Theorem (Serre)

X simply connected finite complex.
Then $\pi_n(X)$ is finitely generated for all n.

So $\quad \pi_n(X) = \mathbb{Z}^r \oplus T$

T finite abelian. unknown in general.

$\pi_n(X) \otimes \mathbb{Q} \cong \mathbb{Q}^r$. often computable!

Goal of second part:

- Develop tools to compute $\pi_k(X) \otimes \mathbb{Q}$ following Sullivan.
- Today: Serre's theory.

Serre classes

A non-empty class $\mathcal{C}$ of abelian groups is called a Serre class if it satisfies:

(i) If $0 \to A' \to A \to A'' \to 0$ is exact,

then $A \in \mathcal{C} \Leftrightarrow A', A'' \in \mathcal{C}$.

Examples: • Trivial groups
- Finite groups
- Finitely generated groups
- Torsion groups

Def: A homomorphism $f: A \to B$ is called a

- $\mathcal{C}$-monomorphism if $\ker(f) \in \mathcal{C}$
- $\mathcal{C}$-epimorphism if $\operatorname{coker}(f) \in \mathcal{C}$
- $\mathcal{C}$-isomorphism if $\ker(f), \operatorname{coker}(f) \in \mathcal{C}$.

We will also assume

(ii) $A, B \in \mathcal{C} \Rightarrow A \otimes B, \operatorname{Tor}_1^{\mathbb{Z}}(A,B) \in \mathcal{C}$

(iii) $A \in \mathcal{C} \Rightarrow H_k(K(A,1)) \in \mathcal{C}$ for all $k > 0$.

(This holds in the examples above)

Proposition $F \to E \to B$ fibration of path-connected spaces, and $\pi_1 B$ acting trivially on $H_*(F)$.
If $\tilde{H}_*(F) \in \mathcal{C}$, then $\tilde{H}_*(B) \in \mathcal{C} \iff \tilde{H}_*(E) \in \mathcal{C}$.

Proof Two observations:

(1) $\tilde{H}_{\leq p}(B) \in \mathcal{C} \iff E^2_{\leq p,q} \in \mathcal{C}$ for all q, because

$E^2_{p,q} \cong H_p(B; H_q(F))$ (universal coefficient theorem)

$\cong H_p(B) \otimes H_q(F) \oplus \mathrm{Tor}_1^{\mathbb{Z}}(H_{p-1}(B), H_q(F))$

(2) $H_n(E) \in \mathcal{C} \iff E^\infty_{p,q} \in \mathcal{C}$ for all $p+q=n$, because

$E^\infty_{p,q} \cong F_p/F_{p-1}$ for a filtration

$H_n(E) = F_n \supset F_{n-1} \supset \ldots \supset F_0 \supset F_{-1} = 0$.

So we just need to show $E^2_{p,q} \in \mathcal{C}$ for all $(p,q) \neq (0,0) \iff$

$E^\infty_{p,q} \in \mathcal{C}$ — " —

$\Rightarrow$ clear since $E^\infty_{p,q}$ is a subquotient of $E^2_{p,q}$.

$\Leftarrow$ Induction on p. $E^2_{0,q} = H_q(F) \in \mathcal{C}$ $q>0$.

Assume $p>0$ and $E^2_{k,q} \in \mathcal{C}$ for all $k<p$, ($\Rightarrow E^r_{k,q} \in \mathcal{C}$ $k<p$) subquotient

$0 \to E^{r+1}_{p,0} \to E^r_{p,0} \xrightarrow{d^r} E^r_{p-r,r-1}$ $\in \mathcal{C}$ by induction

so we get $\mathcal{C}$-isomorphisms

$E^2_{p,0} \xleftarrow{\mathcal{C}\text{-iso}} E^3_{p,0} \xleftarrow{\mathcal{C}\text{-iso}} \ldots \xleftarrow{\mathcal{C}\text{-iso}} E^\infty_{p,0} \in \mathcal{C}$

Hence $H_p(B) \in \mathcal{C}$ so $E^2_{p,q} \in \mathcal{C}$ by (1).

Remark: $\Rightarrow$ requires only (1)'

Corollary $A \in \mathcal{C} \Rightarrow \tilde{H}_*(K(A,n)) \in \mathcal{C}$ for all $n \geq 1$.

Proof: Induction on n starting with (iii) and using proposition on $\underset{\simeq K(A,n-1)}{\Omega K(A,n)} \to PK(A,n) \to K(A,n)$

Theorem X simple.

If $\pi_k X \in \mathcal{C}$ for all k, then $H_k X \in \mathcal{C}$ for all $k > 0$.

Proof: Use Postnikov tower

$$\begin{tikzcd} & \vdots \arrow[d] \\ & X_2 \arrow[d] \\ X \arrow[ur] \arrow[r] \arrow[dr] & X_1 \arrow[d] \\ & X_0 \end{tikzcd}$$

$X \to X_n$ iso. on π_k for $k \leq n$

$\Rightarrow$ iso. on H_k for $k \leq n$

Fibration $K_n \to X_n \to X_{n-1}$ with $K_n = K(\pi_n X, n)$.

By the above, have $\tilde{H}_*(K_n) \in \mathcal{C}$, since $\pi_n X \in \mathcal{C}$.

Induction on n, starting with $X_0 = *$ for $n = 0$,

shows $\tilde{H}_*(X_n) \in \mathcal{C}$ for all n.

Since $H_k(X) \xrightarrow{\cong} H_k(X_n)$ for $k \leq n$ we are done.

<u>Hurewicz theorem mod $\mathcal{C}$</u> $\quad X$ simple.

If $\pi_k X \in \mathcal{C}$ for $k<n$ then $H_k(X) \in \mathcal{C}$ for $0<k<n$ and $\pi_n X \to H_n X$ is a $\mathcal{C}$-isomorphism.

<u>Proof</u>: The hypothesis implies $\pi_k(X_{n-1}) \in \mathcal{C}$ for all k

Hence, $H_k(X) \underset{k<n}{\overset{\cong}{\longrightarrow}} \underbrace{H_k(X_{n-1}) \in \mathcal{C}}_{\text{previous theorem}}$.

Next, consider

$$\begin{tikzcd}
\underset{}{\overset{=0}{\pi_{n+1}(X_{n-1})}} \arrow[r] \arrow[d] & \pi_{n+1}(X_{n-1},X) \arrow[r, "\cong"] \arrow[d, "\cong \text{ relative Hurewicz}"] & \pi_n(X) \arrow[r] \arrow[d, "\therefore\ \mathcal{C}\text{-iso}"] & \overset{=0}{\pi_n(X_{n-1})} \arrow[r] \arrow[d] & \overset{=0}{\pi_n(X_{n-1},X)} \arrow[d, "\cong"] \\
H_{n+1}(X_{n-1}) \arrow[r] & H_{n+1}(X_{n-1},X) \arrow[r, "\therefore\ \mathcal{C}\text{-iso}"'] & H_n(X) \arrow[r] & H_n(X_{n-1}) \arrow[r] & H_n(X_{n-1},X) \\
\in \mathcal{C} & & & \in \mathcal{C} & =0
\end{tikzcd}$$

$H_{n+1}(X) \to H_{n+1}(X_{n-1})$, $\quad H_{n+1}(X) \in \mathcal{C}$

Here we use the (classical) relative Hurewicz theorem for the pair (X_{n-1}, X); $\pi_k(X_{n-1}, X) = 0$ for $k<n+1$, follows from long exact sequence in relative homotopy and the fact that $\pi_k(X) \overset{\cong}{\to} \pi_k(X_{n-1})$ for $k<n$. $\square$

<u>Corollary</u> X simple
If $H_k(X) \in \mathcal{C}$ for $0<k<n$, then $\pi_k(X) \in \mathcal{C}$ for $k<n$.

<u>Proof</u>: $\pi_1 X \overset{\cong}{\to} \underset{\in\mathcal{C}}{H_1 X}$ by classical Hurewicz $\Rightarrow \pi_1 X \in \mathcal{C}$

$\Rightarrow \pi_2 X \to H_2 X$ $\mathcal{C}$-iso. $\Rightarrow \pi_2 X \in \mathcal{C}$

$\Rightarrow \pi_3 X \to \underset{\in\mathcal{C}}{H_3 X}$ $\mathcal{C}$-iso. $\Rightarrow \pi_3 X \in \mathcal{C}$ $\square$

$\vdots$

<u>Corollary</u>: X simply connected finite complex $\Rightarrow \pi_n X$ f.g. for all n.

Pf: $\mathcal{C} = \{\text{f.g. groups}\}$.

Whitehead theorem mod $\mathcal{C}$

$f: X \to Y$ map between simple spaces.

Then the following are equivalent:

(1) $\pi_k X \to \pi_k Y$ $\mathcal{C}$-iso. for $k < n$
$\mathcal{C}$-epi for $k = n$

(2) $H_k X \to H_k Y$ — " —

Proof May assume WLOG X is a subspace of Y.
Use relative Hurewicz mod $\mathcal{C}$:

(1) $\Leftrightarrow$ $\pi_k(Y,X) \in \mathcal{C}$ for $k \leq n$

$\Leftrightarrow$ $H_k(Y,X) \in \mathcal{C}$ for $k \leq n$

$\Leftrightarrow$ (2)

Rational homotopy equivalences

Theorem The following are equivalent for a map $f: X \to Y$ between simple spaces:

(1) $\pi_k X \otimes \mathbb{Q} \xrightarrow[f_*]{\cong} \pi_k Y \otimes \mathbb{Q}$ for all k

(2) $H_k(X;\mathbb{Q}) \xrightarrow[f_*]{\cong} H_k(Y;\mathbb{Q})$ for all k

f is called a rational homotopy equivalence if this holds.

Proof: Use Whitehead mod $\mathcal{C} = \{\text{torsion groups}\}$. For this $\mathcal{C}$, $A \in \mathcal{C} \Leftrightarrow A \otimes \mathbb{Q} = 0$,

and $A \to B$ $\mathcal{C}$-iso. $\Leftrightarrow A \otimes \mathbb{Q} \xrightarrow{\cong} B \otimes \mathbb{Q}$.

$\square$

$\mathbb{Q}$-local spaces

Def An abelian group A is called uniquely divisible, or $\mathbb{Q}$-local, if the following equivalent conditions hold:

(1) $A \xrightarrow{\cdot n} A$ is an isomorphism for all $n \in \mathbb{Z} \setminus \{0\}$

(2) $A \to A \otimes \mathbb{Q}$ is an isomorphism.

Theorem The following are equivalent for a simple space X:

(1) $\pi_k X$ is $\mathbb{Q}$-local for all k

(2) $H_k(X)$ is $\mathbb{Q}$-local for all $k > 0$.

Proof: Hurewicz mod $\mathcal{C} = \{\mathbb{Q}\text{-local groups}\}$

⚠ This is not a Serre class, but it satisfies

$$A \to B \to C \to D \to E \text{ exact and } A, B, D, E \in \mathcal{C} \Rightarrow C \in \mathcal{C}$$

which is enough.

$\mathbb{Q}$-localization

Theorem The following are equivalent for a map $r: X \to X_{\mathbb{Q}}$ of simple spaces, $X_{\mathbb{Q}}$ $\mathbb{Q}$-local

(1) $\pi_k(X) \otimes \mathbb{Q} \xrightarrow{\cong} \pi_k(X_{\mathbb{Q}})$

(2) $H_k(X;\mathbb{Q}) \xrightarrow{\cong} H_k(X_{\mathbb{Q}})$

(3) r is universal for maps into $\mathbb{Q}$-spaces: in the sense that to any $\mathbb{Q}$-space Y

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "r"'] & Y \\ X_{\mathbb{Q}} \arrow[ur, dashed] & \end{tikzcd} \qquad f \simeq g\circ r$$

unique factorization in the homotopy category

Proof: (1)$\Leftrightarrow$(2) by Hurewicz mod $\mathcal{C} = \{A \mid A \otimes \mathbb{Q} = 0\}$.

(2)$\Rightarrow$(3): The obstructions to extending f to $X_{\mathbb{Q}}$ lie in $H^{*+1}(X_{\mathbb{Q}}; X; \pi_*(Y))$

Since $X \to X_{\mathbb{Q}}$ is a $H_*(-;\mathbb{Q})$-iso. and $\pi_*(Y)$ is a $\mathbb{Q}$-vector space vanishes.

The obstruction to finding a homotopy between two lifts lie in $H^*(X_{\mathbb{Q}}, X; \pi_*(Y)) = 0$.

(3) ⇒ (2): Note that (3) is equivalent to

$$r^*: [X_{\mathbb{Q}}, Y] \to [X, Y] \quad \text{bijection}$$

for every $\mathbb{Q}$-local space Y.

Consider $Y = K(\mathbb{Q}, n)$; then

$$\begin{tikzcd}
{[X_{\mathbb{Q}}, K(\mathbb{Q},n)]} \arrow[r, "\cong"', "r"] \arrow[d, "\cong"] & {[X, K(\mathbb{Q},n)]} \arrow[d, "\cong"] \\
H^n(X_{\mathbb{Q}};\mathbb{Q}) \arrow[r, "r^*"] & H^n(X;\mathbb{Q})
\end{tikzcd}$$

shows $X \to X_{\mathbb{Q}}$ is an iso in $H_*(-;\mathbb{Q})$. □

<u>Def</u>: A map $r: X \to X_{\mathbb{Q}}$ with

- r rational homotopy equivalence
- $X_{\mathbb{Q}}$ $\mathbb{Q}$-local

is called a <u>$\mathbb{Q}$-localization</u> of X.