STOCKHOLM UNIVERSITY
Department of Mathematics
Salvador Rodríguez-López
Lefteris Theodosiadis

Examination in
Mathematics for Economic and Statistical Analysis
MM1005, Fall term; 7,5 ECTS
Wednesday 10 November, 2021

Instructions:
- During the exam you may not use any textbook, class notes, or any other supporting material.
- Non-graphical calculators will be provided for the exam by the department. Other calculators may not be used.
- In all solutions, justify your answers and communicate your reasoning.
- Use natural language, not just mathematical symbols. Write clearly and legibly
- Mark your final answer to each question clearly by putting a a box around it.

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1. (a) **[3p]** Determine the value of the parameter b such that

$$\lim_{x\to 0}\frac{\sqrt{1-16bx}-\sqrt{1-8x}}{4x}=1/2.$$

(b) **[2p]** Calculate the limit

$$\lim_{x\to 0}\frac{e^{-4x}-1}{2x}.$$

Solution (a) An algebraic manipulation yields

$$\frac{\sqrt{1-16bx}-\sqrt{1-8x}}{4x}=\frac{(1-16bx)-(1-8x)}{4x(\sqrt{1-16bx}+\sqrt{1-8x})}=\frac{2-4b}{\sqrt{1-16bx}+\sqrt{1-8x}},$$

which implies that

$$\lim_{x\to 0}\frac{\sqrt{1-16bx}-\sqrt{1-8x}}{4x}=\frac{2-4b}{2}=1-2b=1/2\Leftrightarrow b=\frac{1}{4}.$$

(b) Applying L'Hopital's gives us that

$$\lim_{x\to 0}\lim_{x\to 0}\frac{e^{-4x}-1}{2x}=\lim_{x\to 0}-2e^{-4x}=-2.$$

2. Calculate the integrals

$$\text{a)}[\mathbf{3p}]\quad \int_{-1}^{0}15x\sqrt{1+x}\,\mathrm{d}x \qquad\qquad \text{b)}[\mathbf{2p}]\quad \int_{0}^{+\infty}3^{-t}\mathrm{d}t$$

Solution

$$\int_{-1}^{0}15x\sqrt{1+x}\,\mathrm{d}x=6(1+x)^{5/2}-10(1+x)^{3/2}|_{-1}^{0}=-4.$$

$$\int_{0}^{+\infty}3^{-t}\mathrm{d}t=\frac{1}{\ln 3}.$$

3. The expression

$$\sqrt{x^2+2y}+xy^2=2,$$

defines y as a function of x: $y=y(x)$.

(a) **[1p]** Find the values of $y(0)$.

(b) **[3p]** Find the equation of the tangent line to $y(x)$ at the point $P = (0, y(0))$.

(c) **[1p]** Is $y(x)$ increasing or decreasing at the point $(0, y(0))$. Argument your answer.

Solution Setting $y(0)$ one sees that the equations is satisfied, if and only if

$$\sqrt{2y(0)} = 2 \Leftrightarrow y(0) = 2.$$

Implicit differentiation gives

$$(x^2+2y)^{-1/2}(x+y') + y^2 + 2xyy' = 0.$$

Hence

$$\frac{y'(0)}{2} + y(0)^2 = 0 \Leftrightarrow y'(0) = -2y(0)^2 = -8.$$

So the equation of the tangent line at the point P is

$$Y = 2 - 8x.$$

4. **[5p]** Determine for which x is the series $\sum_{n=0}^{\infty}(1+x)^{-2n}$ convergent , and when does the sum equals 2.

Solution The series converges if and only if

$$|1+x| > 1 \Leftrightarrow 1+x > 1 \quad \text{or} \quad 1+x < -1 \Leftrightarrow x > 0 \quad \text{or} \quad x < -2.$$

Note that for those values of x that the series converges, the sum of the series equals 2 if and only if

$$\frac{1}{1-(1+x)^{-2}} = \frac{(1+x)^2}{(1+x)^2-1} = 2 \Leftrightarrow (1+x)^2 = 2(1+x)^2 - 2 \Leftrightarrow (1+x)^2 = 2$$
$$\Leftrightarrow x+1 = \pm\sqrt{2} \Leftrightarrow x = -1 \pm \sqrt{2}.$$

Nb. $-1+\sqrt{2} \approx 0.414214$ and $-1-\sqrt{2} \approx -2.41421$.

5. **[5p]** Determine the maximum and minimum value of the function

$$f(x,y) = -x^2 + xy - 1$$

on the quadrilateral with corners

$$(0,0), (3,0), (3,5) \quad \text{and} \quad (0,2).$$

Solution The partial derivatives of f are

$$f_x(x,y) = -2x + y \qquad f_y(x,y) = x.$$

So the only stationary point is

$$x = (0,0),$$

which does not lie inside the quadrilateral.

The boundary of the domain consist of four parts

$$L_1 := \{(x,y) : y = 0, x \in [0,3]\},$$
$$L_2 := \{(x,y) : x = 3, y \in [0,5]\},$$
$$L_3 := \{(x,y) : y = x+2, x \in [0,3]\},$$
$$L_4 := \{(x,y) : x = 0, y \in [0,2]\}.$$

On the last one we have

$$f(0,y) = -1 \quad y \in [0,6],$$

On the first part of the boundary we have

$$f(x,0) = -x^2 - 1, \quad x \in [0,3],$$

which has no critical point inside the interval. Evaluating gives us

$$f(0,0) = -1 \quad f(3,0) = -10.$$

On L_2 we have that

$$f(3,y) = -10 + 3y, \quad y \in [0,5]$$

which has no critical points inside the interval. Evaluating the function on the extremes gives

$$f(3,0) = -10, \qquad f(3,5) = 5.$$

On L_3, we have that

$$\begin{aligned} h(x) = f(x,x+2) &= -x^2 + x(x+2) - 1 \\ &= 2x - 1 \quad x \in [0,3]. \end{aligned}$$

which has no critical points inside the interval. Evaluating yields

$$h(0) = f(0,2) = -1, \qquad h(3) = f(3,5) = 5.$$

Then, we obtain that the maximum and minimum values of f with the given constrains are respectively

$$5 \qquad \text{and} \qquad -10.$$

6. **[5p]** Determine for which values of the parameter a, the system

$$\begin{cases} ax + y + 3z = 2 \\ 2x + y + az = 2 \\ 2x + y + 3z = a \end{cases}$$

has exactly one solution, no solution, or infinitely many solutions.

Solution The determinant of the matrix of coefficients is $-6 + 5a - a^2 = (2-a)(a-3)$.

If $a \notin \{2,3\}$ the system has a unique solution.

If $a = 3$, the last two equations of the system read

$$\begin{cases} 2x + y + 3z = 2 \\ 2x + y + 3z = 2 \end{cases}$$

which has no solution. Therefore, the whole system has no solution.

If $a = 2$, the system is equivalent to

$$\begin{cases} 2x + y + 2z = 2 \\ 2x + y + 3z = 2 \end{cases}$$

which implies that the system has an infinite number of solutions of the form $(x,y,z) = (1,0,0) + t(1,-2,0)$, where $t \in \mathbb{R}$.

GOOD LUCK!
