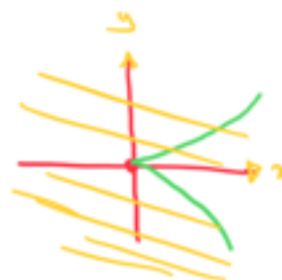
Recall: $\rightarrow$ $\mathrm{Dim}(R) = \boxed{\sup_{n\in\mathbb{N}}} \{ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n \mid P_i \text{ Prime ideal} \}$

$\mathrm{ht}(P) = \sup_{n\in\mathbb{N}} \{ P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n = P \mid P_i \text{ Prime ideal} \}$

Ex: $R = k[x,y]$ $\quad \mathrm{Dim}(R) = 2$



$\boxed{P_0 = (0)} \subsetneq \boxed{P_1 = (y^2 - x^3)} \subsetneq \boxed{P_2 = (x,y)}$

$\mathrm{ht}(P_2) = 2$	$\dim V(P_2) = 0$
$\mathrm{ht}(P_1) = 1$	$\dim V(P_1) = 1$
$\mathrm{ht}(P_0) = 0$	$\dim V(P_0) = 2$

$\rightarrow \boxed{\dim V(P_i) = 2 - \mathrm{ht}(P_i)}$

$\rightarrow$ in general $\boxed{\dim V(P_i) = n - \mathrm{ht}(P_i)}$ in $k[x_1,\ldots,x_n]$

$\rightarrow$ Thm: [Generalized Krull Thm]. Let A be a Noetherian ring and let $\boxed{I = \langle a_1,\ldots,a_r\rangle \subsetneq A}$. Then for

$\rightarrow$ every $P \in \underline{\mathrm{Min}(I)}$; $\boxed{\mathrm{ht}(P) \leq r.}$

Obs: If A is a Polynomial ring $k[x_1,\ldots,x_n]$ then

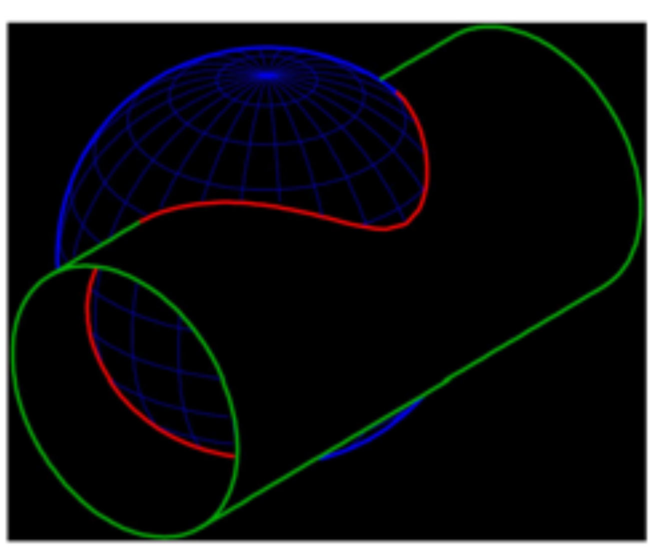
$\rightarrow \boxed{\dim(V(P)) = n - \mathrm{ht}(P) \boxed{\geq} n - r}$

$\rightarrow$ Ex: $a_1 = x^2+y^2+z^2 - 1$, $a_2 = x^2+y^2 - \alpha^2$ $\quad \boxed{A = \mathbb{R}[x,y,z]}$

Careful about other fields e.g., $\mathbb{R}$

$I = (x^2+y^2) \subseteq \mathbb{R}[x,y]$

$\boxed{\mathrm{ht}(I) = 1}$



$\rightarrow \boxed{\dim A = 3}$

$1 \geq \mathrm{ht}((a_1)) = 1 \longleftrightarrow \dim(V(a_1)) = 3 - 1 = 2$

$1 \geq \mathrm{ht}((a_2)) = 1 \longleftrightarrow \dim(V(a_2)) = 3 - 1 = 2$

$\rightarrow$ By Krull: $\mathrm{ht}(\overset{=2}{(a_1,a_2)}) \leq 2 \leftarrow \boxed{(0) \subsetneq (a_1) \subsetneq (a_1,a_2)}$

in here is exactly 2 !! So $\boxed{\dim V((a_1,a_2)) = \overset{3-2=1}{1}}$

$\boxed{\text{The red curve !!}}$

Let $A = C([0,1]) = \{ f: [0,1] \to \mathbb{R} \mid f \text{ is Continuous} \}$

What are the maximal ideal of A?

→ claim: $m \subseteq A$ is maximal ideal $\longleftrightarrow$ $\exists\, x \in [0,1]$

such that $m = m_x = \{ f \in A \mid f(x) = 0 \}$. ←

→ Proof: If $f > 0$ or $f < 0$ then f^{-1} exists.

→ So $f \in m \longleftrightarrow \exists\, y \in [0,1]$ s.t $f(y) = 0$.

→ Now, assume that elements in m doesn't have a Common zero! So, for $x \in [0,1]$ then exists f_x such that $f_x(x) \neq 0$. Also there exists

→ $D_x := (x - \varepsilon_x, x + \varepsilon_x)$ such that f_x is non-zero

→ over D_x. Now, note that $[0,1]$ is Compact in Euclidean toPology so $[0,1] = \bigcup_{i=1}^{n} D_{x_i}$, and each D_{x_i} CorresPond to an $f_i \in m$.

→ Take $f := \underbrace{f_1^2 + \dots + f_n^2}_{\in m}$. check that f is Positive every where, → $1 \in m$ ✗.

So $\boxed{m = m_x}$ Some $x \in [0,1]$. ✓

Hw 8: 1) Ass(a) Consists $P_1 \subsetneq P_2 \subsetneq P_3$

$A := k[x,y,z]$ $q_1 = (x)$ $q_2 = (x^2, y)$ $q_3 = (x^3, y^2, z)$

$a = \bigcap_{i=1}^{3} q_i$ → $P_1 = (x) \subsetneq P_2 = (x,y) \subsetneq P_3 = (x,y,z)$

→ 3) $V(a) = \emptyset \to a = (1)$.

→

Proof: $I(V(a)) = I(\emptyset) = A$ ←

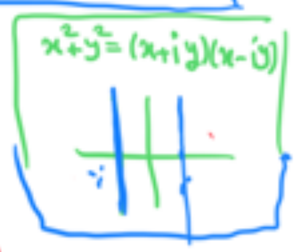
Nullstellensatz → $\sqrt{a}$ $=$ $1 \in \sqrt{a} \to \exists n \;\; 1^n \in a$, $1^n = 1$

→ Also if $m \subseteq A$ maximal ideal, $\boxed{V(m) \ni m}$

So $\boxed{m \subseteq m_x} \longrightarrow \boxed{m = m_x}$ ✓

4) $(t^2+1) \subseteq \mathbb{R}[t]$ $\quad V(t^2+1) = \emptyset$.

$x^2+y^2-r^2 \in \mathbb{R}[x,y]$ $\quad r \in \mathbb{R}$



5) $E := k(t_1, \dots, t_r)$

$= \{ \frac{f}{g} \mid f,g \in k[t_1,\dots,t_r] \}$

$\longrightarrow$ let $\alpha_i = \dfrac{f_i(t_1,\dots,t_r)}{g_i(t_1,\dots,t_r)} = \dfrac{f_i}{g_i}$

$\alpha = \dfrac{1}{g_1 \cdots g_m + 1} \notin k[\alpha_1, \dots, \alpha_m]$

$\alpha \notin k[\alpha_1, \dots, \alpha_m]$

$\boxed{\alpha \neq F(\alpha_1, \dots, \alpha_m)}$

$\sum \prod_i \left(\frac{f_i}{g_i}\right)^{\alpha_i}$

$p_1, \dots, p_m \in \mathbb{Z}$

$1 + p_1 \cdots p_m$