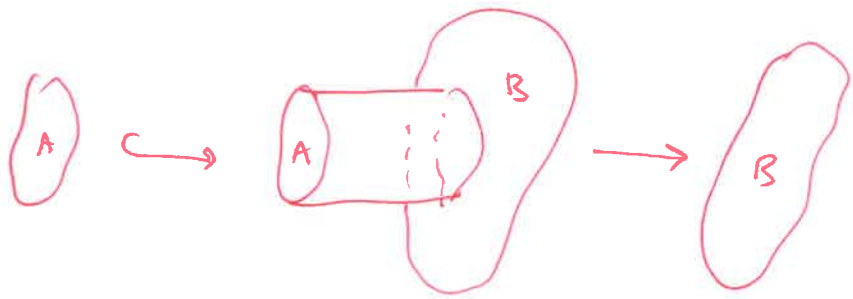
Path space fibration.

Recall mapping cylinder of $f: A \to B$:

$$A \xhookrightarrow[\text{cofibration}]{} M_f \xrightarrow{\simeq} B$$



and $M_f = (A \times I) \bigsqcup_{A \times \{0\}} B$ is a pushout.

We now define a dual construction, the <u>path space fibration</u>.

Define $P_f = \text{map}(I, B) \times_{\text{map}(\{0\}, B)} A$.

We have a factorization

$$A \xhookrightarrow[\text{constant maps}]{\simeq} P_f \longrightarrow B$$

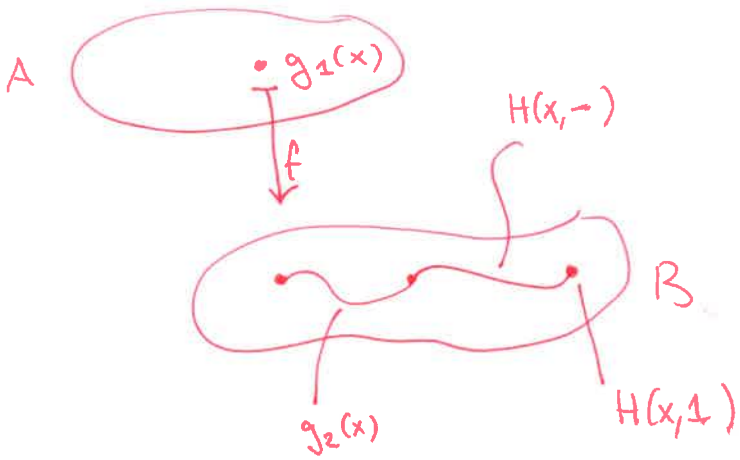
and I claim that $P_f \to B$ is always a fibration.

Consider

$$\begin{tikzcd} X \times \{0\} \arrow[r, "g"] \arrow[d] & P_f \arrow[d] \\ X \times I \arrow[r, "H"'] \arrow[ur, dashed, "\tilde{H}"] & B \end{tikzcd}$$

write $g = (g_1, g_2) \in P_f \subset A \times \mathrm{map}(I, B)$

<u>Note</u> end point of path $g_2(x)$ is start point of path $H(x,-)$.



$\tilde{H} = (\tilde{H}_1, \tilde{H}_2)$. $\tilde{H}_1 = g_1$.

idea: $\tilde{H}_2 \in \mathrm{map}(I, B)$ is a path that goes along $g_2(x)$ and then part of the path $H(x,-)$, ending at $H(x,s)$.

So $s = 0$: runs only along $g_2(x)$
$s = 1$: runs full length of both paths.

Then diagram commutes.

How to do this continuously?

$$\tilde{H}_2(x,s)(t) = \begin{cases} g_2(x)\big((1+s)t\big) & 0 \le t \le \frac{1}{1+s} \\ H\big(x, (1+s)t - 1\big) & \frac{1}{1+s} \le t \le 1 \end{cases}$$

$\boxtimes$

Example X based space, $i : \{x_0\} \hookrightarrow X$ inclusion of base point.

$$P_i = \{ \gamma : I \to X \mid \gamma(0) = x_0 \}$$

is w.e. to $\{*\}$ and $P_i \to X$ fibration.

Usually denoted path space, PX.

fiber is ΩX. Get fibration

$$\begin{array}{ccccc} \Omega X & \longrightarrow & PX & \longrightarrow & X. \\ & & \wr\!| & & \\ & & \{*\} & & \end{array}$$

Long exact sequence: see again that $\pi_n(X) \cong \pi_{n-1}(\Omega X)$ $\forall n$.

Def If $f : A \to B$ any map, define homotopy fiber as fiber of $P_f \to B$.

Explicitly, $\mathrm{hofib}(A \to B) = \left\{ (a, \gamma) \in A \times \mathrm{map}(I, B) \;\middle|\; \begin{array}{l} f(a) = \gamma(0) \\ \gamma(1) = * \end{array} \right\}$

basepoint of B.

Prop If $f : A \to B$ is a fibration, then

$$\begin{array}{ccc} A & \longrightarrow & P_f \\ & \searrow \;\; \swarrow & \\ & B & \end{array}$$

is a fiber homotopy equivalence.

(Note that the obvious map $P_f \to A$ is not a map of fibrations.)

Consider lifting problem

$$\begin{tikzcd} P_f \times \{0\} \arrow[r] \arrow[d] & A \arrow[d] \\ P_f \times I \arrow[r, "\gamma"'] \arrow[ur, dashed, "\tilde{\gamma}"] & B \end{tikzcd}$$

$\gamma(x, \alpha, t) = \alpha(t)$

Define $P_f \xrightarrow{g} A$ by $g(x, \alpha) = \tilde{\gamma}(x, \alpha, 1)$.

Then

$$\begin{tikzcd} A \arrow[r, shift left, "\text{const}"] \arrow[dr] & P_f \arrow[l, shift left, "g"] \arrow[d] \\ & B \end{tikzcd}$$

commutes and the maps are homotopy inverses. We need also that they are homotopy inverse $/B$.

Define $F: A \times I \to I$

by $F(x,t) = \tilde{\gamma}(x, \text{const}_{f(x)}, t)$.

then $F(x,0) = x$

$F(x,1) = g \circ h(x)$ by definition of g.

This is a vertical homotopy. $\boxtimes$

Cor f fibration with fibre F

$\Rightarrow \operatorname{hofib}(f) \simeq F$.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\sim"'] & B \arrow[d, "\sim"] \\ A' \arrow[r, "f'"] & B' \end{tikzcd} \quad \Rightarrow \quad \operatorname{hofib}(f) \overset{w.e.}{\simeq} \operatorname{hofib}(f')$$

by long exact sequence and five lemma.

Observation. $A \overset{i}{\hookrightarrow} B$ subspace.

$\operatorname{hofib}(i)$ = paths starting at A, ending at $x_0 \in A$.

hence: $(I^n, \partial I^n, J^{n-1}) \to (B, A, x_0)$

$$\updownarrow$$

$(I^{n-1}, \partial I^{n-1}) \to (\operatorname{hofib}(i), x_0)$

const. path.

and in particular

$\pi_n(B, A, x_0) \cong \pi_{n-1}(\operatorname{hofib} i)$.

Again we see the long exact sequence in homotopy of a fibration from the long exact sequence in relative homotopy.

Let $F \to E \to B$ fibration.

$$\begin{tikzcd}
 & \mathrm{hofib}(j) \arrow[r] & \mathrm{hofib}(i) \arrow[r, "j"] & \mathrm{hofib}(p) \arrow[r, "i"] \arrow[d] & E \arrow[r, "p"] \arrow[d, equal] & B \arrow[d, equal] \\
\cdots \arrow[r] & \Omega E \arrow[r] \arrow[u, "\simeq"] & \Omega B \arrow[r] \arrow[u, "\simeq"] & F \arrow[r] & E \arrow[r] & B
\end{tikzcd}$$

$F \overset{\simeq}{\hookrightarrow} \mathrm{hofib}(p)$.

Note i fibration since pullback of $\underset{P_{id}}{\overset{\shortparallel}{\mathrm{map}(I,B) \to B}}$ along p.

"Actual" fiber of i is ΩB, so get $\Omega B \xrightarrow{\simeq} \mathrm{hofib}(i)$

iterate.

Apply π_0 to get l.e.s. in homotopy.

dual sequence:

$$A \hookrightarrow X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \Sigma^2 A \to \cdots$$

Apply H^i / H_i to get l.e.s. in homology/cohomology.

Spectral sequences.

A <u>spectral sequence</u> is a collection

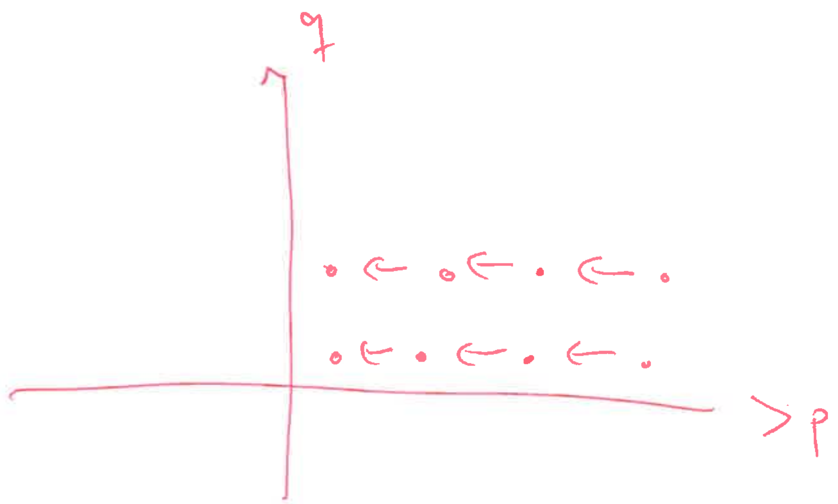
$$E^r_{pq} \qquad \begin{matrix} p,q \in \mathbb{Z} \\ r \geq r_0 \end{matrix} \qquad \text{of abelian groups / } R\text{-modules}$$

$+ \quad \partial : E^r_{pq} \longrightarrow E^r_{p-r,\, q+r-1}$ s.t. $\partial^2 = 0$

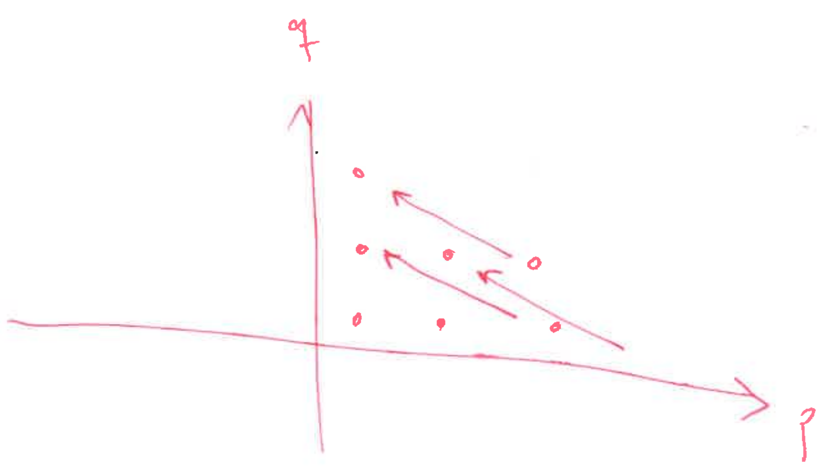
$+$ isomorphisms

$$H_{p,q}(E^r_{\bullet,\bullet}) \cong E^{r+1}_{p,q}.$$

E^1 :



E^2



we say that a class lives in bidegree (p,q)

total degree $p+q$.

Note All differentials lower total degree by 1.

There are also cohomological spectral sequences.

$$\{E_r^{pq}\} \text{ and } \partial : E_r^{pq} \to E_r^{p+r,\, q-r+1}.$$

raises total degree by 1.

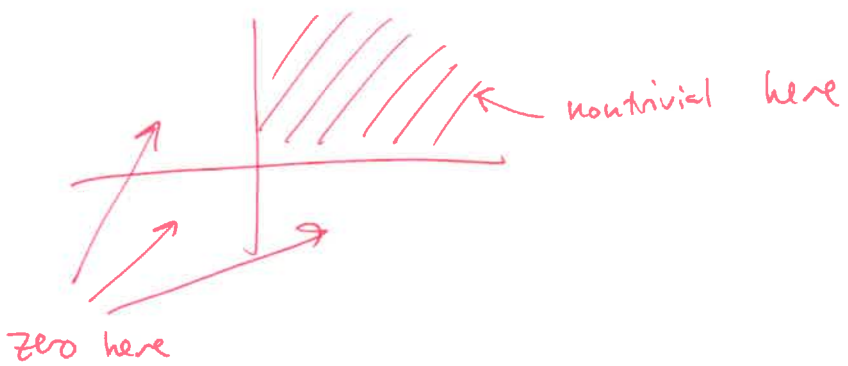
Suppose for some p,q, that all differentials into/out of E^r_{pq} vanish, for $r \geq N$.

Then $E^N_{pq} \cong E^{N+1}_{pq} \cong \dots$

and we denote this group E^∞_{pq}.

If this happens for all p,q, we say that the spectral sequence converges.

Lemma If $E^r_{pq} = 0$ when $p<0$ or $q<0$ then $\{E^r_{pq}\}$ converges. (first quadrant s.s.)



Proof Eventually all arrows to/from (p,q) slot stretch out of first quadrant. $\boxtimes$

Let $H_\bullet$ graded abelian group.

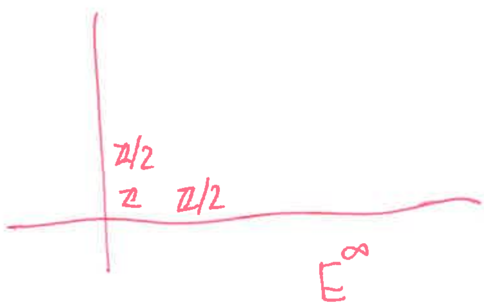
$\{E^r_{pq}\}$ <u>converges to $H_\bullet$</u> if $\exists$ filtration L on $H_\bullet$ s.t. $E^\infty_{pq} \cong Gr^L_p H_{p+q}$.

$$\cdots \subseteq L_i H_k \subseteq L_{i+1} H_k \subseteq \cdots$$

Typically we want to know $H_\bullet$, it is homology of something.
This may be hard.
If we can write down a spectral sequence and compute all differentials, we may be able to determine $Gr^L H_\bullet$ (almost $H_\bullet$).

<u>Ex</u> $\mathbb{Z}/4$ has two-step filtration with $Gr^L(\mathbb{Z}/4) \cong \mathbb{Z}/2 \oplus \mathbb{Z}/2$.

So we may be unable to see the difference between $\mathbb{Z}/4$ and $\mathbb{Z}/2 \oplus \mathbb{Z}/2$.



what is $H_1(-)$?

Example Suppose E^2_{pq} nonzero for $p \in \{0,1\}$

$\forall n$: $$0 \to E^2_{0,n} \to H_n \to E^2_{1,n-1} \to 0$$

and no differentials.

Example: Suppose E^2_{pq} nonzero for $q \in \{0,1\}$.

Get

$$\cdots \to H_{n+1} \to E^2_{n+1,0} \to E^2_{n-1,1} \to H_n \to E^2_{n,0} \to E^2_{n-2,1} \to \cdots$$

Theorem (Serre)

Let $F \to E \to B$ fibration.

Suppose B connected, $\pi_1(B)$ acts trivially on $H_*(F)$ / $H^*(F)$.

Then $\exists$ spectral sequences

$$E^2_{pq} = H_p(B, H_q(F)) \Rightarrow H_{p+q}(E)$$

$$E_2^{pq} = H^p(B, H^q(F)) \Rightarrow H^{p+q}(E).$$

Example Hopf bundle $S^1 \to S^3 \to S^2$.

E^2 :

$$\begin{tikzcd} \mathbb{Z} & \cdot & \mathbb{Z} \\ \mathbb{Z} & \cdot & \mathbb{Z} \arrow[ull] \end{tikzcd}$$

differential must be isomorphism since $H_1(S^3) = 0$.

E^3 :

$$\begin{array}{ccc} \cdot & \cdot & \mathbb{Z} \\ \mathbb{Z} & \cdot & \cdot \end{array}$$

which fits with

$H_0(S^3) \cong H_3(S^3) \cong \mathbb{Z}$.

Example

$$\begin{array}{ccccc} \mathbb{C}^* & \to & \mathbb{C}^{n+1} \setminus \{0\} & \to & \mathbb{C}P^n \\ \cup & & \cup & & \parallel \\ S^1 & \longrightarrow & S^{2n+1} & \longrightarrow & \mathbb{C}P^n \end{array}$$

from knowing homology of fiber, we must have the E^2-page

$$\begin{tikzcd} H_0 & H_1 & H_2 & \cdots & H_{2n} \\ H_0 & H_1 \arrow[ul] & H_2 \arrow[ul] & \cdots \arrow[ul] & H_{2n} \end{tikzcd}$$

$H_i = H_i(\mathbb{C}P^n)$

vanishes above dim $\mathbb{C}P^n$.

From our knowledge of $H_*(E) = H_*(S^{2n+1})$

we can deduce that $H_i = 0$; odd

$H_k \to H_{k-2}$ isomorphism

$k = 2, 4, \dots, 2n$.

so $H_i(\mathbb{C}P^n) = \begin{cases} \mathbb{Z} & i = 0, 2, \dots, 2n \\ 0 & \text{otherwise}. \end{cases}$

Could also have been seen from

$$\mathbb{C}P^n = \mathbb{C}^n \cup \mathbb{C}P^{n-1}$$
$$= \mathbb{C}^n \cup \mathbb{C}^{n-1} \cup \dots \cup \mathbb{C}^0$$

CW decomposition with no odd dim cells, one cell in each even dimension.