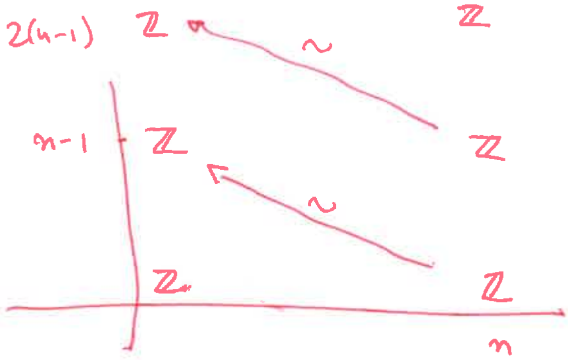
More examples of Serre s.s.

Consider path-loop fibration: $\Omega S^n \longrightarrow \underset{\underset{*}{\simeq}}{PS^n} \longrightarrow S^n \qquad n \geq 2$

Working inductively up the rows there is only one possibility for the entries in column 0 consistent with $PS^n \simeq *$.



$$\Rightarrow \quad H_q(\Omega S^n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & q = k(n-1), \quad k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

With more care we can also get the multiplicative structure on cohomology.

<u>Def</u> A cohomological spectral sequence $\{E_r^{pq}\}$ is <u>multiplicative</u> if $E_r^{\bullet,\bullet}$ has the structure of a bigraded cdga d_r, and $H(E_r^{\bullet,\bullet}) \cong E_{r+1}^{\bullet,\bullet}$ is an isomorphism of algebras.

It converges to a graded algebra $(H^\bullet, \cdot)$ if $\exists$ filtration $L^\cdot$ on H s.t.

$$\bigoplus_{p,q} E_\infty^{pq} \cong \bigoplus_{p,q} Gr_L^p H^{p+q}$$

isomorphism of algebras

we assume $L^i H^p \otimes L^j H^q \to L^{i+j} H^{p+q}$ ie $(H^\bullet, \cdot, L)$ filtered algebra.

<u>rmk</u> this means $xy = (-1)^{|y||x|}\, yx$

$$\partial(xy) = \partial(x)y + (-1)^{|x|} x\,\partial(y)$$

where $|\cdot|$ denotes total degree, i.e. $p+q$.

Consider $\mathbb{CP}^n$: $\quad H^p(\mathbb{CP}^n, H^q(S^1,\mathbb{Z})) \Rightarrow H^{p+q}(S^{2n+1},\mathbb{Z})$.

$$\begin{array}{|ccccc}
\mathbb{Z}a & \cdot & \mathbb{Z}ax & \cdot & \mathbb{Z}ax^2 \\
\mathbb{Z} & \cdot & \mathbb{Z}x & \cdot & \mathbb{Z}x^2 \\
\hline
\end{array}$$

$\partial(a) = x \qquad \partial(x) = 0$

$\partial(ax) = x^2$

$\partial(ax^2) = x^3$

$\vdots$

conclusion: $H^*(\mathbb{CP}^n,\mathbb{Z}) = \mathbb{Z}[x]/(x^{n+1}) \qquad |x| = 2$

$H^*(\mathbb{CP}^\infty,\mathbb{Z}) = \mathbb{Z}[x]$.

Consider $\Omega S^n \to PS^n \to S^n \qquad n$ odd.

$$\begin{array}{r|cc}
3(n-1) & \mathbb{Z}\frac{a^3}{6} & \mathbb{Z}\frac{a^3x}{6} \\
2(n-1) & \mathbb{Z}\frac{a^2}{2} & \mathbb{Z}\frac{a^2x}{2} \\
n-1 & \mathbb{Z}a & \mathbb{Z}ax \\
 & \mathbb{Z} & \mathbb{Z}x \\
\hline
 & & n
\end{array}$$

$\partial(a) = x$

$\partial(a^2) = ax + ax = 2ax$

$\partial(a^3) = 3a^2\partial(a) = 6\cdot \underbrace{\frac{a^2x}{2}}_{\text{generator}}$

Definition The <u>divided power algebra</u> $\Gamma(a)$ is the $\mathbb{Z}$-algebra generated by $\frac{a^k}{k!}$ $k \geq 1$.

<u>Conclude</u> n odd $\Rightarrow$ $H^*(\Omega S^n, \mathbb{Z}) \cong \Gamma(a)$ $|a| = n-1$.

(rationally we just get a polynomial ring.)

Even spheres similar:

$$
\begin{array}{l|ll}
 & \mathbb{Z}\frac{e^2}{2} & \mathbb{Z}\frac{e^2x}{2} \\
 & \mathbb{Z}ae & \mathbb{Z}aex \\
 & \mathbb{Z}e & \mathbb{Z}ex \\
 & \mathbb{Z}a & \mathbb{Z}ax \\
 & \mathbb{Z} & \mathbb{Z}x \\
\hline
\end{array}
$$

$\partial(a) = x$

$\partial(a^2) = 0$, so get new generator e.

$\partial(e) = ax$

$\partial(ae) = x \cdot e - a \cdot ax = xe$

$\partial(e^2) = 2e\,\partial(e) = 2eax$

$\rightsquigarrow$ $H^*(\Omega S^n, \mathbb{Z}) \cong \Gamma(e) \otimes \mathbb{Z}[a]/(a^2)$

$|a| = n-1$

$|e| = 2(n-1)$

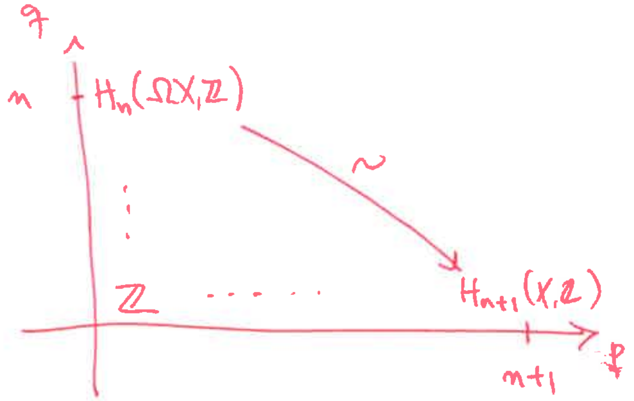
Hurewicz theorem.

Recall that if X is path-connected, then
$$\pi_1(X)^{ab} \cong H_1(X,\mathbb{Z}). \qquad (*)$$

The "higher" Hurewicz theorem says that if X is n-connected, then $\pi_{n+1}(X) \to H_{n+1}(X,\mathbb{Z})$ is an isomorphism.

We will prove it using Serre spectral sequence.

Let X n-connected, $n \geq 1$.
Consider $\Omega X \to PX \to X$.



Using that X is n-connected (and so ΩX is $(n-1)$-connected). the indicated differential is an isomorphism since SS converges to $H_\circ(PX) = 0$.

So
$$\begin{aligned} H_{n+1}(X,\mathbb{Z}) &\cong H_n(\Omega X,\mathbb{Z}) \\ &\cong \pi_n(\Omega X) \qquad [\text{by induction on } n] \\ &\cong \pi_{n+1}(X). \end{aligned}$$

Base case of induction is "usual" Hurewicz thm $(*)$.

We are ignoring the issue of how to identify $\pi_{n+1}(X) \to H_{n+1}(X)$ with the above differential.

$\boxtimes$

Relative Hurewicz.

Thm Let $A \hookrightarrow X$, $\pi_1(A)=0$, (X,A) is n-connected.
Then $\pi_{n+1}(X,A) \xrightarrow{\sim} H_{n+1}(X,A;\mathbb{Z})$.

Proof Use Serre SS in relative homology.

in general for a fibration $F \to E \to B$
and subspace $A \hookrightarrow B$, we can pull back fibration to A
to get

$$\begin{tikzcd} F \arrow[r] & E \arrow[r] & B \\ F \arrow[r] \arrow[u, equal] & D \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

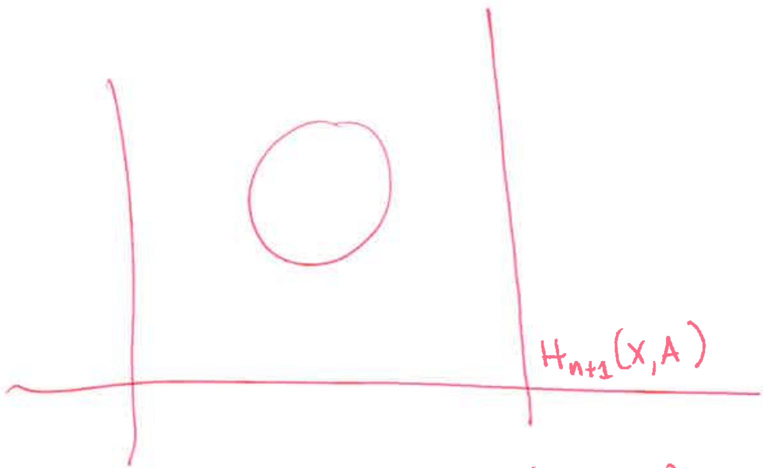
and $\exists$ spectral sequence

$$E^2_{pq} = H_p(B,A; H_q(F)) \Rightarrow H_{p+q}(E,D), \text{ if}$$

$\pi_1 A$ and $\pi_1 B$ act trivially on $H_*(F)$.
Consider

$$\begin{tikzcd} \Omega X \arrow[r] & PX \arrow[r] & X \\ \Omega X \arrow[r] \arrow[u, equal] & L \arrow[r] \arrow[u] & A \arrow[u] \end{tikzcd}$$

as in theorem. Since (X,A) is n-connected, first
n columns are zero:



so we get $H_{n+1}(X,A) \cong H_{n+1}(PX,L)$ since nothing
can kill this entry in the spectral sequence.
But $H_{n+1}(PX,L) \cong H_n(L)$ [PX contractible]
$\cong \pi_n(L)$ [absolute Hurewicz]
$\cong \pi_{n+1}(X,A)$

where in the last step we used that L is in fact our definition of $\mathrm{hofib}(A \hookrightarrow X)$, so L is $(n-1)$-connected and its homotopy groups are the relative homotopy of (X,A) shifted by 1. $\boxtimes$

<u>Corollary</u> (Homology Whitehead theorem)
Let $X \to Y$ homology isomorphism of simply connected spaces. Then $X \to Y$ is a weak equivalence.

<u>Proof</u> By CW approximation & mapping cone, we can assume that (Y,X) is a CW pair.
Then we get a long exact sequence in relative homology and relative homotopy.
All relative homology groups vanish implies by rel. Hurewicz that all relative homotopy groups vanish. $\boxtimes$

Another consequence is Freudenthal suspension theorem for spheres.
For any based X there is natural map $X \to \Omega \Sigma X$, unit of adjunction. For $X = S^n$ we get $S^n \to \Omega S^{n+1}$ and in particular $\pi_k(S^n) \to \pi_k(\Omega S^{n+1}) = \pi_{k+1}(S^{n+1})$ $\forall k,n$.

<u>Thm</u> (Freudenthal) $\pi_k(S^n) \to \pi_{k+1}(S^{n+1})$ isomorphism for $k \leq n-2$.

<u>Proof</u> Apply relative Hurewicz to $S^n \to \Omega S^{n+1}$.
From our calculation of $H_\bullet(\Omega S^{n+1})$ we have a homology isom. up to degree $2n$, so $H_{2n}(\Omega S^{n+1}, S^n; \mathbb{Z})$ first nonzero relative homology group. Then $\pi_{2n}(\Omega S^{n+1}, S^n)$ first nonzero relative homotopy group. Result follows from l.e.s. in relative homotopy. $\boxtimes$

Remark Argument also shows that

$$\pi_{2n-1}(S^n) \to \pi_{2n-1}(\Omega S^{n+1})$$

is surjective, with kernel generated by a single element.

Serre spectral sequence, construction.

For the construction of the Serre spectral sequence we use the spectral sequence of a filtered complex.

If A is a chain complex with an increasing filtration

$$\cdots \subseteq F_pA \subseteq F_{p+1}A \subseteq \cdots \quad , \quad d(F_pA) \subseteq F_pA$$

then there is a spectral sequence

$$E^1_{pq} = H_{p+q}(Gr^F_p A) \Rightarrow H_{p+q}(A)$$

which converges for degreewise bounded below and exhaustive filtrations.

We postpone the proof.

If X is a space filtered by subspaces,

$$\cdots \subseteq X_p \subseteq X_{p+1} \subseteq \cdots$$

then we get a filtration on $C_*(X,\mathbb{Z})$ by $F_p C_*(X,\mathbb{Z}) = C_*(X_p,\mathbb{Z})$.

In this case the spectral sequence of a filtered complex becomes

$$E^1_{pq} = H_{p+q}(X_p, X_{p-1}; \mathbb{Z}) \Rightarrow H_{p+q}(X,\mathbb{Z}).$$

Let us now construct Serre's spectral sequence.

Let $p: E \longrightarrow B$ fibration.

Let B_k : k-skeleton of B. Set $E_k = p^{-1}(B_k)$.

We get a filtration of the space E, hence a spectral sequence

$$E^1_{pq} = H_{p+q}(E_p, E_{p-1}; \mathbb{Z}) \Rightarrow H_{p+q}(E,\mathbb{Z}).$$

<u>claim</u> $E^1_{pq} \cong C_p(B, H_q(F,\mathbb{Z}))$.

why? Let σ be an open p-cell in B.

$p^{-1}(\sigma) \to \sigma$ trivial fibration, and $p^{-1}(\bar\sigma)/p^{-1}(\partial\sigma)$

$$\simeq D^p \times F / S^{p-1} \times F = \Sigma^p F_+$$

disjoint basepoint

now $E^1_{pq} = \tilde H_{p+q}(E_p/E_{p-1}, \mathbb{Z})$

$$= \tilde H_{p+q}\Big(\bigvee_{\sigma\ p\text{-cell}} p^{-1}(\bar\sigma)/p^{-1}(\partial\sigma), \mathbb{Z}\Big)$$

$$= \bigoplus_{\sigma\ p\text{-cell}} \tilde H_{p+q}\big(p^{-1}(\bar\sigma)/p^{-1}(\partial\sigma), \mathbb{Z}\big) = \bigoplus_{\sigma\ p\text{-cell}} \tilde H_{p+q}(\Sigma^p F_+, \mathbb{Z}) = \bigoplus_{\sigma\ p\text{-cell}} H_q(F,\mathbb{Z}).$$

We want to argue that E^1- differential of this spectral sequence equals cellular differential, so that the E^2-page is given by

$$E^2_{pq} = H_p(B, H_q(F)).$$

It is now we must assume $\pi_1 B$ acts trivially on $H_*(F)$. (otherwise we get homology with coefficients in a nontrivial local system.)

To be canonical on previous page we should write

$$E^1_{pq} = \bigoplus_{\substack{\sigma\ p\text{-cell} \\ \text{in } B}} H_q(F_x, \mathbb{Z})$$

where x is a point inside σ, for all σ.

Homology of fibers over different fibers are identified by monodromy: $x \mapsto F_x$ is a functor

$$\Pi_1(B) \longrightarrow \mathrm{Ho}(\mathrm{Top}),$$

composing with $H_*(-)$ gives $\Pi_1(B) \to Ab$.

If monodromy is trivial then this is equivalent to a constant functor and we can canonically identify all $H_*(F_x)$ for various x.

Result follows.

$\boxtimes$