Cellular approximation

Thm $f: X \to Y$ map of CW complexes.

Then f is homotopic to a cellular map, i.e. $\tilde{f}$ such that $\tilde{f}(X^k) \subset Y^k$ $\forall k$.

If $A \subset X$ and f cellular on A, homotopy can be take a to be identity on A.

Cor $\pi_k(S^n) = 0$ for $k < n$.

Even $\pi_1(S^n) = 0$ for $n \geq 2$ is nontrivial (usually via Seifert-van Kampen).

But you might expect it to be trivial to prove $\pi_1(S^n) = 0$, as follows: take a loop $\gamma: S^1 \to S^n$, choose $p \in S^n \setminus \mathrm{im}(\gamma)$, project away from p, in the direction of the antipode, to homotope γ to a constant map.

Problem: γ may be surjective.

("space-filling curves")

Need to lift homotope γ to not be surjective.

idea smooth functions are dense in
$map_*(S^1, S^n) = \Omega S^n$.

convergence in $map(-,-)$ is uniform convergence on compacts.

Convolve with a sequence of bump functions

Smooth maps satisfy Sard's theorem:

$f : M^d \to N^e$ smooth map.

critical points of f are $x \in M$ s.t. rank of Jacobian at x is $< e$.

critical values of f are $f(x)$, x critical point.

(familiar from ordinary calculus in case of $f : \mathbb{R} \to \mathbb{R}$.)

Sard Set of critical values has Lebesgue measure zero.

Cor $d < e \Rightarrow f(M)$ has measure zero.

Proof of cellular approximation.

Suppose f cellular on X^{n-1}.

let $X' = X^{n-1} \coprod_{\Phi} D^n = X^{n-1} \cup e^n$.

$f(D^n)$ meets a finite # of cells by compactness, let e^m be one of maximal dimension.

WLOG $m > n$.

<u>Claim</u> $f|_{X'}$ homotopic rel X^{n-1} to map which is not onto e^m.

$\Rightarrow$ $f|_{X'}$ homotopic rel X^{n-1} to map which does not meet e^m at all.

Do this for all n-cells: $f|_{X^n}$ homotopic rel X^{n-1} to cellular map.

HEP: get homotopy on all of X

Do n^{th} step on interval $\left[1 - \frac{2}{2^n}, 1 - \frac{1}{2^{n+1}}\right]$.

$\boxed{X}$

Proof of claim.

We are given $f: D^n \longrightarrow Y$,

want homotopy rel ∂D^n s.t. not onto $e^m \subset Y$.

compactly contained in e^m.

choose $\mathcal{U} \Subset e^m$ open.

"make f smooth" on $f^{-1}(\mathcal{U})$ by convolving with sequence of bump functions.

"mollify" this homotopy near ∂D^n to make this homotopy rel ∂D^n.

$\boxtimes$

Cor Every map $(X,A) \to (Y,B)$ of CW pairs can be homotoped through maps $(X,A) \to (Y,B)$ to be cellular.

Cor A CW pair (X,A) is n-connected if all cells in $X \setminus A$ are of dimension $> n$.

[(X,A) is n-connected if $\pi_i(X,A,x_0) = 0$ for $i \leq n$.]

Thus there is (up to homotopy) no nontrivial way of mapping a lower-dimensional sphere to a higher-dimensional one.

By contrast there are highly nontrivial ways of e.g. mapping $S^3 \to S^2$. (Hopf)

<u>Thm</u> Ivanov-Mikhailov-Wu 16: $\pi_q(S^2) \neq 0 \quad \forall q \geq 2$.

Whitehead theorem.

<u>Thm</u> $f: X \to Y$ map of connected CW complexes. If $\pi_n X \xrightarrow{\sim} \pi_n Y \quad \forall n$, then f homotopy equivalence.

<u>Lemma</u> (X, A) and (Y, B) CW pairs. Suppose for all n s.t. $X \setminus A$ has n-cells that $\pi_n(Y, B) = 0$. Then any $f: (X, A) \to (Y, B)$ is homotopic rel A to a map into B.

<u>Proof</u> Induct over skeleton.

Assume $f(X^{k-1}) \subseteq B$.

let $X' = X^{k-1} \underset{\Phi}{\sqcup} D^k = X^{k-1} \cup e^k$.

map $(D^k, S^{k-1}) \to (X', X^{k-1}) \to (Y, B)$

is nullhomotopic by assumption, i.e.

represents zero in $\pi_k(Y,B)$. So it

can be homotoped rel S^{k-1} into B.

$\Longrightarrow$ get homotopy of $f|X'$ rel X^{k-1} into B.

Do same for all k-cells, get

homotopy of $f|X^k$ rel X^{k-1} into B.

HEP $\Rightarrow$ homotopy of f rel A s.t. $f(X^k) \subseteq B$.

do kth homotopy in $[1-\frac{1}{2^k}, 1-\frac{1}{2^{k+1}}]$.

$\boxtimes$

Proof of Whitehead.

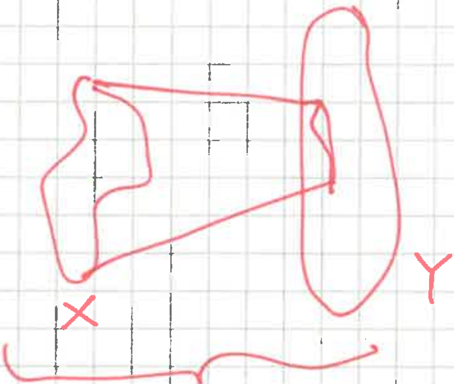
Case 1. $X \overset{f}{\hookrightarrow} Y$ inclusion of CW complexes.

LES in relative homotopy $\Rightarrow \pi_n(Y,X) = 0 \ \forall n$.

Lemma just shown: Y deformation retracts to X.

In general:

$$X \hookrightarrow M_f \xrightarrow{\sim} Y$$



M_f mapping cylinder

and (M_f, X) is a CW pair if f cellular. We are done by cellular approx. ☒

<u>NB</u> Thm does <u>not</u> say that $\pi_n X \cong \pi_n Y$ $\forall n$ implies $X \simeq Y$. It is important that there is a map $X \to Y$ inducing iso. on homotopy.

<u>Ex</u> $\mathbb{RP}^2$ and $S^2 \times \mathbb{RP}^\infty$ have isomorphic homotopy groups but are not homotopy equivalent.

Ex π group.

Let $K(\pi,n)$ be a space s.t.

$$\pi_i K(\pi,n) = \begin{cases} \pi & i=n \\ 0 & i\neq n. \end{cases}$$

(Suppose π abelian if $n>1$.)

We will see that such spaces exist and are unique/$\sim$ homotopy.

If spaces w/ same homotopy groups were htpy equivalent, then any space would be product of $K(\pi,n)$ spaces.

What is true is (roughly) that any space is an iterated fiber bundle with $K(\pi,n)$ fibers.

Def A <u>weak homotopy equivalence</u> is a map $f: X \to Y$ of top-spaces s.t. $\pi_n X \xrightarrow{\sim} \pi_n Y$ $\forall n$. and any basepoint.

Thm For any space X $\exists$ CW cx Z and a weak equivalence $Z \xrightarrow{\sim} X$.

Proof Suppose inductively we have constructed Z_k and $Z_k \to X$ s.t. $\pi_i(Z_k) \to X$ is onto $i \leq k$, injective $i < k$.

We will attach $(k+1)$cells to Z_k to get space Z_{k+1} w/ same property.

$$\begin{tikzcd} & Z_{k+1} \arrow[dr] & \\ Z_k \arrow[ur] \arrow[rr] & & X \end{tikzcd}$$

(1) choose generators of $\pi_{k+1}(X)$

For each generator attach trivially a $(k+1)$-cell to Z_k, extend map via chosen generator.

(2) choose generators of $\ker(\pi_k Z_k \to \pi_k X)$

WLOG cellular.

Use these as attaching maps for $(k+1)$-cells at each step we do not destroy previous surjectivity (clear) and injectivity (by cellular approximation). $\boxtimes$

Remark Construction is not functorial, depends on choice of generators at each step. To remedy this, make MAXIMAL choice: attach a $(k+1)$-cell for EVERY continuous map $S^k \to X$
result: geom. realization of singular set.

Prop X n-connected $\Leftrightarrow$ X w.e. to CW cx with n-skeleton a point.

Proof $\Leftarrow$ cellular approx.
$\Rightarrow$ build CW approximation by preceding argument. Note that for the first n steps one does not need to attach any cell!

For any map

$f: A \longrightarrow B$, any $n \in \mathbb{N}$

can construct

$$\begin{tikzcd} & X \arrow[dr] & \\ A \arrow[ur, hook] \arrow[rr] & & B \end{tikzcd}$$

where X is built from A by attaching cells, such that

$\pi_i(A) \to \pi_i(X)$ iso $i<n$, onto $i=n$

$\pi_i(X) \to \pi_i(B)$ iso $i>n$, injective $i=n$.

Again uses same technique.