Orlando Marigliano
Tim Hosgood
Vahid Shahverdi
Sjoerd Wijnand de Vries

Fall Term 2022

Commutative algebra and algebraic geometry – Exercises

The deadline for handing in exercises is on Wednesdays at 11:59 PM, one week after the exercise was released. Exercises are to be handed in *online* via the course website. Please write your solutions in LaTeX if you can.

Grading: At the exam, you will get a percentile score. Regularly handing in solutions to these exercises will *improve* this score, adding an amount of percentage points on top of it. Your final grade will be determined by this combined score. The specifics of this system are not yet fixed, but will be communicated around the middle of the term.

TA session exercise. Let K be a field and $f \in K[t_1, \dots, t_n]$ be a polynomial in n variables. Then we can see f as a function $K^n \to K$ by sending a point $x = (x_1, \dots, x_n)$ to $f(x) \coloneqq f(x_1, \dots, x_n)$. The *vanishing set* of f is $V(f) \coloneqq \{x \in K^n \mid f(x) = 0\}$.

(1) Let K be infinite. Show that $V(f) = K^n$ if and only if $f = 0$.
(2) Let K be algebraically closed. Show that $V(f) = \emptyset$ if and only if f is a unit.

In the following, k shall always denote an algebraically closed field.

Let $\mathfrak{a}$ be an ideal of a ring A. A *set of generators* for $\mathfrak{a}$ is a subset $E \subseteq \mathfrak{a}$ such that $\mathfrak{a} = (E) \coloneqq \{\sum_{i=1}^n x_i a_i \mid x_i \in A, a_i \in E\}$. For a subset $E \subseteq A$, it can be shown that the set (E) is the smallest ideal of A that contains E. The ideal (E) is called the *ideal generated by* E. If $E = \{a_1, \dots, a_k\}$ is finite, we write $(E) = (a_1, \dots, a_k)$.

Exercise 1. Let $A \coloneqq k[t_1, \dots, t_n]$ and let $x \in k^n$ be a point.

(1) Show that the evaluation map $\varphi_x : A \to k$ defined by $\varphi_x(f) = f(x)$ is a surjective ring homomorpism.
(2) Let $\mathfrak{m}_x \coloneqq \ker(\varphi_x)$. Give a finite set of generators for $\mathfrak{m}_x$.
(3) Show that $\mathfrak{m}_x$ is a maximal ideal of A.
(4) Let $y \in k^n$ be another point. Show that if $x \neq y$ then $\mathfrak{m}_x \neq \mathfrak{m}_y$.
(5) Let

$$A_x \coloneqq \left\{ \frac{f}{g} \mid f, g \in A, g(x) \neq 0 \right\}.$$

This is a subring of the function field $k(t_1, \dots, t_n)$. Show that A_x is a local ring.

Exercise 2. Let $A \coloneqq k[t_1, \dots, t_n]$. The *variety* of a subset $E \subseteq A$ is the set

$$V(E) \coloneqq \{x \in k^n \mid f(x) = 0 \text{ for all } f \in E\}.$$

(1) Let $E \subseteq A$ be a subset and $\mathfrak{a}$ the ideal generated by E. Show that

$$V(E) = V(\mathfrak{a}) = V(\operatorname{rad} \mathfrak{a}).$$

(2) Show that $V((0)) = k^n$ and $V((1)) = \emptyset$.
(3) Let $(\mathfrak{a}_i)_{i \in I}$ be a family of ideals of A. Show that $V(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_{i \in I} V(\mathfrak{a}_i)$.
(4) Let $\mathfrak{a}, \mathfrak{b}$ be ideals of A. Show that $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
(5) Let $\mathfrak{a}$ be an ideal of A and $x \in k^n$. Show that $x \in V(\mathfrak{a})$ if and only if $\mathfrak{m}_x \supseteq \mathfrak{a}$. Deduce that $V(\mathfrak{m}_x) = \{x\}$.

Exercise 3. Let M be a module over $A \coloneqq k[t_1, \dots, t_n]$. For all $x \in k^n$, define

$$M|_x \coloneqq M/\mathfrak{m}_x M.$$

(1) Show that $M|_x$ can be made into a k-vector space in two equivalent ways, one using the inclusion $k \to A$ and one using the evaluation morphism $\varphi_x : A \to k$.
(2) Find $n \in \mathbb{N}$, an A-module M, and $x, y \in k^n$ such that $\dim M|_x \neq \dim M|_y$.
(3) Find $n \in \mathbb{N}$ and A-modules $M \not\simeq N$ such that $\dim M|_x = \dim N|_x$ for all $x \in k^n$.
(4) Define

$$\mathrm{Supp}(M) := \{x \in k^n \mid M|_x \neq \{0\}\}.$$

Show that $\mathrm{Supp}(M) \subseteq V(\mathrm{ann}(M))$.
(5) Now let M be finitely generated. Show that $\mathrm{Supp}(M) = V(\mathrm{ann}(M))$.

Exercise 4. In this exercise, you'll compute examples of tensor products in various ways.
(1) Show that $\mathbb{Z}/m\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$.
(2) Let K be a field and V, W finite-dimensional K-vector spaces. Show that

$$\dim(V \otimes_K W) = \dim(V)\dim(W).$$

(3) Let K be a field. Show that

$$K[t_1, \dots, t_n] \otimes_K K[s_1, \dots, s_m] \simeq K[t_1, \dots, t_n, s_1, \dots, s_m]$$

as K-algebras.
(4) Let M be an A-module and $\mathfrak{a} \subseteq A$ an ideal. Show that

$$A/\mathfrak{a} \otimes_A M \simeq M/\mathfrak{a}M$$

as A-modules.
(5) Let M, N be modules over $A := k[t_1, \dots, k_n]$. Show that for all $x \in k^n$,

$$(M \otimes_A N)|_x \simeq M|_x \otimes_k N|_x$$

as k-vector spaces.

Exercise 5. (1) Let A be a ring. Show that the set of units $S := A^\times$ is a multiplicative subset and that the structure map $A \to S^{-1}A$ is an isomorphism.
(2) Let A be a ring and S, $T \subseteq A$ multiplicative subsets such that $S \subseteq T$. Show that there exists a unique ring homomorphism $f : S^{-1}A \to T^{-1}A$ that commutes with the structure maps. That is, if $s : A \to S^{-1}A$ and $t : A \to T^{-1}A$ are the structure maps, then $f \circ s = t$.
(3) Now let A be an integral domain and S, T as in (2), where additionally $0 \notin T$. Show that the homomorphism f from (2) is *injective*. Deduce that if A is an integral domain, then the structure morphism $A \to S^{-1}A$ is injective and $S^{-1}A$ can be regarded as a subring of $\mathrm{Frac}(A)$.
(4) Let $A = k[t_1, \dots, t_n]$, $f \in A$, and $x \in k^n$. Show that

$$\bigcup_{g \in A \setminus \mathfrak{m}_x} A_g = A_{\mathfrak{m}_x}$$

as subrings of $\mathrm{Frac}(A)$, and that if $f^k = ag$ for some $a, g \in A$ $(k \geq 0)$ then $A_g \subseteq A_f$.
(5) Let A be a ring, $f \in A$, and $S := \{f^k\}_{k \geq 0}$. Show that

$$A_f \simeq A[t]/(tf - 1).$$

Exercise 6. Recall that if $f : A \to B$ is a ring homomorphism and $\mathfrak{a} \subseteq A$, $\mathfrak{b} \subseteq B$ ideals, then $\mathfrak{b}^c$ is the ideal $f^{-1}(\mathfrak{b})$ and $\mathfrak{a}^e$ is the ideal generated by the set $f(\mathfrak{a}) \subseteq B$.
(1) Let A be a ring and M a finitely-generated A-module. Let $\mathfrak{p} \subseteq A$ be a prime ideal. Show that $M_{\mathfrak{p}} = 0$ if and only if there exists $f \in A \setminus \mathfrak{p}$ such that $M_f = 0$.

(2) Let A be a ring and $S \subseteq A$ a multiplicative subset. Show that $\mathrm{nil}(S^{-1}A) = \mathrm{nil}(A)^e$, where the extension is taken with respect to the structure map $A \to S^{-1}A$.
(3) Let A be a ring. We call A *reduced* if $\mathrm{nil}(A) = 0$. Show that A is reduced if and only if $A_\mathfrak{p}$ is reduced for every prime ideal $\mathfrak{p} \subseteq A$.
(4) Let A be a ring and $\mathfrak{p}', \mathfrak{p} \subseteq A$ prime ideals such that $\mathfrak{p}' \subseteq \mathfrak{p}$. Find a ring B and a ring homomorphism $f : A \to B$ such that $\mathfrak{q} \mapsto \mathfrak{q}^c$ gives a bijection

$$\{\mathfrak{q} \subseteq B \mid \mathfrak{q} \text{ prime}\} \xrightarrow{1:1} \{\mathfrak{q}' \subseteq A \mid \mathfrak{q}' \text{ prime}, \mathfrak{p}' \subseteq \mathfrak{q}' \subseteq \mathfrak{p}\}.$$

(5) Let $f : M \to N$ be an A-module homomorphism and $S \subseteq A$ a multiplicative subset. Show that $\ker(S^{-1}f) \simeq S^{-1}\ker(f)$ and $\mathrm{coker}(S^{-1}f) \simeq S^{-1}\mathrm{coker}(f)$.

In particular, $\ker(f_\mathfrak{p}) \simeq \ker(f)_\mathfrak{p}$ *and* $\mathrm{coker}(f_\mathfrak{p}) \simeq \mathrm{coker}(f)_\mathfrak{p}$ *for every prime ideal* $\mathfrak{p} \subseteq A$.

Exercise 7. For this exercise, let $A := k[x, y, z]$ and let $\mathfrak{a}$ denote the ideal $(yz, xz^2) \subseteq A$.
(1) Show that the ideals (z) and (y, x) are prime and that $V(\mathfrak{a}) = V(z) \cup V(y, x)$.
(2) Show that the ideal (y, z^2) is primary and find its radical.
(3) Show that $\mathfrak{a} = (z) \cap (y, x) \cap (y, z^2)$.
(4) Show that the relation in (3) is a minimal primary decomposition of $\mathfrak{a}$.
(5) Determine the set $\mathrm{Ass}(\mathfrak{a})$ of prime ideals associated to $\mathfrak{a}$, together with its partial order $\subseteq$. Indicate which associated primes are isolated and which are embedded.

Exercise 8. In this exercise, K denotes an arbitrary field. Recall that the *variety* of an arbitrary subset $E \subseteq K[t_1, \dots, t_n]$ is

$$V(E) := \{x \in K^n \mid f(x) = 0 \text{ for all } f \in E\}.$$

Also recall the following convention: if $A \subseteq B$ are rings and $b_1, \dots, b_m \in B$, then $A[b_1, \dots, b_m]$ denotes the smallest subring of B that contains A and all the b_i. It equals the set of all polynomial expressions $f(b_1, \dots, b_m)$ where $f \in A[t_1, \dots, t_m]$.
(1) Find an ideal $\mathfrak{a} \subseteq K[x, y, z]$ such that $\mathrm{Ass}(\mathfrak{a})$ consists of three distinct prime ideals $\mathfrak{p}_1, \mathfrak{p}_2, \mathfrak{p}_3$ with $\mathfrak{p}_1 \subset \mathfrak{p}_2 \subset \mathfrak{p}_3$.
(2) Let $E \subseteq K[t_1, \dots, t_n]$ be any subset. Show that there exist $f_1, \dots, f_k \in K[t_1, \dots, t_n]$ such that $V(E) = V(\{f_1, \dots, f_k\})$.
(3) Let K be algebraically closed and let $\mathfrak{a} \subseteq K[t_1, \dots, t_n]$ be an ideal. Show that $V(\mathfrak{a}) = \emptyset$ implies $\mathfrak{a} = (1)$. Deduce that every maximal ideal $\mathfrak{m} \subseteq K[t_1, \dots, t_n]$ is of the form $\mathfrak{m}_x$ for some $x \in K^n$. (Or, if you prefer, prove the second statement and deduce the first from the second).
(4) Find an ideal $\mathfrak{a} \subseteq \mathbb{R}[t]$ such that $V(\mathfrak{a}) = \emptyset$ but $\mathfrak{a} \neq (1)$. Compute $V(\mathfrak{a}^e)$, where the extension is taken with respect to the inclusion $\mathbb{R}[t] \to \mathbb{C}[t]$.
(5) Let $r \geq 1$, $E := K(t_1, \dots, t_r)$, and $\alpha_1, \dots, \alpha_m \in E$. Show that there exists $\alpha \in E$ such that $\alpha \notin K[\alpha_1, \dots, \alpha_m]$. Deduce that E is not a finitely-generated K-algebra.

Exercise 9. *Deadline: 9th of November*
Recall that we write $\mathbb{A}^n$ to denote $\mathbb{C}^n$ with the Zariski topology.
(1) Prove that the non-trivial (i.e. not equal to $\varnothing$ or $\mathbb{C}$ itself) closed subsets of $\mathbb{A}^1$ are exactly the *finite* subsets. (We say that the Zariski topology on $\mathbb{C}$ is exactly the *cofinite* topology).
(2) Prove that the Zariski topology on $\mathbb{V}(y - x^2) \subset \mathbb{A}^2$ is also exactly the cofinite topology.
(3) Is the Zariski topology on any plane curve exactly the cofinite topology? What about for $\mathbb{V}(y^2 - x^2)$? If this statement is not true for *all* plane curves, can you characterise those for which it is true?
(4) By considering the diagonal $\Delta = \{(z, z) \mid z \in \mathbb{C}\} \subset \mathbb{C}^2$, show that the Zariski topology on $\mathbb{A}^2$ is *not* the product topology $\mathbb{A}^1 \times \mathbb{A}^1$.

Exercise 10. *Deadline: 23rd of November*

(1) Are $\mathbb{V}(x^3, y)$ and $\mathbb{V}(x^2, y^2)$ isomorphic as complex affine varieties? Prove your answer.

(2) Let $F\colon X \to Y$ be a morphism of affine varieties.

  (a) Show that the pullback $F^\sharp\colon \mathbb{C}[Y] \to \mathbb{C}[X]$ is *injective* if and only if F is *dominant* (i.e. if and only if $\mathrm{Im}(F)$ is *Zariski dense* in Y, i.e. if and only if the closure of $\mathrm{Im}(F) \subseteq Y$ in the Zariski topology is all of Y).

  (b) Show that the pullback $F^\sharp\colon \mathbb{C}[Y] \to \mathbb{C}[X]$ is *surjective* if and only if F is a *closed embedding* (i.e. if and only if F gives an isomorphism between X and an affine subvariety of Y).

(3) Let $F = (F_1, \ldots, F_n)\colon \mathbb{A}^n \to \mathbb{A}^n$ be an isomorphism. Show that the *Jacobian*

$$J(F) = \det \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{pmatrix}$$

is a non-zero constant polynomial.

Exercise 11. *Deadline: 7th of December*

(1) Prove that, if V is an irreducible affine variety, then its projective closure $\overline{V}$ is also irreducible. (*Hint: this is a purely topological question, in that it doesn't rely on any special properties about affine or projective varieties.*)

(2) Is the converse to the above statement true? If so, prove it; if not, give a counter-example.

(3) Prove that the homogenisation of a radical ideal is a radical ideal.