Lecture 1

You already know:

$$\pi_1(X,x_0) = [S^1, X] = [(I,\partial I),(X,x_0)].$$

makes group structure more transparent.

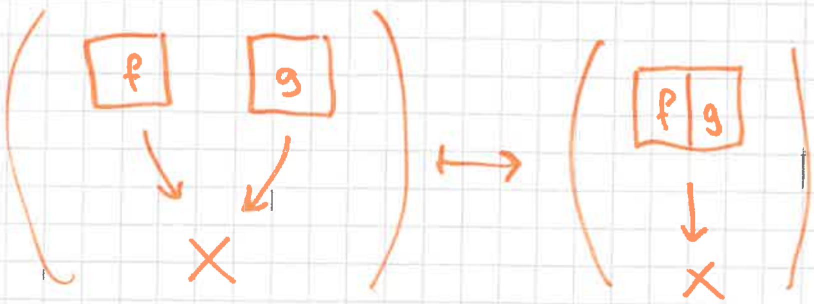
Definition. For all $n \geq 0$, let

$$\pi_n(X,x_0) = [S^n, X] = [(I^n,\partial I^n),(X,x_0)].$$

Prop if $n \geq 1$, $\pi_n(X,x_0)$ is a group.

(π_0 is a pointed set).

$$(f+g)(x_1,\dots,x_n) = \begin{cases} f(2x_1,\dots,x_n) & 0 \leq x_1 \leq \frac{1}{2} \\ g(2x_1-1,\dots,x_n) & \frac{1}{2} \leq x_1 \leq 1 \end{cases}$$



Proof is same as for $n=1$, since group operation only uses first coordinate.

E.g. associativity follows from a homotopy

Inverses are defined by

$$(f^{-1})(x_1,\dots,x_n) = f(1-x_1,\dots,x_n).$$

Since this figure is symmetric about the middle, we can homotope to a constant map by "cutting out" larger and larger slabs from the middle and rescaling the remainder.

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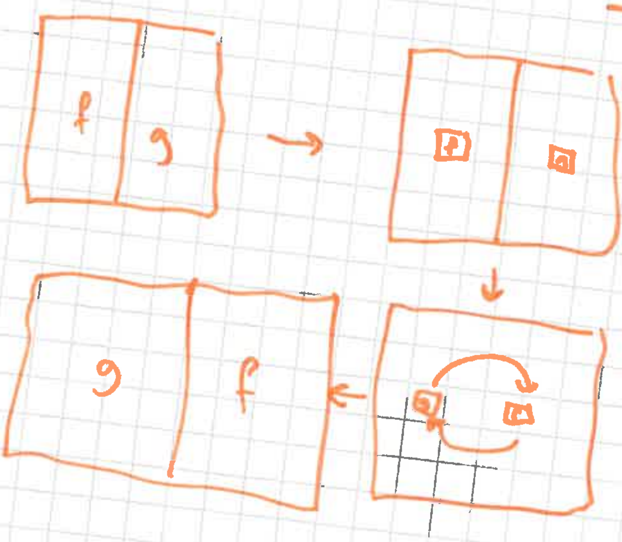
Prop $\pi_n(-)$ is a functor from pointed spaces to groups.

Prop If $n \geq 2$, $\pi_n(X, x_0)$ is abelian

Proof



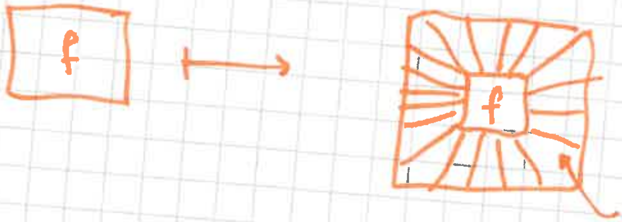
(Eckmann-Hilton) $\boxtimes$

Basepoint dependence.

Recall if X is path-connected then $\pi_1(X, x_0) \cong \pi_1(X, x_1)$ for any two $x_0, x_1 \in X$.

More generally: if γ path from x_0 to x_1, get

$\gamma^* : \pi_n(X, x_1) \to \pi_n(X, x_0)$ by



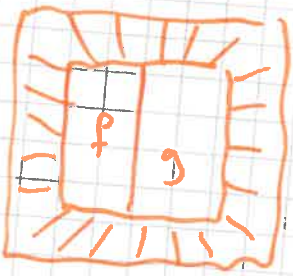
each line here is a little γ.

Prop This defines action of $\pi_1(X, x_0)$ on $\pi_n(X, x_0)$ $\forall n \geq 1$.

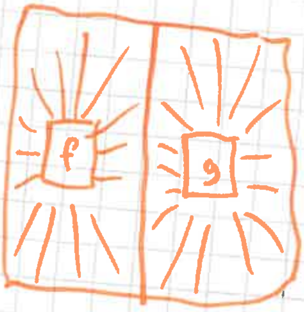
[when $n=1$, π_1 acts on itself by conjugation.]

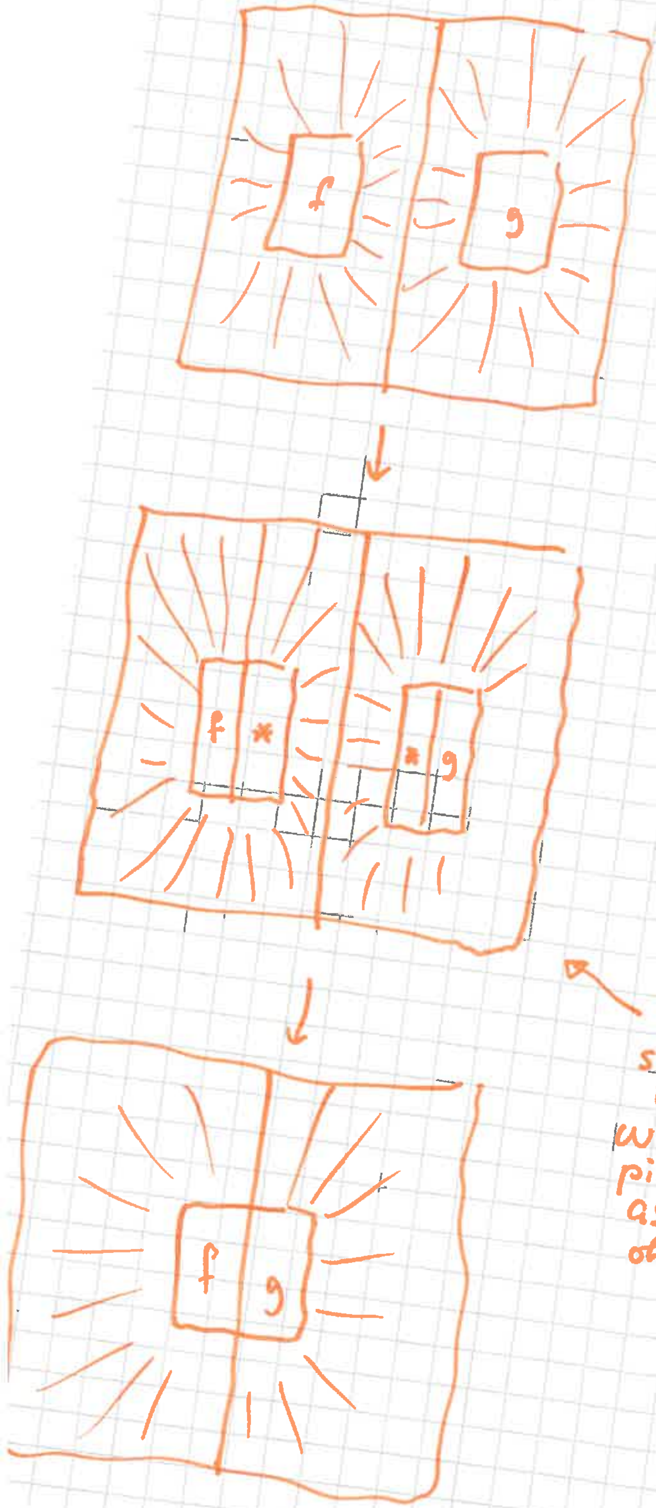
We check $\gamma^*(f+g) = \gamma^* f + \gamma^* g$.

Want homotopy between



and





$*$ = const. at basepoint.

this picture is symmetric across the middle, so we can cut out larger and larger pieces and rescale as in construction of inverses in π_n.

Alternative description, when $n \geq 2$.

$\tilde{X} \to X$ universal cover,

$\tilde{x}_0 \mapsto x_0$.

$$\pi_n(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_n(X, x_0)$$

is an isomorphism for $n \geq 2$: any $S^n \to X$ lifts to $\tilde{X}$, since S^n simply connected. also homotopies lift.

Any path γ based at x_0 lifts to a deck transformation $\gamma_* : \tilde{X} \to \tilde{X}$, and

$$\begin{tikzcd}
\pi_n(\tilde{X}, \tilde{x}_0) \arrow[rr, "\gamma_*"] \arrow[dr, "\cong"'] & & \pi_n(\tilde{X}, \tilde{x}_0) \arrow[dl, "\cong"] \\
& \pi_n(X, x_0) &
\end{tikzcd}$$

Relative homotopy groups.

$J^{n-1} \subset I^n$ is $\overline{(\partial I^n \setminus I^{n-1})}$.

$J^0 = \bullet$ $\quad J^1 =$  $\quad J^2 =$ 

missing bottom

<u>Def</u> $\pi_n(X, A, x_0) = \left[(I^n, \partial I^n, J^{n-1}), (X, A, x_0)\right]$

$$= \left[(D^n, S^{n-1}, pt), (X, A, x_0)\right]$$

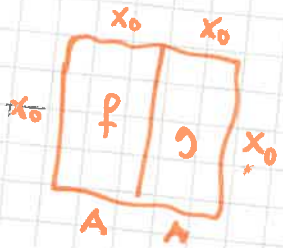
$n \geq 1$.

$x_0 \in A \subset X$.

So $\pi_1(X, A, x_0)$ is the set of paths starting at x_0, ending anywhere in A. (not a group)

$\pi_2(X, A, x_0)$: loop in A based at x_0 and null homotopy in X.

(a group!)



Prop $\pi_n(X,A,x_0)$ group if $n \geq 2$
abelian if $n \geq 3$.

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Rmk $A = \{x_0\}$ get back usual homotopy groups.

Lemma $f : (D^n, S^{n-1}, x_0) \to (X, A, x_0)$ represents zero
iff f is homotopic rel S^{n-1} to a map with image in A.

$\Leftarrow$ D^n is contractible

$\Rightarrow$ we have nullhomotopy $D^n \times I \to X$.
want to modify it to be constant on S^{n-1}.

$\exists$ homotopy $D^n \times I \to D^n \times I$

taking bottom & outside to bottom.
precompose with this.

$\boxtimes$

Thm $\exists$ long exact sequence

$$\cdots \to \pi_n(A,x_0) \to \underset{(1)}{\pi_n(X,x_0)} \to \underset{(2)}{\pi_n(X,A,x_0)} \to \underset{(3)}{\pi_{n-1}(A,x_0)} \to \cdots$$

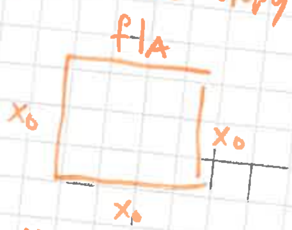
(of abelian groups, then groups, then pointed sets)

Pf Exactness at (1): lemma just proved.

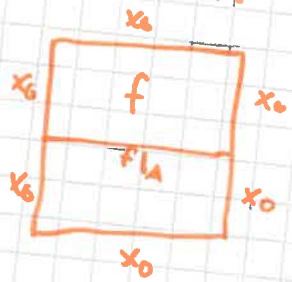
Exactness at (2): clear composition $= 0$.

If f goes to zero in $\pi_{n-1}(A,x_0)$

represent nullhomotopy as



$\rightsquigarrow$



Exactness at (3): clear.

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Hurewicz map.

$$\pi_n(X, x_0) \longrightarrow H_n(X, \mathbb{Z})$$

[$n=1$: Hurewicz theorem says

$$\pi_1(X, x_0) \to H_1(X, \mathbb{Z})$$

exhibits H_1 as abelianization of π_1.]

In general neither injective nor surjective.

For relative groups there is

$$\pi_n(X, A, x_0) \longrightarrow H_n(X, A; \mathbb{Z})$$

and a map of long exact sequences.

$\boxtimes$

Mapping spaces.

$map(X,Y)$ set continuous maps $X \to Y$

We give $map(X,Y)$ the compact-open topology: coarsest topology s.t. all sets

$$\{ f \in map(X,Y) \mid f(K) \subseteq \mathcal{U} \}$$

are open, $\forall K \subseteq X$ compact, $\forall \mathcal{U} \subseteq Y$ open.

Fact $Hom(X \times Y, Z) \cong Hom(X, map(Y,Z))$ (*)

Y locally compact, all spaces Hausdorff

[or CGWH, if $X \times Y$ has k-topology...]

Need also version for based maps.

$map_*(X,Y) \subset map(X,Y)$

subspace of based maps, X, Y have basepoints.

To state the analogue of (*) for based spaces we need to modify cartesian product on LHS.

To see what to do, easiest to first think of what happens for pointed sets.

In set:

$$\mathrm{Hom}_{\underline{set}}(X\times Y, Z) \cong \mathrm{Hom}_{\underline{set}}(X, \mathrm{Func}(Y,Z))$$

In $\underline{Set_\bullet}$

$$\mathrm{Hom}_{\underline{Set_\bullet}}(X\wedge Y, Z) \cong \mathrm{Hom}_{\underline{Set_\bullet}}(X, \mathrm{Func}_\bullet(Y,Z))$$

$$X\wedge Y = X\times Y/\sim \qquad (x_0,y)\sim(x_0,y') \quad \forall y,y'\in Y$$
$$(x,y_0)\sim(x',y_0) \quad \forall x,x'\in X$$

i.e. $X\wedge Y = X\times Y / X\vee Y$.

Fact same hypotheses:

$$\mathrm{map}_\bullet(X\wedge Y, Z) \cong \mathrm{map}_\bullet(X, \mathrm{map}_\bullet(Y,Z)).$$

Examples $[X,Y] = \pi_0\, \mathrm{map}_\bullet(X,Y)$

$CX \cong I\wedge X$ (reduced) cone

$\Sigma X \cong S^1\wedge X$ (reduced) suspension.

[for non-pathological spaces, reduced $\simeq$ nonreduced]

$\Omega X \cong \mathrm{map}_\bullet(S^1, X)$

$S^n\wedge S^m \cong S^{n+m}$

$$\Omega^n X = \mathrm{map}_\bullet(S^n, X) = \overbrace{\Omega\Omega\Omega\cdots\Omega}^{n} X$$

$\pi_n(X, x_0) = \pi_0 \Omega^n X$

Another way to think about group structure on homotopy groups:

S^1 is a cogroup object in pointed Spaces$/\sim$

$\mathrm{map}_\circ(-, X)$ takes cogroup objects to group objects.

$[Y, \Omega Z]$

Similarly $[\Sigma Y, Z]$ is a group $\forall Y, Z$.

$\cong$

$\pi_1 \mathrm{map}_\circ(Y, Z)$