Home Work Problems 3

The problems are part of the examination of the course. This is the third of a total of five sets of problems. They are to be solved individually and are due by November 24 (2014). Please write clearly and present the solutions well.

1. A large cardinal axiom inspired by finite combinatorics. A cardinal $\kappa$ is said to be weakly compact if firstly it is uncountable, and secondly for every map $f : [\kappa]^2 \to \{0, 1\}$ there is a set $S$ of cardinality $\kappa$ which is homogeneous for $f$. For any set $X$, $[X]^2$ the denotes the set of two-element subsets of $X$. A subset $S$ of $\kappa$ is homogeneous for $f$ if and only if $f([S]^2) = \{0\}$ or $f([S]^2) = \{1\}$. (For $\kappa = \omega$, the second part of the condition is known as the infinite Ramsey theorem, and is provable in ZFC.)

(a) Suppose that $\kappa$ is weakly compact cardinal. Prove that $\kappa$ is regular, by assuming to the contrary that it can be written as a union

$$\kappa = \bigcup_{\lambda < \gamma} A_\lambda$$

where $|A_\lambda| < \kappa$, $\gamma < \kappa$. Then show that $f : [\kappa]^2 \to \{0, 1\}$ defined by $f(\{\alpha, \beta\}) = 1$ iff $\alpha, \beta \in A_\lambda$ for some $\lambda$, has no homogeneous set of size $\kappa$. Thus $\kappa$ is regular.

(b) It may be proved that any weakly compact cardinal is a strong limit. Can you prove the existence of a weakly compact cardinal in ZFC?

(2 p)
2. **An application of the regularity axiom.** In ZF the we can define the following equivalence relation (equipollence) on the class $\mathcal{V}$ of all sets:

$$A \simeq B \iff \text{there is a bijection } f : A \to B.$$  

Now the equivalence class $[A] = \{B : A \simeq B\}$ is a proper class. Consider the alternative definition

$$[A]_m = \{B : A \simeq B \land (\forall C)(B \simeq C \Rightarrow \text{rank}(B) \leq \text{rank}(C))\}.$$

Prove that $[A]_m$ is a set, and that $[A]_m = [A']_m$ iff $A \simeq A'$. Now we can form the class $\{[A]_m : A \in \mathcal{V}\}$.  

(3 p)

3. **Gödel’s Constructible Hierarchy of Sets.**

(a) Verify that $L_n = V_n$ för $n \leq \omega$.

(b) Prove that $L_{\omega+1} \subsetneq V_{\omega+1}$.

(c) Prove within the theory ZF that for every set $x \subseteq L$, there is $y \in L$ such that $x \subseteq y$.  

(5 p)

4. **Kripke-Platek set theory.** Prove that the set-theoretic axioms in (a) – (c) below are valid in $L_{\omega_1}$.

(a) Extensionality, Fundation, Pairing, Union, Infinity

(b) $\Delta_0$-separation: for restricted formulas $\varphi$

$$\exists z \forall y [y \in z \leftrightarrow y \in u \land \varphi(x_1, \ldots, x_n, y)].$$

(c) $\Delta_0$-collection:

$$(\forall x \in u) \exists y \varphi(x_1, \ldots, x_n, x, y) \to \exists v (\forall x \in u) (\exists y \in v) \varphi(x_1, \ldots, x_n, x, y)$$

for restricted formulas $\varphi$.  

(5 p)