Fibrations with Eilenberg-MacLane fibre.

Consider a fibration $K(G,n) \to E \to B$.
We say that it is a <u>principal fibration</u> if there is a diagram

$$\begin{tikzcd}
K(G,n) \arrow[r, equal] \arrow[d] & K(G,n) \arrow[d] \\
E \arrow[r] \arrow[d] & * \arrow[d] \\
B \arrow[r] & K(G,n+1)
\end{tikzcd}$$

making $E \to B$ the pullback of the universal path-loop fibration over $K(G,n+1)$.

<u>Thm</u> $E \to B$ is principal if and only if $\pi_1(B)$ acts trivially on G.

[$\pi_1(B)$ acts on $H_n(K(G,n),\mathbb{Z}) \cong G$ by monodromy.]

<u>Proof</u> Only if: the action factors through $\pi_1 B \to \pi_1 K(G,n+1) = *$.
if: let B/E denote the homotopy cofiber of $E \to B$.
We have a relative Hurewicz map

$$\pi_{*-1} K(G,n) \xrightarrow{\cong} \pi_*(B,E) \to H_*(B,E) \cong \widetilde{H}_*(B/E)$$

and the relative Hurewicz theorem implies that B/E is n-connected and $\pi_{n+1}(B/E) \cong H_{n+1}(B/E) \cong G$.
(Here is where we use that $\pi_1 B$ acts trivially!)

Now we can build a $K(G,n+1)$ from B/E by attaching cells of dim $\geq n+3$. We get a commutative diagram (up to homotopy)

$$\begin{tikzcd}
E \arrow[r] \arrow[d] & B \arrow[d] \arrow[r] & B/E \arrow[dl] \\
* \arrow[r] & K(G,n+1) &
\end{tikzcd}$$

and the result follows by comparing the long exact sequences in homotopy.

$\boxtimes$

To me this argument is a bit mysterious. Here is an alternative argument which is more conceptual but uses material outside the scope of the course.

Let X be a based space. There is a fiber sequence

$$hAut_\circ(X) \to hAut(X) \to X$$

where $hAut(-)$ resp $hAut_\circ(-)$ denote the topological monoids of unpointed resp. pointed self-equivalences of X, and $hAut(X) \to X$ evaluate a self-equivalence at the base point.

Fibrations over B with fiber X are classified by maps $B \to BhAut(X)$.

Now take $X = K(G,n)$. Then $hAut_\bullet(X) \simeq Aut(G)$ considered as a discrete group. Indeed $hAut_\bullet(X) \subset map_\bullet(X,X)$ and

$$\pi_0\, map_\bullet(X,X) = [K(G,n), K(G,n)] = Hom(G,G)$$
$$\pi_k\, map_\bullet(X,X) = [\Sigma^k K(G,n), K(G,n)] = H^n(\Sigma^k K(G,n), G) = 0 \text{ for } k>0.$$

Taking classifying spaces gives

$$BAut(G) \to BhAut(K(G,n)) \to K(G,n+1)$$

so the nontrivial homotopy groups of $BhAut\, K(G,n)$ are

$$\pi_1 \cong Aut(G)$$
$$\pi_{n+1} \cong G$$

with the obvious action of π_1 on π_{n+1}.

Now a fibration $K(G,n) \to E \to B$ is classified by $B \to BhAut(K(G,n))$ and if $\pi_1 B$ acts trivially on G this map lifts to the universal cover of $BhAut(K(G,n))$ which is $K(G,n+1)$.

<u>Conclusion</u> Principal fibrations $K(G,n) \to E \to B$ are classified by $H^{n+1}(B,G)$.

Moore-Postnikov towers and obstruction theory.

The subject of obstruction theory is the following question. Suppose given a cofibration $A \hookrightarrow X$ and a fibration $E \to B$, and a diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

When is there a lift?

$$\begin{tikzcd} A \arrow[r] \arrow[d] & E \arrow[d] \\ X \arrow[r] \arrow[ur, dashed, "\exists ?"] & B \end{tikzcd}$$

We already know some answers:

- If $A \hookrightarrow X$ is of the form $Z \times \{0\} \cup W \times I \hookrightarrow Z \times I$ for a CW pair (Z, W), then a lift always exists.
- More generally, some of you may have seen that a lift exists if $A \to X$ or $E \to B$ is a weak equivalence. This is part of the axioms of a model category.
- Hatcher's "compression lemma" (Hatcher Lemma 4.6) can be seen as answering this type of lifting problem, at least up to homotopy. He shows that if for all n such that $X \setminus A$ has n-cells, the group $\pi_{n-1}(F)$ vanishes, then a lift exists (where $F = \mathrm{fib}(E \to B)$.)

The general technique of obstruction theory subsumes the above examples.

Obstruction theory can be developed either by filtering $A \hookrightarrow X$ by relative skeleta, or by a so called Moore-Postnikov tower of $E \to B$.

We take the second approach.

Moore-Postnikov tower.

Let $f: X \to Y$ a map. A Moore-Postnikov tower of f is a diagram

$$\begin{tikzcd} & Z_3 \arrow[d] \arrow[dr] & \\ & Z_2 \arrow[d] \arrow[dr] & \\ X \arrow[r] \arrow[ur] \arrow[uur] & Z_1 \arrow[r] & Y \end{tikzcd}$$

s.t. vertical maps are fibrations, with $\pi_i X \to \pi_i Z_n$ iso $i<n$, onto $i=n$; $\pi_i Z_n \to \pi_i Y$ iso $i>n$, injective $i=n$

and $\mathrm{fib}(Z_{n+1} \to Z_n) = K(\pi_n F, n)$ with $F = \mathrm{hofib}(X \to Y)$.

[in fact last condition is implied by the previous part.]

Thm Moore-Postnikov towers exist and are unique up to homotopy.

Proof let us first build a system of maps $X \to Z_n \to Y$ satisfying correct conditions on homotopy groups. we use same technique as when constructing CW approximations:

- attach $(n+1)$-cells to X to kill $\ker(\pi_n X \to \pi_n Y)$
- attach more $(n+1)$-cells to X to make the map on π_{n+1} onto
- attach $(n+2)$-cells to kill kernel of map on π_{n+1}

$\vdots$

the result is a space with correct behavior on homotopy groups, a kind of "interpolation" of X and Y along f.

Now we need to fill in vertical arrows.

We set up lifting problem

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Z_n \arrow[d] \\ Z_{n+1} \arrow[r] & Y \end{tikzcd}$$

and Hatcher's compression lemma gives existence of lift and since $Z_{n+1} \setminus X$ has cells in dim $\geq n+2$ and $\pi_i(\mathrm{hofib}(Z_n \to Y)) = 0$ for $i \geq n$.

Same argument shows that Z_n independent of choices (given two Z_n, Z_n' set up lifting problem $\begin{tikzcd} X \arrow[r] \arrow[d] & Z_n' \arrow[d] \\ Z_n \arrow[r] & Y \end{tikzcd}$)

and that vertical maps are unique up to homotopy

$$\left(\begin{tikzcd} X \times I \cup Z_{n+1} \times \partial I \arrow[r] \arrow[d] & Z_n \arrow[d] \\ Z_{n+1} \times I \arrow[r] & Y \end{tikzcd} \right).$$

Now turn vertical maps into fibrations.
To see that fibers of vertical maps are Eilenberg-MacLane we consider diagram

$$\begin{tikzcd} \pi_{n+1}(Z_{n+1}) \arrow[r] \arrow[d, equal] & \pi_{n+1}(Z_n) \arrow[r] \arrow[d, equal] & \pi_{n+1}(Z_n, Z_{n+1}) \arrow[r] \arrow[d] & \pi_n(Z_{n+1}) \arrow[r] \arrow[d, equal] & \pi_n(Z_n) \arrow[d] \\ \pi_{n+1}(Z_{n+1}) \arrow[r] & \pi_{n+1}(Y) \arrow[r] & \pi_{n+1}(Y, Z_{n+1}) \arrow[r] & \pi_n(Z_{n+1}) \arrow[r] & \pi_n(Y) \\ \pi_{n+1}(X) \arrow[r] \arrow[u, two heads] & \pi_{n+1}(Y) \arrow[r] \arrow[u, equal] & \pi_{n+1}(Y, X) \arrow[r] \arrow[u] & \pi_n(X) \arrow[r] \arrow[u, equal] & \pi_n Y \arrow[u, equal] \end{tikzcd}$$

where both middle vertical arrows are iso by five lemma. $\boxtimes$

When Y is point, we call this a <u>Postnikov tower</u> of X. So any space is "cofiltered" by Eilenberg-MacLane spaces canonically. This is dual to a filtration by skeleta, in a sense.

When X is a point this is the <u>Whitehead tower</u> of Y, and Z_n is called the <u>n-connective cover</u> of Y.

<u>Example</u> Z_1 is the universal cover of Y (kills π_1).

<u>Example</u> the 2-connected cover of S^2 is S^3.

This follows from the Hopf fibration $S^3 \to S^2$, and the long exact sequence in homotopy.

From our earlier discussion of Eilenberg-MacLane fibrations we also see that if $\pi_1(X)$ acts trivially on $\pi_n(\text{hofib } X \to Y)$ $\forall n$ then we may assume all fibrations in Moore-Postnikov tower to be <u>principal</u>.

In particular, a simply connected space X, or more generally a space X s.t. $\pi_1 X$ acts trivially on $\pi_n X$ $\forall n \geq 1$ (in particular π_1 is abelian) always admits a Postnikov tower of principal fibrations.

In this case, X can (at least in principle) be completely described by its k-invariants.

Set
$$\begin{array}{lll} X_1 = K(\pi_1 X, 1) & \longleftarrow & \text{(note the shift in indexing — this would previously be } Z_2.) \\ \uparrow & & \\ X_2 \leftarrow K(\pi_2 X, 2) & & \\ \uparrow & & \\ X_3 \leftarrow K(\pi_3 X, 3) & & \end{array}$$

to be its Postnikov tower. Each successive fibration is classified by $X_n \to K(\pi_{n+1} X, n+2)$ which defines a class
$$k_n \in H^{n+2}(X_n, \pi_{n+1} X).$$

We have seen that a space is not determined by its homotopy groups, but it IS determined by its homotopy groups and the data of all successive k-invariants. However for this to be made precise we need to discuss convergence of the Postnikov tower.

Suppose in general that $X_1 \leftarrow X_2 \leftarrow X_3 \leftarrow \cdots$ is a tower of fibrations. Define $\varprojlim_n X_n$ to be the subspace of $\prod_n X_n$ of tuples (x_n) s.t. $x_n \mapsto x_{n-1}$ $\forall n$.

Prop The natural map $\pi_i(\varprojlim X_n) \to \varprojlim \pi_i(X_n)$ is onto. It is injective if $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ onto for $n \gg 0$.

<u>Proof</u> If $f \in \varprojlim \pi_i(X_n)$ represented by elements $[f_n] \in \pi_i(X_n)$ $\forall n$

s.t. $[f_n] \mapsto [f_{n-1}]$ then by homotopy lifting

we may successively replace f_2, f_3, etc to get

$f_n \mapsto f_{n-1}$ (i.e. not just up to homotopy).

Hence surjectivity.

For injectivity, wlog $\pi_{i+1}(X_n) \to \pi_{i+1}(X_{n-1})$ onto $\forall n$.

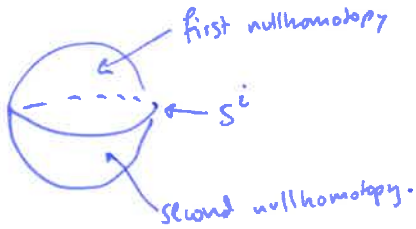
Suppose $f: S^i \to \varprojlim X_n$ goes to zero in $\varprojlim \pi_i(X_n)$.

We have nullhomotopies $F_n: D^{i+1} \to X_n$ $\forall n$.

We have $p_n F_n \simeq F_{n-1}$ on S^i (where $p_n: X_n \to X_{n-1}$)

and we may consider the two nullhomotopies as defining a map

$S^{i+1} \to X_{n-1}$.



This map now lifts to X_n, so we can rechoose F_n

so that $p_n F = F_{n-1}$ on the whole disk. Do this

inductively.

$\boxtimes$

<u>Cor</u> The Moore-Postnikov tower converges: For $X \to \{Z_n\} \to Y$

we have $X \to \varprojlim Z_n$ a weak equivalence.

$\boxtimes$

Let us now return to obstruction theory.

Consider
$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & E \arrow[d] \\ X \arrow[r] & B \end{tikzcd}$$

and suppose $\pi_1 E$ acts trivially on $fib(E \to B)$.

Replace $E \to B$ with a Moore-Postnikov tower of principal fibrations. Enough to construct a lift one step at a time. Hence

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & Z_{n+1} \arrow[r] \arrow[d] & * \arrow[d] \\ X \arrow[r] & Z_n \arrow[r] & K(\pi_n, n+1) \end{tikzcd}$$

it suffices to find a lift in this diagram.

But this exists if and only if $X \to K(\pi, n+1)$ is nullhomotopic rel A, i.e. if the class represented by $X \to K(\pi, n+1)$ vanishes in

$$H^{n+1}(X, A; \pi_n(fib(E \to B))).$$

The consequence is that there is a sequence of obstruction classes $\omega_n \in H^{n+1}(X, A; \pi_n(F))$

s.t. if $\omega_n = 0$ then ω_{n+1} can be constructed.

(Unfortunately ω_{n+1} is not necessarily unique!)

The criterion is that if all ω_n vanish then a lift exists.

Immediately implies e.g. Hatcher's compression lemma.