

**Instructions:**

- During the exam you MAY NOT use textbooks, class notes, or any other supporting material.
- Use of calculators is permitted for performing calculations. The use of graphic or programmable features is NOT permitted.
- Start every problem on a new page, and write at the top of the page which problem it belongs to. (But in multiple part problems it is not necessary to start every part on a new page)
- In all of your solutions, give explanations to clearly show your reasoning. Points may be deducted for unclear/wrong reasoning even if the answer is correct.
- Write clearly and legibly.
- Where applicable, indicate your final answer clearly by putting A BOX around it.

Note: There are six problems, some with multiple parts. The problems are not ordered according to difficulty

- (1) (5pt) Compute the degree 3 Taylor polynomial of the function  $f(x) = e^{x^2}$ , around the point  $x_0 = 1$ , and use it to give an approximation of  $f(1.1)$ .

**Solution:** We begin by computing the first and second derivative of  $f(x)$  in  $x = 1$ :

$$f'(x) = 2xe^{x^2}.$$

We plug in  $x = 1$  and get  $f'(1) = 2e$ . For the second derivative we have

$$f''(x) = 2e^{x^2} + 4x^2e^{x^2} = (2 + 4x^2)e^{x^2}.$$

In  $x = 1$  we get  $f''(1) = 6e$ . For the third derivative we have

$$f'''(x) = 8xe^{x^2} + (4x + 8x^3)e^{x^2} = (12x + 8x^3)e^{x^2}.$$

In  $x = 1$  we get  $f'''(1) = 20e$ . Now we have all the ingredients to compute the Taylor polynomial:

$$p_3(x) = e^1 + 2e(x - 1) + 3e(x - 1)^2 + \frac{20}{3!}e(x - 1)^3.$$

Then we have that

$$f(1.1) \simeq p_3(1.1) = e + \frac{2e}{10} + \frac{3e}{100} + \frac{2e}{600} = \frac{739}{600}e \approx 3.34801711872$$

- (2) The equation

$$xe^y - y^2 = 2x^3 - e^{x-1}$$

defines a curve.

- (a) (1pt) Verify that point  $(1, 0)$  belongs to the curve.
- (b) (3pt) Compute the slope of the tangent line to the curve in  $(1, 0)$ .
- (c) (1pt) Give the equation of the tangent line to the curve in  $(1, 0)$ .

**Solution:** For the first point we must check that the RHS and the LHS are the same when  $(x, y) = (1, 0)$ . We get

$$RHS = 1 - 0^2 = 1;$$

$$LHS = 2 - e^0 = 1.$$

As they are the same  $(1, 0)$  belongs to the curve.

The first thing to do in order to compute the slope of the curve in the given point is to derive the equation implicitly. We get

$$-2y'y + e^y + xy'e^y = 6x^2 - e^{x-1}.$$

We impose again  $(x, y) = (1, 0)$  and get:

$$0 + e^0 + y'(1)e^0 = 6 - e^0.$$

We solve for  $y'(1)$  and get  $y'(1) = 4$ , which is exactly the slope of the curve in  $(1, 0)$ .

Finally, for the equation of the tangent line, we now that

$$y = 4x + m,$$

since the slope is exactly the slope of the curve. We have to find  $m$  given that the tangent passes through  $(1, 0)$ . We get  $m = -4$  so the equation we are looking for is

$$y = 4x - 4.$$

Consider the function  $f(x) = \frac{x^2-1}{x^2-4}$ .

- (2pt) Give the maximum and minimum value of the function in the interval  $[-1, 1]$ .
- (2pt) Find where the function is increasing or decreasing and concave or convex.
- (1pt) Compute  $\lim_{x \rightarrow +\infty} f(x)$ .

**Solution:** Observe that the function is continuous on its domain  $D = \mathbb{R} \setminus \pm 2$ , therefore it has max and min on  $[-1; 1]$ . We look for the critical points, by computing the first derivative:

$$f'(x) = \frac{2x(x^2 - 4) - (x^2 - 1)2x}{(x^2 - 4)^2} = \frac{2x(x^2 - 4 - x^2 + 1)}{(x^2 - 4)^2} = \frac{-6x}{(x^2 - 4)^2}.$$

We have that  $f'(x) = 0$  if, and only if,  $x = 0$ , which is an element of the interval  $[-1, 1]$ . The candidates for extreme points are  $x = \pm 1, 0$ . We evaluate the functions in these points and we get:

$$f(1) = f(-1) = 0, \quad f(0) = \frac{1}{4}.$$

We deduce that the maximum value of the function in  $[-1, 1]$  is  $\frac{1}{4}$  while the minimum value is 0.

As  $(x^2 - 4)^2$  is always positive, the function is increasing when  $x \leq 0$ ,  $x \neq -2$  and decreasing when  $x \geq 0, x \neq 2$ . For the concavity we have to compute the second derivative:

$$\begin{aligned} f''(x) &= \frac{-6(x^2 - 4)^2 + 24x^2(x^2 - 4)^2}{(x^2 - 4)^4} \\ &= \frac{(x^2 - 4)(-6x^2 - 4) + 24x^2}{(x^2 - 4)^4} \\ &= \frac{24 + 18x^2}{(x^2 - 4)^3} \end{aligned}$$

As  $24 + 18x^2$  is always positive, we have that the concavity of the functions depends on the sign of  $(x^2 - 4)^3$ . Thus  $f$  is concave when  $x \in (-2, 2)$  and convex when  $x < -2$  or  $x > 2$ .

For the last part we have

$$\lim_{x \rightarrow +\infty} f(x) = \lim_{x \rightarrow +\infty} \frac{x^2(1 - 1/x^2)}{x^2(1 - 4/x^2)} = \lim_{x \rightarrow +\infty} \frac{1 - 1/x^2}{1 - 4/x^2} = \frac{1}{1} = 1$$

(3) Compute the following integrals:

(a) (3pt)  $\int (x^2 \ln(x^3 + 1) + 5x \ln(x)) dx,$

(b) (2pt)  $\int_4^{\ln(8)} \frac{x}{6\sqrt{x^2 - 15}} dx.$

**Solution:** We begin with the first

$$\begin{aligned} \int (x^2 \ln(x^3 + 1) + 5x \ln(x)) dx &= \int x^2 \ln(x^3 + 1) dx + \int 5x \ln(x) dx \\ &= \frac{1}{3} \int \ln(u) du + \int 5x \ln(x) dx \end{aligned}$$

Where in the first summand I used the substitution  $u = x^3 + 1$ , so  $du = 3x^2 dx$ . Going forward we get

$$\begin{aligned} \int (x^2 \ln(x^3 + 1) + 5x \ln(x)) dx &= \frac{1}{3}(\ln(u)u - u) + C_1 + \frac{5}{2}x^2 \ln(x) - \frac{5}{4}x^2 + C_2 \\ &= \frac{1}{3}(\ln(x^3 + 1)(x^3 + 1) - (x^3 + 1)) + \frac{5}{2}x^2 \ln(x) - \frac{5}{4}x^2 + C \end{aligned}$$

For the second integral we set  $u = x^2 - 15$ . Thus  $du = 2x dx$ . We need to recompute the extremes of integration. In order to do that we have that if  $x = 4$  then  $u = 1$ , and if  $x = \ln(8)$ , the  $u = 2 \ln(8) - 15$ . Thus we get

$$\begin{aligned} \int_4^{\ln(8)} \frac{x}{6\sqrt{x^2 - 15}} dx &= \frac{1}{12} \int_1^{\ln(8)} \frac{du}{\sqrt{u}} dx \\ &= \frac{1}{24} [u^{\frac{1}{2}}]_1^{\ln(8)} = \frac{1}{24} (\ln(8)^{\frac{1}{2}} - 1). \end{aligned}$$

(4) Consider the matrix

$$A = \begin{pmatrix} 3 & 2 & 1 \\ k-1 & k-10 & -2 \\ 3 & k+2 & 1 \end{pmatrix}$$

- (a) (2 pt) Compute the determinant of  $A$ ,  $|A|$  as a function of  $k$ .  
 (b) (1 pt) Find all the values of  $k$  for which  $A$  is not invertible.  
 (c) (2 pt) Determine whether the following linear system has 1, 0, or infinite many solutions:

$$\begin{cases} 3x & +2y & +z & = & 1 \\ -1x & -10y & -2z & = & -3 \\ 3x & 2y & +z & = & 1 \end{cases}$$

**Solution:** We perform elementary operation on the rows of  $A$  to make the computations of the determinant easier:

$$\begin{aligned} \begin{vmatrix} 3 & 2 & 1 \\ k-1 & k-10 & -2 \\ 3 & k+2 & 1 \end{vmatrix} &= \begin{vmatrix} 3 & 2 & 1 \\ k-1 & k-10 & -2 \\ 0 & k & 0 \end{vmatrix} \\ &= (-1)(k)(-6 - (k-1)) = (k)(k+5) \end{aligned}$$

We see immediately that  $\det(A) = 0$  if and only if  $k = 0$  or  $k = -5$ , so we deduce that  $A$  is invertible if  $k \neq 0, -5$ .

We attempt to solve the system we perform elementary row operations to the matrix

$$\left( \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ -1 & 10 & -2 & -3 \\ 3 & 2 & 1 & 1 \end{array} \right)$$

We subtract the first row to the third row:

$$\left( \begin{array}{ccc|c} 3 & 2 & 1 & 1 \\ -1 & 10 & -2 & -3 \\ 3 & 2 & 1 & 1 \end{array} \right) \sim \left( \begin{array}{ccc|c} 5 & 4 & 1 & 1 \\ -7 & -4 & -2 & -3 \\ 0 & 0 & 0 & 0 \end{array} \right)$$

As we have obtained a row with all zero we know that the system is underdetermined. Therefore, it has infinite many solutions.

- (5) Consider the two variables function

$$f(x, y) = 3x^3 - x^2 - y^2.$$

- (a) (2pt) Find all the critical points of  $f(x, y)$  and determine their type.  
 (b) (1pt) Consider now the set  $D = \{(x, y) \mid x \geq 0, \text{ and } x^2 + y^2 \leq 1\}$ . Draw a sketch of  $D$  and determine which critical points of  $f(x, y)$  lie in the interior of  $D$ .  
 (c) (2pt) Determine the minimum and the maximum value of  $f(x, y)$  on  $D$ .

**Solution:** We first compute the partial derivatives of  $f$  and set them to zero to find critical points for  $f$  in the interior of  $D$ . We get

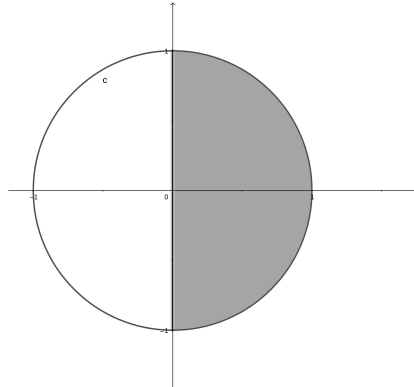
$$\frac{\partial}{\partial x} f(x, y) = 9x^2 - 2x = 0$$

$$\frac{\partial}{\partial y} f(x, y) = -2y = 0.$$

From the second equation we find  $y = 0$ . In the first equation we find  $x(9 - 2x) = 0$  Which has as solution  $x = 0$  and  $x = 2/9$ . This yields two critical points:  $(0, 0)$  and  $(2/9, 0)$ . To determine the type of the critical points we have to compute the Hessian matrix

$$\mathcal{H} = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \begin{pmatrix} 18x - 2 & 0 \\ 0 & -2 \end{pmatrix}$$

The determinant  $H$  of  $\mathcal{H}$  is  $-2(18x - 2)$  we evaluate this in the two critical points. In  $(0, 0)$   $H$  takes value 4 which is positive. As in  $(0, 0)$  we have that  $f_{xx} = -2$  and  $f_{yy} = -2$ , we deduce that  $(0, 0)$  is a local maximum. In  $(2/9, 0)$ ,  $H$  has value 2, so this is neither a minimum or a maximum point (it is a saddle point). The domain  $D$  consist in the interior of the unit circle in the first and fourth quadrant (see the gray are in the picture)



Both critical points lies in  $D$  as  $4/81 + 0 < 1$ . However  $(0, 0)$  is not in the interior of  $D$  and should not, a priori, be considered as a candidate for max or min point. In order to evaluate the min and max value of  $f$  in  $D$  we have to investigate the restriction of  $f$  to the boundary of  $D$ . Note that the boundary has two corners  $(0, -1)$  and  $(0, +1)$  these have to be investigate separately. The boundary consist of two curves, the curve  $x = 0$  and the curve  $x^2 + y^2 = 1$ . See the picture.

*Curve  $x = 0$ .* We have that  $f(0, y) = -y^2$ .  $\frac{d}{dy}f(0, y) = -2y$  has only one critical point when  $y = 0$  this yields the candidate  $(0, 0)$ .

*Curve  $x^2 + y^2 = 1$ .* If we restrict  $f$  to this curve we obtain the function  $g(x) = 3x^3 - 1$ . We have that  $g'(x) = 9x^2 - 1$  which yields the two critical points  $x = \pm\frac{1}{3}$ . As  $x$  needs to be nonnegative, we will consider only  $x = 1/3$  which yields  $y = \pm\frac{\sqrt{8}}{3}$ . Thus we get two further candidates for the max and min points  $(1/3, \sqrt{8}/3)$  and  $(1/3, -\sqrt{8}/3)$ .

*Conclusion:* From the reasoning above we get 6 candidates for max and min points of  $f$ ,  $(0, 0)$ ,  $(2/9, 0)$ ,  $(1/3, \pm\sqrt{8}/3)$ , and  $(0, \pm 1)$ . We compute the values that  $f$  takes at these points (as the point  $(2/9, 0)$  is a saddle point we could avoid testing  $f$  in it):

$$f(0, 0) = 0$$

$$f(2/9, 0) = -4/243.$$

$$f(1/3, \pm\sqrt{8}/3) = 1/9 - 1 = -8/9$$

$$f(0, \pm 1) = -1$$

Of these the biggest is surely  $f(0, 0)$  - it is the only one which is non-negative- and the smallest is  $f(0, \pm 1)$ , so these are respectively the maximum and the minimum value of  $f$  in  $D$ .

### FORMULAS

- Taylor polynomial of degree  $n$  of  $f(x)$  around the point  $x_0 = a$ :

$$p_n(x) = f(a) + f'(a)(x - a) + \frac{f^{(2)}(a)}{2}(x - a)^2 + \dots + \frac{f^{(n)}(a)}{n!}(x - a)^n$$

GOOD LUCK!!!

**Texten på svenska**

- (1) (5pt) Bestäm Taylor polynomet av grad 2 till funktionen  $f(x) = e^{-\sqrt{x+3}}$  kring punkten  $x_0 = 2$ .
- (2) Betrakta funktionen  $f(x) = \frac{x^2+x+3}{x+1}$ .
- (a) (2pt) Antar att  $f$  defineras på  $[0, 2]$ : bestäm max och min värden av  $f$  samt ge var dessa värden anta.
- (b) (2pt) Antar nu att  $f$  defineras på  $[0, +\infty)$ . Har funktionen nu en max värde? Förklara din mening.
- (c) (1pt) Bestäm  $\lim_{x \rightarrow +\infty} \frac{f(x)}{6x}$ .
- (3) Bestäm följande integraler:
- (a) (3pt)  $\int (x \ln(x^2 + 1) + 3\sqrt{x^5}) dx$ ,
- (b) (2pt)  $\int_0^1 \frac{5e^{3x}}{1 - 3e^{3x}} dx$ .
- (4) Betrakta serien  $2 + \frac{4}{x} + \frac{8}{x^2} + \frac{16}{x^3} + \dots$ . Bestäm
- (a) (2pt) för vilka  $x$  konvergerar serien;
- (b) (3pt) för vilka  $x$  är seriens värden 1.
- (5) Låt

$$A = \begin{pmatrix} 5 & 4 & 1 \\ k-7 & k-4 & -2 \\ 5 & k-2 & 1 \end{pmatrix}$$

- (a) (2 pt) Räkna determinanten till  $A$ ,  $|A|$  som en funktion av  $k$ .
- (b) (1 pt) Hitta alla värden  $k$  sådana att  $A$  inte är invertierbar.
- (c) (2 pt) Använd Gausselimination för att lösa det linjär systemet nedanför

$$\begin{cases} 5x & +4y & +1z & = & 1 \\ -7x & -4y & -2z & = & -3 \\ 5x & -2y & +z & = & 1 \end{cases}$$

- (6) Vi definierar området  $D$  i  $\mathbb{R}^2$  som kvadraten med hörnen  $(-1, -1)$ ,  $(-1, 1)$ ,  $(1, -1)$ ,  $(1, 1)$ .

Låt  $f(x, y) = xe^{3y} - e^x$ .

- (a) (1pt) Skissa området  $D$ .
- (b) (4pt) Bestäm max och min värde av  $f(x, y)$  i  $D$ .