Homework set 2

HW1. Suppose that $F \to E \to B$ is a fibration with $\pi_1(B)$ acting trivially on $H_\bullet(F)$. Suppose that $H_i(B) = 0$ for $0 < i < m$ and $H_j(F) = 0$ for $0 < j < n$. Show that there is an exact sequence

$$H_{n+m-1}(F) \to H_{n+m-1}(E) \to H_{n+m-1}(B) \to H_{n+m-2}(F) \to \ldots \to H_1(E) \to H_1(B) \to 0.$$

Apply this result to the path-loop fibration $\Omega\Sigma X \to P\Sigma X \to \Sigma X$ of an $(n-1)$-connected space X, to show that $\pi_k(X) \to \pi_k(\Omega\Sigma X)$ is an isomorphism for $k \leq 2n-2$, generalizing the statement we proved for $X = S^n$ in class.

(As in class, you should feel free to take on faith that the differentials in the spectral sequence agree with any geometrically defined homomorphisms you can cook up.)

HW2. Consider the fibration $S^1 \to S^{2n+1} \to \mathbb{CP}^n$ and take the quotient by the antipodal map on the total space. We obtain a fibration sequence

$$S^1 \to \mathbb{RP}^{2n+1} \to \mathbb{CP}^n,$$

since S^1 modulo its antipodal involution is S^1. Determine the differentials in the spectral sequence to compute the cohomology groups $H^\bullet(\mathbb{RP}^{2n+1}, \mathbb{Z})$ and $H^\bullet(\mathbb{RP}^{2n+1}, \mathbb{Z}/2)$. (Hint: use that $\pi_1(\mathbb{RP}^n) = \mathbb{Z}/2$ and the universal coefficient theorem to determine H^2, which gives you the first differential, and then use multiplicativity.)

Does the ring structure you get on the E_∞ page coincide with the ring structure on $H^\bullet(\mathbb{RP}^{2n+1})$?

Deadline: 2022–10–18. If you have used any resources outside the course literature/lecture notes, please indicate this in your solution. Similarly if you have discussed the problems with another student. Hand in your solutions by e-mail to: dan.petersen@math.su.se