**Problem 1: Poisson Processes**

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate $\lambda$. Let $\{S_i, i = 1, 2, \cdots\}$ be the points of the Poisson Process, such that $S_1 \leq S_2 \leq S_3 \leq \cdots$. Let $k$ be a strictly positive integer.

**a)** Provide the distribution of $S_{k+1} - S_k$. (4p)

Solution: For a Poisson process interarrival times are exponentially distributed with mean $1/\lambda$.

**b)** Assume that $n \geq 1$. Compute $\mathbb{E}[S_{n+1} - S_n | N(s) = n]$. (4p)

Solution: Since $N(s) = n$ we know that on the interval $(0, s)$ there were $n$ events and the times of those events are (by the order statistic property) distributed as $n$ i.i.d. uniforms on $(0, s)$. The probability that all those points are in $(0, s - t)$ (and thus $s - S_n > t$) is given by $\left(\frac{s-t}{s}\right)^n$. Noting that

$$\mathbb{E}[s - S_n | N(s) = n] = \int_0^s \mathbb{P}(s - S_n > t) dt = \int_0^s \left(\frac{s-t}{s}\right)^n dt = \frac{s}{n+1}[-\left(\frac{s-t}{s}\right)^{n+1}]_{t=0}^s = \frac{s}{n+1}.$$

Since the exponential distribution is memoryless the time until the next event after time $s$ is exponentially distributed with expectation $1/\lambda$. Therefore, $\mathbb{E}[S_{n+1} - s | N(s) = n] = 1/\lambda$ and

$$\mathbb{E}[S_{n+1} - S_n | N(s) = n] = \mathbb{E}[S_{n+1} - s | N(s) = n] + \mathbb{E}[s - S_n | N(s) = n] = \frac{s}{n+1} + 1/\lambda$$

**c)** Still assume that $n \geq 1$. Compute $\mathbb{E}[\sum_{i=1}^n S_i | N(s) = n]$ and $Var[\sum_{i=1}^n S_i | N(s) = n]$. (4p)

Solution: Using the order statistic property

$$\mathbb{E}[\sum_{i=1}^n S_i | N(s) = n] = \mathbb{E}[\sum_{i=1}^n U_{(i)}] = \mathbb{E}[\sum_{i=1}^n U_i],$$

where the $U_i$'s are i.i.d. uniforms on $(0, s)$ (with expectation $s/2$ and variance $s^2/12$) and the $U_{(i)}$'s are the same random variables in increasing order. So $\mathbb{E}[\sum_{i=1}^n S_i | N(s) = n] = ns/2$. Similarly,

$$Var[\sum_{i=1}^n S_i | N(s) = n] = Var(\sum_{i=1}^n U_i) = ns^2/12$$

by independence.

**Problem 2: Renewal Theory**

A traffic police officer is using the following strategy at an alcohol control for car drivers. At the start of the inspection period (say time $t = 0$) the drivers of all passing cars have to take an alcohol test, this continues until $k$ consecutive drivers test negative. From that moment on passing drivers are independently tested, each with probability $\alpha$. This continues until a driver with too much alcohol in his blood is caught by testing positive. From that moment on the officer starts his strategy anew and 100% inspection is implemented again until $k$ consecutive drivers test negative, etc. Assume that tested drivers test positive with probability $\beta$, independently of one another. For $j \geq 1$, denote the $j$-th driver (since time 0) to pass on the road by $x_j$. Let

$$J = \min\{j \geq k; \text{drivers } x_{j-k+1} \text{ up to } x_j \text{ test negative}\}$$

be the number of drivers tested until for the first time $k$ consecutive tests are negative. Let

$$L = \min\{j \geq 1; \text{driver } x_j \text{ tests positive}\}$$

be the label of the first driver to test positive.

**a)** Show that $\mathbb{E}[J] = \frac{1-(1-\beta)^k}{\beta(1-\beta)^k}$. (4p)

Solution: Conditioning on $L$ gives and using the hint in the second line gives

$$\mathbb{E}[J] = \sum_{i=1}^{\infty} \mathbb{E}[J|L=i]\mathbb{P}(L=i) = \sum_{i=1}^{k} \mathbb{E}[J|L=i]\mathbb{P}(L=i) + \sum_{i=k+1}^{\infty} \mathbb{E}[J|L=i]\mathbb{P}[L=i].$$

Note that for $i \leq k$, we have that $\mathbb{E}[J|L=i] = i + \mathbb{E}[J]$, since after $i$ we have to start anew waiting for $k$ consecutive negative tests. So, now using the hint:

$$\begin{aligned}\mathbb{E}[J] &= \sum_{i=1}^{k}(i+\mathbb{E}[J])\beta(1-\beta)^{i-1} + k\mathbb{P}[L>k] \\ &= \frac{1-(1-\beta)^k}{\beta} - k(1-\beta)^k + \sum_{i=1}^{k}\mathbb{E}[J]\beta(1-\beta)^{i-1} + k(1-\beta)^k \\ &= \frac{1-(1-\beta)^k}{\beta} + \mathbb{E}[J](1-\mathbb{P}(L>n)) = \frac{1-(1-\beta)^k}{\beta} + \mathbb{E}[J](1-(1-\beta)^k).\end{aligned}$$

And it follows that $(1-\beta)^k\mathbb{E}[J] = \frac{1-(1-\beta)^k}{\beta}$.

**b)** After the first $k$ consecutive negative tests, i.e. after time $J$, which driver will have the first positive test, i.e. find $\min\{j > J; x_j \text{ tests positive}\}$? (4p)

Solution: A fraction $\alpha$ of the drivers will be tested of which a fraction $\beta$ will be positive. So the first driver to test positive is $x_j$ with probability $\beta\alpha(1-\beta\alpha)^{j-J-1}$ for $j > J$.

**c)** In the long run, what fraction of the drivers will get tested? (4p)

Solution: Let a cycle start when 100% testing starts again. The expected cycle length is $\mathbb{E}[J]$ plus the expected number asked for in part *b*. The expected number of people tested is $\mathbb{E}[J]$ plus the expected number of tests until there is a positive test, which is Geometric with expectation $1/\beta$. So the asymptotic fraction tested is (using renewal reward theorem).

$$\frac{\frac{1-(1-\beta)^k}{\beta(1-\beta)^k} + \frac{1}{\beta}}{\frac{1-(1-\beta)^k}{\beta(1-\beta)^k} + \frac{1}{\alpha\beta}} = \frac{\frac{1-(1-\beta)^k}{(1-\beta)^k} + 1}{\frac{1-(1-\beta)^k}{(1-\beta)^k} + \frac{1}{\alpha}} = \frac{\alpha}{\alpha + (1-\alpha)(1-\beta)^k}.$$

**Problem 3: Queueing Theory**

Consider an $M/M/1$ queue in which customers arrive at rate $\lambda$, workloads are exponential with mean $1/\mu$ (i.e. customers in service depart at rate $\mu$), and there is 1 server serving customers. Assume $\mu > \lambda$.

**a)** Provide the asymptotic distribution of the number of customers in the system (i.e. the number of customers in the queue + the number of customers in service). (4p)

Solution: Let $\rho_i$ be the asymptotic probability of being in state $i$. Since in the long run the number of steps from $i$ to $i+1$ is at most 1 different from the number of steps from $i+1$ to $i$ and every state is arbitrary many times (because $\mu > \lambda$) the balance equations should be satisfied. Balance equations give for $i \geq 0$, we have $\lambda\rho_i = \mu\rho_{i+1}$, which implies $\rho_i = (\lambda/\mu)^i\rho_0$. In order to obtain that $\sum_{i=0}^{\infty}\rho_i = 1$, we have $\rho_i = (1-\lambda/\mu)(\lambda/\mu)^i$.

**b)** If the $k$-th customer stays for a time $t$ in the system, what is the expected time the $k+1$-st customer stays in the system. (4p)

Solution: Say that the $k$-th customer arrives at time $t_k$, then he leaves at time $t_k + t$. Let $a$ be the time between the arrivals of customers $k$ and $k+1$ and $s$ be the service time needed for customer $k+1$. Note that the $k+1$-st customer arrives at time $t_k + a$ If $t > a$, then the $k+1$ customer departs at $t_k + t + s$, while if $t < a$ then the $k+1$-st customer departs at $t_k + a + s$. So the time in the system for the $k+1$st customer is $\max(t-a+s, s)$ Let $A$ be exponentially distributed with mean $1/\lambda$ and $S$ exponentially distributed with mean $1/\mu$. Then the expected time we are after is

$$\mathbb{E}[S] + \mathbb{E}[\max(0, t-a)] = \frac{1}{\mu} + \int_0^t (t-a)\lambda e^{-\lambda a}da = \frac{1}{\mu} + te^{-\lambda t} - \frac{1}{\lambda}(1-e^{-t}),$$

where we have use partial integration.

**c)** What is the expected duration of a busy period? That is, what is the expected time from the arrival of a customer in a further empty system to the first time the system is empty again? (4p)

Solution: See P 504 of book.

**Problem 4: Brownian Motion and Stationary Processes**

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion with $B(0) = 0$. For $a, b > 0$, let $T_a = \inf\{t \geq 0; B(t) \geq a\}$ be the hitting time of $a$ and let $T_{ab}$ be the first hitting time of $-b$ after time $T_a$, i.e. $T_{ab} = \inf\{t \geq T_a; B(t) \leq -b\}$.

**a)** Compute $\mathbb{P}(T_{ab} \leq t)$. (4p)

Solution: By the reflection principle we know that $T_{ab}$ is distributed as $T_{2a+b}$ (from $T_a$ the time until reaching $-b$ is distributed as the time until reaching $2a+b$, by symmetry). We know that $\mathbb{P}(T_{ab} \leq t) = \mathbb{P}(T_{2a+b} \leq t)$, which by the cheat-sheet is $2\mathbb{P}(X(t) \geq 2a+b) = \sqrt{\frac{2}{\pi t}} \int_{2a+b}^{\infty} e^{-x^2/(2t)} dx$.

**b)** Compute the probability that $B(t)$ does not hit 0 in $t \in (1,2)$, that is, compute $\mathbb{P}(\{t \in (1,2); B(t) = 0\} = \emptyset)$. (4p)

Solution: Conditioning on $B(1)$ gives

$$\mathbb{P}(\{t \in (1,2); B(t) = 0\} = \emptyset) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathbb{P}(\{t \in (1,2); B(t) = 0\} = \emptyset | B(1) = x) dx.$$

By using independent and stationary increment property this is

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \mathbb{P}(\{t \in (0,1); B(t) = -x\} = \emptyset) dx$$

Note that $x < 0$ we have that $\mathbb{P}(\{t \in (0,1); B(t) = -x\} = \emptyset) = \mathbb{P}(T_{-x} > 1)$ and for $x > 0$ we have by symmetry that $\mathbb{P}(\{t \in (0,1); B(t) = -x\} = \emptyset) = \mathbb{P}(T_x > 1)$. By the cheat-sheet $\mathbb{P}(T_x > 1) = \sqrt{\frac{2}{\pi}} \int_x^{\infty} e^{-y^2/2} dy$. So,

$$\mathbb{P}(\{t \in (1,2); B(t) = 0\} = \emptyset) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \sqrt{\frac{2}{\pi}} \int_{|x|}^{\infty} e^{-y^2/2} dy dx$$

$$= \int_0^{\infty} \int_x^{\infty} \frac{2}{\pi} e^{-(x^2+y^2)/2} dy dx = \frac{1}{2} \left( \int_0^{\infty} \int_x^{\infty} \frac{2}{\pi} e^{-(x^2+y^2)/2} dy dx + \int_0^{\infty} \int_y^{\infty} \frac{2}{\pi} e^{-(x^2+y^2)/2} dx dy \right)$$

$$= \frac{1}{\pi} \int_0^{\infty} \int_0^{\infty} e^{-(x^2+y^2)/2} dx dy = 2 \left( \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-x^2/2} dx \right)^2 = 1/2.$$

**c)** Provide the definition of a Gaussian process. (4p)

Solution: A stochastic process $\{X(t), t \geq 0\}$ is a Gaussian or Normal process if $X(t_1), X(t_2), \cdots X(t_n)$ have a multivariate Normal distribution for all $n \geq 1$ and all $t_1, \cdots, t_n$

**Problem 5: Simulation**

Let $Z$ be a standard normal distributed random variable with expectation 0 and variance 1. We want to simulate from the absolute value of $Z$ i.e. from $|Z|$, which has density function $f(x) = \sqrt{2/\pi}e^{-x^2/2}$, for $x \geq 0$.
Suppose that we have cheap methods to simulate an exponentially distributed random variable with mean 1 (i.e. with density function $g(x) = e^{-x}$ for $x \geq 0$) and a uniformly distributed random variable on $(0, 1)$. We are going to use the rejection method.

**a)** Show that the maximum of $f(x)/g(x)$ for $x \geq 0$ is given by $c = \sqrt{2e/\pi}$.

Solution: $\frac{f(x)}{g(x)} = \sqrt{2/\pi}e^{-x^2/2}e^x = \sqrt{2/\pi}e^{-(x-1)^2/2}e^{-1/2} = \sqrt{2e/\pi}e^{-(x-1)^2/2}$. Note that $e^{-(x-1)^2/2} \leq 1$, while this value is taken in $x = 1$. So the required result follows.

**b)** Argue that the following method can be used to simulate from $|Z|$.
(i) Generate independent random variables $Y$ with density function $g(x)$ and $U$, which is uniformly distributed on $(0, 1)$.
(ii) If $U < e^{-(Y-1)^2/2}$ set $|Z| = Y$, otherwise return to step $(i)$. (4p)

Solution: This is the rejection method. We propose a random variable $Y$ with density function $g(x)$ and accept it with probability $f(Y)/[cg(y)] = e^{-(Y-1)^2/2}$. Otherwise reject and return to step 1. Accepting with probability $e^{-(Y-1)^2/2}$ is equivalent with accepting if an independent Uniform on (0,1) is less than $e^{-(Y-1)^2/2}$.

**c)** What is the probability of acceptance in step (ii) of part b), i.e. what is the probability that in this step $|Z| = Y$? (2p)

Solution: We accept with probability

$$\mathbb{E}[e^{-(Y-1)^2/2}] = \int_0^\infty e^{-y}e^{-(y-1)^2/2}dy = \frac{1}{\sqrt{e}}\int_0^\infty e^{-y^2/2}dy = \sqrt{\frac{\pi}{2e}} = \frac{1}{c}.$$

**d)** Argue that the following method can also be used to simulate from $|Z|$.
(i) Generate independent exponential random variables $Y_1$ and $Y_2$ both with expectation 1.
(ii) If $Y_2 > (Y_1 - 1)^2/2$ set $|Z| = Y_1$, otherwise return to step $(i)$. (3p)

Solution: This is the same method as in question b) only observing that $-\log[U]$ is distributed as an exponential random variable with mean 1.