

# Homework 3 of 3

Logic, Stockholm University, Autumn 2014

Peter LeFanu Lumsdaine / Håkon Gylterud

<http://kurser.math.su.se/course/view.php?id=186>

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Due Thursday 23 October, in class (or by email before class). Problems are marked with the per milles they count for on the final grade. This homework contains 4 problems, and is worth 35 % of the final grade.

Make sure to state clearly whenever you are using the Soundness or Completeness theorems.

N.B. There will only be 3 homeworks, not 4 as originally planned.

1. (10 %) Give natural deduction proofs showing each of the following (assuming appropriate arity types):

(a)  $\vdash \forall x_0 \exists x_1 (x_1 \doteq x_0)$

(b)  $\vdash_{x_0, x_1} \exists x_2 (f_1(x_0, x_1) \doteq x_2)$

(c)  $\vdash \forall x_0, x_1 ((x_0 \doteq x_1) \rightarrow (f_1(x_0) \doteq f_1(x_1)))$

(d)  $\exists x_0 \forall x_1 (x_1 \doteq x_0) \vdash \forall x_0 \exists x_1 (f(x_1) = x_0)$

2. (8 %) Which of the following entailments hold? (Justify your answers with proofs, countermodels, etc. as appropriate.)

(a) For all  $\varphi, \psi$  in  $\text{Form}(S \cup \{i\})$ ,  $\exists x_i (\varphi \vee \psi) \models_S (\exists x_i \varphi) \vee (\exists x_i \psi)$ .

(b) For all  $\varphi, \psi$  in  $\text{Form}(S \cup \{i\})$ ,  $(\exists x_i \varphi) \vee (\exists x_i \psi) \models_S \exists x_i (\varphi \vee \psi)$ .

(c) For all  $\varphi, \psi$  in  $\text{Form}(S \cup \{i\})$ ,  $\exists x_i (\varphi \wedge \psi) \models_S (\exists x_i \varphi) \wedge (\exists x_i \psi)$ .

(d) For all  $\varphi, \psi$  in  $\text{Form}(S \cup \{i\})$ ,  $(\exists x_i \varphi) \wedge (\exists x_i \psi) \models_S \exists x_i (\varphi \wedge \psi)$ .

3. (7 %) Show that the following is provable:

$$\vdash (\forall x_0, x_1, x_2 (x_0 \doteq x_1 \vee x_1 \doteq x_2 \vee x_0 \doteq x_2)) \\ \vee (\forall x_0, x_1 \exists x_2 (\neg(x_2 \doteq x_0) \wedge \neg(x_2 \doteq x_1)))$$

4. (10 %) Define closed formulas  $\varphi_{\text{inj}}, \varphi_{\text{surj}}, \varphi_{\text{invol}} \in \text{Form}(\emptyset)$  as follows:

$$\varphi_{\text{inj}} := \forall x_0, x_1 (f_1(x_0) \doteq f_1(x_1) \rightarrow x_0 \doteq x_1) \\ \varphi_{\text{surj}} := \forall x_0 \exists x_1 (f_1(x_1) \doteq x_0) \\ \varphi_{\text{invol}} := \forall x_0 (f_1(f_1(x_0)) \doteq x_0)$$

Which of the following theories are consistent? (Justify your answers appropriately.)

- (a)  $\{\varphi_{\text{inj}}, \neg\varphi_{\text{surj}}\}$
- (b)  $\{\neg\varphi_{\text{inj}}, \varphi_{\text{surj}}\}$
- (c)  $\{\varphi_{\text{invol}}, \neg\varphi_{\text{surj}}\}$
- (d)  $\{\varphi_{\text{invol}}, \varphi_{\text{inj}}\}$