SOLUTION TO HOMEWORK SET 3, PROBLEM 2.

HW2. Consider the Postnikov tower of S^2. Show that the k-invariant

$$k_2 \in H^4(K(\pi_2(S^2),2),\pi_3(S^2)) = H^4(\mathbb{CP}^\infty,\mathbb{Z})$$

is a generator for this abelian group.

Suggested solution. Consider the diagram of fiber sequences

$$\begin{array}{ccccc} K(\mathbb{Z},3) & \longrightarrow & S^2[3] & \xrightarrow{f} & S^2[2]=K(\mathbb{Z},2) \\ \| & & \downarrow & & \downarrow \\ K(\mathbb{Z},3) & \longrightarrow & * & \xrightarrow{g} & K(\mathbb{Z},4). \end{array}$$

There is an induced map of cohomological Serre spectral sequences $E_r^{pq}(g) \to E_r^{pq}(f)$.

From our knowledge of the cohomologies of the base spaces and fibers involved, the only differential in low degrees can happen on the E_4-page. On the E_4-page we have a commuting diagram:

$$\begin{array}{ccccccc} \mathbb{Z} \cong H^3(K(\mathbb{Z},3),\mathbb{Z}) & = & E_4^{0,3}(g) & \xrightarrow{d} & E_4^{4,0}(g) & = & H^4(K(\mathbb{Z},4),\mathbb{Z}) \cong \mathbb{Z} \\ & & \downarrow & & \downarrow & & \\ \mathbb{Z} \cong H^3(K(\mathbb{Z},3),\mathbb{Z}) & = & E_4^{0,3}(f) & \xrightarrow{d} & E_4^{4,0}(f) & = & H^4(K(\mathbb{Z},2),\mathbb{Z}) \cong \mathbb{Z} \end{array}$$

The top horizontal differential is an isomorphism, since the Serre spectral sequence for g converges to zero. The left vertical arrow is an isomorphism, since the fibration f is the pullback of the fibration g, so the map on fibers is an equivalence. The right vertical arrow by definition takes a generator for $H^4(K(\mathbb{Z},4),\mathbb{Z})$ to the Postnikov invariant k_2. Conclusion: the assertion that k_2 is a generator is equivalent to saying that the lower horizontal differential is an isomorphism. Thus we will be done if we can show that $H^4(S^2[3],\mathbb{Z}) = 0$.

Now we consider the long exact sequence in relative homotopy for the map $S^2 \to S^2[3]$. It is immediate that the first nontrivial relative homotopy group is $\pi_5(S^2[3],S^2)$ (and in fact this group is isomorphic to $\pi_4(S^2) \cong \pi_4(S^3) \cong \mathbb{Z}/2$, but we do not need this). Hence the first nontrivial relative homology group is $H_5(S^2[3],S^2;\mathbb{Z})$, and $H_i(S^2,\mathbb{Z}) \to H_i(S^2[3],\mathbb{Z})$ is an isomorphism for $i < 5$. The result follows (by universal coefficients).