Construction of spectral sequence of filtered complex.

A abelian group

$d: A \to A$ s.t. $d^2 = 0$

$HA = \ker(d)/\operatorname{im}(d)$

Suppose we have a filtration

$$\cdots \subset F_p A \subset F_{p+1} A \subset \cdots$$

$$d(F_p A) \subset F_p A.$$

Define
$$C_p^r = \{ x \in F_p A \mid dx \in F_{p-r} A \}$$
$$B_p^r = \{ x \in F_p A \mid x = dy,\ y \in F_{p+r} A \} = d(C_{p+r}^r).$$

note that

$$B_p^0 \subseteq B_p^1 \subseteq \cdots \subseteq B_p^\infty \subseteq C_p^\infty \subseteq \cdots \subseteq C_p^1 \subseteq C_p^0 = F_p A.$$

Define
$$E_p^r = \frac{C_p^r}{C_{p-1}^{r-1} + B_p^{r-1}}.$$

Since d maps C_p^r into C_{p-r}^r

and $C_{p-1}^{r-1} + B_p^{r-1}$ into B_{p-r}^{r-1}.

we get induced

$$d: E_p^r \to E_{p-r}^r \quad \forall p, r.$$

Let $d_p^r : E_p^r \to E_{p-r}^r$.

$$\text{kernel}(d_p^r) = \frac{(C_p^{r+1} + C_{p-1}^{r-1})}{(C_{p-1}^{r-1} + B_p^{r-1})}$$

$$\text{image}(d_{p+r}^r) = \frac{(C_{p-1}^{r-1} + B_p^r)}{(C_{p-1}^{r-1} + B_p^{r-1})}$$

$$\text{ker/im} = \frac{(C_p^{r+1} + C_{p-1}^{r-1})}{(B_p^r + C_{p-1}^{r-1})} = \frac{C_p^{r+1}}{C_p^{r+1} \cap (B_p^r + C_{p-1}^{r-1})}$$

$$= \frac{C_p^{r+1}}{B_p^r + C_{p-1}^r} = E_p^{r+1}.$$

Now suppose $A = \bigoplus_n A_n$ $\quad dA_n \subseteq A_{n-1}$.

then get additional grading E_{pq}^r where $p+q=n$.

conclusion: $\quad E_{pq}^1 = H_{p+q}(Gr_p^F A) \Rightarrow H_{p+q}(A)$.

This is the spectral sequence of a filtered complex.

Eilenberg MacLane spaces.

We say that a space X is a $K(\pi,n)$
if $\pi_k(X) = \begin{cases} \pi & k=n \\ * & k\neq n. \end{cases}$

Here π is a group, abelian if $n>1$.

$K(\pi,1)$ spaces appear many places in nature:

- $\mathbb{RP}^\infty$
- $\mathrm{Conf}_k(\mathbb{R}^2)$ configurations of points
- Any Riemann surface except $\mathbb{CP}^1$

X is a $K(\pi,1) \iff X$ has contractible universal cover

(since $\tilde{X}\to X$ iso on π_k for $k>1$.)

$K(\pi,n)$ for $n>1$ do not often appear naturally except for $\mathbb{CP}^\infty$, which is a $K(\mathbb{Z},2)$.

Proof Look at l.e.s. in homotopy associated to fibration $S^1 \to S^\infty \to \mathbb{CP}^\infty$, and that S^1 is a $K(\mathbb{Z},1)$.

Thm $K(\pi,n)$ spaces exist for any π, n.

Proof Choose surjection $\mathbb{Z}^N \twoheadrightarrow \pi$

let $X = \bigvee_N S^n$. Hurewicz $\Rightarrow \pi_n(X)\cong\mathbb{Z}^N$ and X is $(n-1)$-connected.

Let $K = \ker(\mathbb{Z}^N\to\pi)$. Attach cells to X to kill $K\subset\pi_n(X)$. Then attach cells to kill π_{n+1}, then π_{n+2}, etc.

(cf. construction of CW approximation.) $\boxtimes$

We will show that Eilenberg-MacLane spaces represent cohomology. But first we need the Puppe sequence.

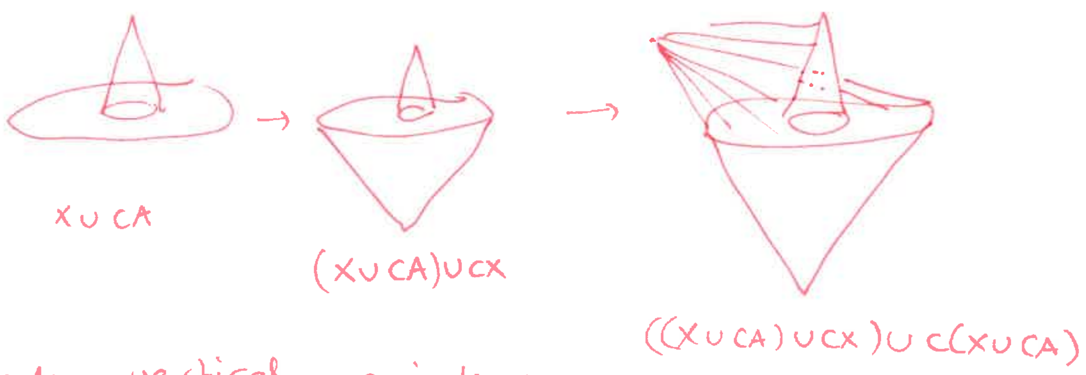
We say that $A \to X \to X/A$ is a cofiber sequence if $A \subset X$ is a CW pair and X/A is the space obtained by collapsing A to a point.

Recall construction for a fibration $F \to E \to B$:

$$\cdots \to \Omega^2 B \to \Omega F \to \Omega E \to \Omega B \to F \to E \to B$$

applying $\pi_i(-)$ gives l.e.s. in homotopy.
More generally taking $[K,-]$ for any space K gives a long exact sequence.
We now dualize construction:

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, equal] & X \arrow[r] \arrow[d, equal] & X \cup CA \arrow[r] \arrow[d, "\sim"] & (X \cup CA) \cup CX \arrow[r] \arrow[d, "\sim"] & ((X \cup CA) \cup CX) \cup C(X \cup CA) \arrow[r] \arrow[d, "\sim"] & \cdots \\
A \arrow[r] & X \arrow[r] & X/A \arrow[r] & \Sigma A \arrow[r] & \Sigma X \arrow[r] & \cdots
\end{tikzcd}$$



where each vertical equivalence is given by collapsing the most recently attached cone to a point.

This is the Puppe sequence.

since $\tilde{H}_i(X) \cong \tilde{H}_{i+1}(\Sigma X)$

and $\tilde{H}_i(X,A) \cong \tilde{H}_i(X/A)$

we see that applying $\tilde{H}_i(-)$ to Puppe sequence gives long exact sequence in homology, dual to l.e.s. in homotopy. More generally applying $[-,K]$ for any K gives a long exact sequence.

<u>Thm</u> For any based CW complex X,
$$[X, K(G,n)] \cong \tilde{H}^n(X,G),$$
naturally in X, G.

<u>Step 1</u> Construct natural transformation
$$\Theta : [-, K(G,n)] \Rightarrow \tilde{H}^n(-,G).$$
Such a natural transformation is an element of
$$\tilde{H}^n(K(G,n), G).$$
now $H_n(K(G,n), \mathbb{Z}) \cong G$ canonically.
Universal coefficients gives (since $H_{n-1} = 0$)
$$\Rightarrow H^n(K(G,n), G) \cong \mathrm{Hom}_{Ab}(G,G)$$
and id is a canonical element of $\mathrm{Hom}(G,G)$.

<u>Step 2</u> Θ iso if X is a sphere.
(direct computation)

<u>Step 3</u> Θ iso if X wedge of spheres.

Proof: both functors take wedges functorially to cartesian products.

Step 4 Θ iso for any k-dim CW complex.

Use induction on k, base case $k=0$ is OK.

$\exists$ cofiber sequence

$$\bigvee S^{k-1} \to \underset{\substack{\uparrow \\ (k-1)\text{-skeleton}}}{X_{k-1}} \to \underset{\substack{\uparrow \\ k\text{-skeleton}}}{X_k}$$

since cone on attaching map is the result of attaching cells.

Now apply Puppe sequence, writing $K = K(G,n)$
$H = \tilde{H}^n(-, G)$

$$\begin{array}{ccccccccc} [\Sigma X_{k-1}, K] & \to & [\vee S^k, K] & \to & [X_k, K] & \to & [X_{k-1}, K] & \to & [\vee S^{k-1}, K] \\ \downarrow \Theta' & & \downarrow \simeq & & \downarrow \Theta & & \downarrow \simeq & & \downarrow \simeq \\ H(\Sigma X_{k-1}) & \to & H(\vee S^k) & \to & H(X_k) & \to & H(X_{k-1}) & \to & H(\vee S^{k-1}) \end{array}$$

maps 2, 4, 5 are iso by induction and by step 3 of proof.

four lemma $\Rightarrow$ Θ onto.

Since X_k can be ANY k-dim CW cx, get also Θ' onto.

four lemma again $\Rightarrow$ Θ injective.

Step 5 induction stops.

claim: $[X, K(G,n)] \to [X_k, K(G,n)]$

and $\tilde{H}^n(X, G) \to \tilde{H}^n(X_k, G)$

are iso for $k \geq n+1$.

This is clear for cohomology.

Now $[X, K(G,n)] \to [X_k, K(G,n)]$ onto: indeed, pick $X_k \to K(G,n)$ and extend it one cell at a time.

Map can be extended to a d-dimensional cell if and only if its attaching map $S^{d-1} \to K(G,n)$ is nullhomotopic, which is automatic for $d > n+1$.

Similarly given $f, g : X \to K(G,n)$ s.t. $f|_{X_k} \simeq g|_{X_k}$ consider diagram

$$\begin{tikzcd}
X_k \times I \cup X \times \partial I \arrow[r] \arrow[d, hook] & K(G,n) \\
X \times I \arrow[ur, dashed]
\end{tikzcd}$$

all cells of dim $\geq k+2$.

$\boxtimes$

<u>Cor</u> (silly) $K(G,n)$ spaces are unique up to weak equivalence.

<u>Proof</u> they represent a functor. $\boxtimes$

<u>Cor</u> $\mathrm{Nat}(\tilde{H}^i(-,A), \tilde{H}^j(-,B)) = \tilde{H}^j(K(A,i),B)$

<u>Proof</u> Yoneda lemma.

<u>Cor</u> Natural transformations between cohomology functors may raise degree, but never lower degree.

<u>Proof</u> $\tilde{H}^j(K(A,i),B) = 0$ for $j < i$.

<u>Ex</u> $H^4(K(\mathbb{Z},2),\mathbb{Z}) \simeq \mathbb{Z}$

generator represents operation $x \mapsto x^2$, $H^2 \to H^4$.

<u>Ex</u> $H^2(\mathbb{RP}^\infty, \mathbb{Z}) \simeq \mathbb{Z}/2$.

represents Bockstein map
$$H^1(-, \mathbb{Z}/2) \Downarrow H^2(-, \mathbb{Z}).$$

Let us use $K(\pi, n)$ spaces to compute

$$\pi_1^S = \pi_4(S^3) \cong \pi_5(S^4) \cong \pi_6(S^5) \cong \cdots$$

Consider map $S^3 \to K(\mathbb{Z},3)$ classified by generator of $H^3(S^3, \mathbb{Z})$ and let F be its homotopy fiber.

$$\Omega K(\mathbb{Z},3) = K(\mathbb{Z},2) \to F \to S^3 \to K(\mathbb{Z},3)$$

$$\Rightarrow \quad H^p(S^3, H^q(K(\mathbb{Z},2),\mathbb{Z})) \Rightarrow H^{p+q}(F, \mathbb{Z}).$$

$$\begin{array}{c|cccc}
\mathbb{Z}a^2 & & & \mathbb{Z}a^2x \\
\cdot & & & \cdot \\
\mathbb{Z}a & & & \mathbb{Z}ax \\
\cdot & & & \\
\mathbb{Z} & \cdot & \cdot & \mathbb{Z}x \\
\hline
\end{array}$$

E_2

$\partial a = x$

$\partial(a^2) = 2a\partial a$
$= 2ax.$

$$\begin{array}{c|cccc}
\cdot & \cdot & \cdot & \mathbb{Z}/3 \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \mathbb{Z}/2 \\
\cdot & \cdot & \cdot & \cdot \\
\mathbb{Z} & \cdot & \cdot & \cdot \\
\hline
\end{array}$$

$E_4 = E_\infty$

So $H^5(F, \mathbb{Z}) \cong \mathbb{Z}/2$

and then $H_4(F, \mathbb{Z}) \cong \mathbb{Z}/2.$

Hurewicz: $H_4(F, \mathbb{Z}) \cong \pi_4(F) \cong \pi_4(S^3)$

via long exact sequence in homotopy associated to $F \to S^3 \to K(\mathbb{Z},3)$.

$\boxtimes$