MATEMATISKA INSTITUTIONEN
STOCKHOLMS UNIVERSITET
Avd. Matematik
Examinator: Alexander Berglund

Tentamensskrivning i
Topologi, MM8002/SF2721
7.5 hp
2015-05-27

(1) Let $\mathbb{Z}$ denote the set of integers. Declare a subset $U \subseteq \mathbb{Z}$ to be open if

$$n \in U \Leftrightarrow -n \in U.$$

- (a) Show that this defines a topology on $\mathbb{Z}$.
- (b) Determine the closure and the interior of the subset $\{-1, 0, 1, 2\} \subset \mathbb{Z}$.
- (c) Is $\mathbb{Z}$ connected in this topology? Compact? Second countable (i.e., does it admit a countable base)? Motivate your answers. [15 points]

(2) Let $f, g \colon X \to Y$ be continuous maps and assume that Y is Hausdorff.

- (a) Show that the set $\{x \in X \mid f(x) = g(x)\}$ is closed in X.
- (b) Give a counterexample if Y is not Hausdorff. [15 points]

(3) Let X and Y be topological spaces and let $A \subseteq X$, $B \subseteq Y$. We can give A and B the subspace topologies and then give $A \times B$ the product topology. On the other hand, we could first give $X \times Y$ the product topology and then give $A \times B \subseteq X \times Y$ the subspace topology. Show that these two ways of putting a topology on $A \times B$ yield the same result. [15 points]

(4) The orthogonal group $O(n)$ acts on $\mathbb{R}^n$ by matrix multiplication.

- (a) Show that the orbit space is homeomorphic to $[0, \infty) \subset \mathbb{R}$.
- (b) The orbits are familiar subspaces of $\mathbb{R}^n$. Can you recognize them? [15 points]

(5) Consider the subspace $X = S^2 \cup D^2$ of $\mathbb{R}^3$ where

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\},$$
$$D^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 1,\ z = 0\right\}.$$

- (a) Calculate the fundamental group of X.
- (b) Calculate the Euler characteristic of X.
- (c) Calculate the Betti numbers of X.
  (Hint: The Euler-Poincaré formula may come in handy.)
- (d) Is X homotopy equivalent to a closed surface? [20 points]

(6) State and prove the Brouwer fixed-point theorem for D^2. [20 points]

Corrected exams will be returned on Monday June 1, 13:00 – 13:15, in room 414. Thereafter exams may be collected in room 204.