slope is $5a$, so in order to have $5a = 20$, we must have $a = 4$.

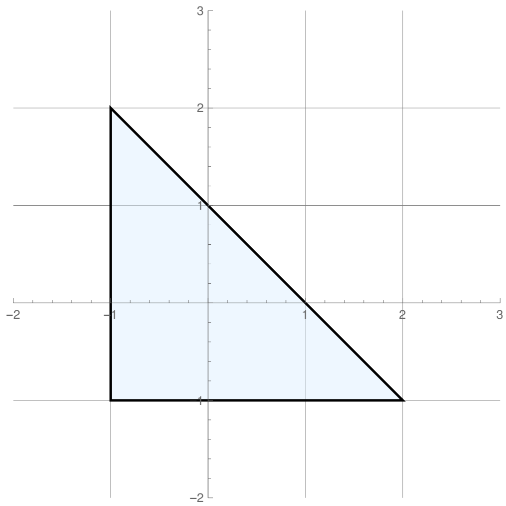
Problem 5. Consider the function $f(x, y) = x^2 + 2y^2 + 2$, and the triangular region with corners $(-1, -1)$, $(2, -1)$ and $(-1, 2)$.

(a) (1p) Draw the region.

(b) (1p) Find all critical points of f.

(b) (3p) Find the maximum of $f(x, y)$ in the region.

Solution. (a)



(b) We find that $f'_x(x,y) = 2x$, $f'_y(x,y) = 4y$, so $(0,0)$ is a critical point. It is inside the region.
(c) There are three lines bounding the region. First, we have $x = -1$, so we must examine $f(-1,t) = 1 + 2t^2 + 2$, where $t \in [-1,2]$. We see that the maximum on this line is when $t = 2$. The second line is $y = -1$, so we examine $f(t,-1) = t^2 + 1 + 2$, $t \in [-1,2]$. Again, this has a maximum when $t = 2$.
Finally, we have the line between the points $(2,-1)$ and $(-1,2)$. The equation for this line is $y = 1 - x$, so we must examine $f(t, 1-t)$ as $t \in [-1,2]$. Now,

$$f(t, 1-t) = t^2 + 2(1-t)^2 + 2 = 3t^2 - 4t + 4,$$

and the derivative of this expression is $6t - 4$. This vanish when $t = \frac{2}{3}$. Since the second derivative is positive, this is a local minimum. Hence, again, the maximum must be at one of the endpoints. Thus, it suffices to compare the values at the corners of the triangle, and the critical point in the interior. We have

$$f(0,0) = 2, \qquad f(-1,-1) = 5, \qquad f(2,-1) = 6, \qquad f(-1,2) = 11,$$

so the answer is 11.

Alternatively, one can also examine the third line, $y = 1 - x$, by the Lagrange method, with $g(x,y) = x + y - 1$, and $L(x,y,\lambda) = x^2 + 2y^2 + 2 + \lambda(x + y - 1)$. This leads to

$$L_x(x,y,\lambda) = 2x + \lambda, \quad L_y(x,y,\lambda) = 4y + \lambda, \quad L_\lambda(x,y,\lambda) = x + y - 1.$$

We want all these to vanish, so we must have $2x = -\lambda$, $4y = -\lambda$. Hence, $2x = 4y$, and $x = 2y$. Substituting this into $x + y - 1 = 0$ gives $2y + y - 1 = 0$, so $y = \frac{1}{3}$ and $x = \frac{2}{3}$. With this method, it is harder to see if this is a local minima or maxima, so one simply must evaluate $f(2/3, 1/3)$ and compare with the other points.

Problem 6. Consider the matrix

$$A = \begin{pmatrix} 2+k & 3 & 1 \\ -1 & k & 1 \\ 0 & 1 & 1+k \end{pmatrix}.$$

Answer the following questions.

(a) (1p) Compute $|A|$.

(b) (1p) If a square matrix A has determinant zero, will the system of equations $AX = 0$ have multiple solutions?

(c) (3p) Find two different solutions to the system of equations below.

$$\begin{pmatrix} 2 & 3 & 1 \\ -1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \iff \begin{cases} 2x + 3y + z & = 0 \\ -x + z & = 0 \\ y + z & = 0. \end{cases}$$

Solution. (a) Sarrus's rule tells us that

$$\begin{aligned} |A| &= ((2+k)k(1+k) + 3 \cdot 1 \cdot 0 + 1 \cdot (-1) \cdot 1) - ((2+k) \cdot 1 \cdot 1 + 3 \cdot (-1)(1+k) + 1 \cdot k \cdot 0) \\ &= (2+k)k(1+k) - 1 - (2+k) + 3(k+1) \\ &= 4k + 3k^2 + k^3 \\ &= k(k^2 + 3k + 4). \end{aligned}$$

(b) Yes, if the determinant is 0, we can always find several solutions to the system, since the right hand side is 0.
(c) One solution is of course $x = y = z = 0$. In order to find another solution, we first do some Gaussian elimination.

$$\left(\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 3 & 3 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right)$$

Thus, we need to find a non-zero solution to $-x + z = 0$ and $y + z = 0$. For example, we can pick $z = 1$, which forces $x = 1$ and $y = -1$. That is, $(x, y, z) = (1, -1, -1)$ is another solution.

Formula for geometric series

We have $1 + r + r^2 + r^3 + \cdots + r^{n-1} = \frac{1-r^n}{1-r}$, and $1 + r + r^2 + r^3 + \cdots = \frac{1}{1-r}$ whenever $-1 < r < 1$. The infinite series does not converge if $r \geq 1$ or $r \leq -1$.

Formula for Taylor polynomials

Taylor polynomial of degree k for $f(x)$ at $x = a$, is

$$f(a) + f'(a)\frac{(x-a)}{1!} + f''(a)\frac{(x-a)^2}{2!} + \cdots + f^{(k)}(a)\frac{(x-a)^k}{k!}.$$

Characterization of critical points

Let $f(x, y)$ be differentiable, and let $H = \begin{vmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{vmatrix}$. If, at critical point (x, y) we have

- $H > 0$ and $f''_{xx} > 0$, $f''_{yy} > 0$ then f has a local minimum at this critical point.
- $H > 0$ and $f''_{xx} < 0$, $f''_{yy} < 0$ then f has a local maximum at this critical point.
- $H < 0$ then f has neither a local maximum or minimum at this critical point.