1. (a)

$$\dfrac{\dfrac{\dfrac{}{(x_0)\quad x_0 \doteq x_0}\,\text{refl}}{(x_0)\quad \exists x_1\ x_1 \doteq x_0}\,\exists I \ \text{(with } t := x_0\text{)}}{\forall x_0\, \exists x_1\ x_1 \doteq x_0}\,\forall I$$

(b)

$$\dfrac{\dfrac{}{(x_0, x_1)\quad f_1(x_0, x_1) \doteq f_1(x_0, x_1)}\,\text{refl}}{(x_0, x_1)\quad \exists x_2\ \ f_1(x_0, x_1) \doteq x_2}\,\exists I \quad \text{(with } t := f_1(x_0, x_1)\text{)}$$

(c)

$$\dfrac{\dfrac{\dfrac{\dfrac{\left[(x_0, x_1)\quad x_0 \doteq x_1\right]^1 \qquad \dfrac{}{(x_0, x_1)\quad f_1(x_0) \doteq f_1(x_0)}\,\text{refl}}{(x_0, x_1)\quad f_1(x_0) \doteq f_1(x_1)}\,\text{subst}}{(x_0, x_1)\quad x_0 \doteq x_1 \to f_1(x_0) \doteq f_1(x_1)}\,\to I_1}{(x_0)\quad \forall x_1 (x_0 \doteq x_1 \to f_1(x_0) \doteq f_1(x_1))}\,\forall I}{\forall x_0, x_1\ (x_0 \doteq x_1 \to f_1(x_0) \doteq f_1(x_1))}\,\forall I$$

(substituting x_0 and x_1 for x_2 in "$f_1(x_0) \doteq f_1(x_2)$")

(d)

$$\dfrac{\dfrac{\left[(x_0, x_2)\ \forall x_1 (x_1 \doteq x_2)\right]^3}{(x_0, x_2)\quad x_0 \doteq x_2}\,\forall E \qquad \dfrac{\left[(x_0, x_2)\ \forall x_1 (x_1 \doteq x_2)\right]^3}{(x_0, x_2)\quad f(x_2) \doteq x_2}\,\forall E}{(x_0, x_2)\quad f(x_0) \doteq x_0}\,\text{subst}$$

$$\dfrac{\dfrac{\dfrac{[(x_0)]^1\ \exists x_0\, \forall x_1\, (x_1 \doteq x_0) \qquad \dfrac{\left[(x_0)\quad \forall x_1 (x_1 \doteq x_0)\right]^2}{(x_0)\quad \exists x_2\, \forall x_1 (x_1 \doteq x_2)}\,\exists I}{(x_0)\quad \exists x_2\, \forall x_1\ (x_1 \doteq x_2)}\,\exists E_2 \qquad \dfrac{}{(x_0, x_2)\quad \exists x_1\ \ f(x_1) \doteq x_0}\,\exists I}{(x_0)\quad \exists x_1\ f(x_1) \doteq x_0}\,\exists E_3}{\forall x_0\, \exists x_1\ f(x_1) \doteq x_0}\,\forall I_1$$

1(d) cont'd Note: the point of the left-hand subtree is to rename the bound variable "$\exists x_0$" to "$\exists x_2$"; we need to do this before the application $\exists E_3$, since x_0 is free in the conclusion of $\exists E_3$.

2(a) $\exists x_i\,(\varphi\vee\psi) \models_S (\exists x_i\,\varphi)\vee(\exists x_i\,\psi)$ does hold; by soundness, to show this, it suffices to derive $\exists x_i\,(\varphi\vee\psi) \vdash_S (\exists x_i\,\varphi)\vee(\exists x_i\,\psi)$.

First, assume that x_i is free for x_i in φ and ψ. (I.e. that x_i does not occur bound in these.)

Then:

$$\dfrac{(S)\ \exists x_i(\varphi\vee\psi) \qquad \dfrac{[(S\cup\{i\})\ \varphi\vee\psi]^1 \quad \dfrac{\dfrac{[(S\cup\{i\})\ \ \varphi]^2}{(S\cup\{i\})\ (\exists x_i\ \varphi)}\,\exists I}{(S\cup\{i\})\ (\exists x_i\varphi)\vee(\exists x_i\psi)}\,\vee I1 \quad \dfrac{\dfrac{[(S\cup\{i\})\ \psi]^2}{(S\cup\{i\})\ (\exists x_i\ \psi)}\,\exists I}{(S\cup\{i\})\ (\exists x_i\varphi)\vee(\exists x_i\psi)}\,\vee I2}{(S\cup\{i\})\,(\exists x_i\varphi)\vee(\exists x_i\ \psi)}\,\vee E_2}{(S)\ \ (\exists x_i\varphi)\vee(\exists x_i\ \psi)}\,\exists E_1$$

Here the two instances of $\exists I$ use the fact that $\varphi = \varphi[x_i/x_i]$, $\psi = \psi[x_i/x_i]$, and require the freeness assumption from above.

If x_i is not free for x_i in φ or ψ, then take some variable x_j not occurring in φ or ψ (and therefore free for x_i in them), replace the hypothesis "$(S)\ \exists x_i\,(\varphi\vee\psi)$" above with a derivation of $\exists x_j\ (\varphi[x_j/x_i]) \vee (\psi[x_j/x_i])$ from $\exists x_i\ \varphi \vee \psi$, ~~as~~ and adapt the rest of the derivation to use x_j where appropriate (as done with x_2, x_0 in 1(d) above).

(b) Again, to show $(\exists x_i\, \varphi) \vee (\exists x_i\, \psi) \models_S \exists x_i\, (\varphi \vee \psi)$, it suffices by soundness to show " $\vdash_S$ " ; and again, we assume that x_i is free for x_i in φ, ψ, and make a change of variables in case it does not.

$$
\dfrac{
(S)\ (\exists x_i \varphi) \vee (\exists x_i \psi)
\qquad
\dfrac{
[(S)\ \exists x_i\, \varphi]
\qquad
\dfrac{\dfrac{[(S \cup \{i\})\ \varphi]^2}{(S \cup \{i\})\ \varphi \vee \psi}\,{\scriptstyle \vee I1}}{(S \cup \{i\})\ \exists x_i\, (\varphi \vee \psi)}\,{\scriptstyle \exists I}
}{(S)\ \exists x_i\, (\varphi \vee \psi)}\,{\scriptstyle \exists E_2}
\qquad
\dfrac{
[(S)\ \exists x_i\, \psi]^1
\qquad
\dfrac{\dfrac{[(S \cup \{i\})\ \psi]^3}{(S \cup \{i\})\ \varphi \vee \psi}\,{\scriptstyle \vee I2}}{(S \cup \{i\})\ \exists x_i\, (\varphi \vee \psi)}\,{\scriptstyle \exists I}
}{(S)\ \exists x_i\, (\varphi \vee \psi)}\,{\scriptstyle \exists E_3}
}{(S)\ \exists x_i\, (\varphi \vee \psi)}\,{\scriptstyle \vee E_1}
$$

c) Again, using soundness:

$$
\dfrac{
\dfrac{
(S)\ \exists x_i\, (\varphi \wedge \psi)
\qquad
\dfrac{\dfrac{[(S \cup \{i\})\ \varphi \wedge \psi]^1}{(S \cup \{i\})\ \varphi}\,{\scriptstyle \wedge E1}}{(S \cup \{i\})\ \exists x_i\, \varphi}\,{\scriptstyle \exists I}
}{(S)\ \exists x_i\, \varphi}\,{\scriptstyle \exists E_1}
\qquad
\dfrac{
(S)\ \exists x_i\, (\varphi \wedge \psi)
\qquad
\dfrac{\dfrac{[(S \cup \{i\})\ \varphi \wedge \psi]^2}{(S \cup \{i\})\ \psi}\,{\scriptstyle \wedge E2}}{(S \cup \{i\})\ \exists x_i\, \psi}\,{\scriptstyle \exists I}
}{(S)\ \exists x_i\, \psi}\,{\scriptstyle \exists E_2}
}{(S)\ (\exists x_i\, \varphi) \wedge (\exists x_i\, \psi)}\,{\scriptstyle \wedge I}
$$

2 (d) "$(\exists x_i\, \varphi) \wedge (\exists x_i\, \psi) \models_S \exists x_i\, (\varphi \wedge \psi)$" does not hold in general. Consider the structure $A := \langle \mathbb{N};\ ;\{n \in \mathbb{N} \mid n \text{ odd}\}, \{n \in \mathbb{N} \mid n \text{ even}\}\rangle$.

I.e., P_1 and P_2 are unary predicates for the odd and even numbers respectively.

Then $A \models (\exists x_0\, P_1(x_0)) \wedge (\exists x_0\, P_2(x_0))$,

but $A \not\models (\exists x_0\, (P_1(x_0) \wedge P_2(x_0)))$

since no natural number is both even and odd.

3. Set $\varphi := \forall x_0, x_1, x_2,\ (x_0 \doteq x_1 \vee x_1 \doteq x_2 \vee x_0 \doteq x_2)$

$\psi := \forall x_0, x_1\ \exists x_2\ (\neg(x_2 \doteq x_0) \wedge \neg(x_2 \doteq x_1))$

By completeness, to show $\vdash \varphi \vee \psi$, it suffices to show $\models \varphi \vee \psi$, i.e. that $[\![\varphi \vee \psi]\!] = 1$ in every interpretation.

So: let A be any interpretation.

Claim 1: if $|A|$ has <3 elements, then $A \models \varphi$.

Proof: unwinding the interpretation,

$$A \models \varphi \iff \langle\rangle \in [\![\ \mid \varphi]\!]$$

$$\iff [\![x_0, x_1, x_2 \mid x_0 \doteq x_1 \vee x_1 \doteq x_2 \vee x_0 \doteq x_2]\!] = A^{\{0,1,2\}}$$

$\iff \forall a_0, a_1, a_2 \in |A|$, either $a_0 = a_1$, $a_1 = a_2$, or $a_0 = a_2$

But if $|A|$ has <3 elements, this holds, since any such a_0, a_1, a_2 cannot all be distinct, so some two must be equal.

3 Cont'd. Claim 2: If $|A|$ has ≥ 3 elements, then $A \models \psi$.

<u>Proof</u>. Similarly to above,

$A \models \psi \iff \forall a_0, a_1 \in |A|.\ \exists a_2 \in |A|$, s.t. $a_2 \neq a_0$ and $a_2 \neq a_1$.

But if $|A|$ has three distinct elements b_0, b_1, b_2 then given any a_0, a_1, at least one of b_0, b_1, b_2 must be distinct from a_0, a_1; so it gives a_2 as above. So $A \models \psi$.

Putting the two claims together, we see that in either case, whether $|A|$ has ≥ 3 or < 3 elements, $A \models \varphi \vee \psi$.

<u>So</u> $\models \varphi \vee \psi$, ~~as~~ so by completeness, $\vdash \varphi \vee \psi$.

4. In any structure A, we have:

$A \models \varphi_{inj}$ iff f_1^A is ~~an~~ injective,

since unwinding the definition gives

$[\![\varphi_{inj}]\!]^A = 1 \iff \forall a_0, a_1 \in |A|$, if $f_1^A(a_0) = f_1^A(a_1)$, then $a_0 = a_1$,

the usual definition of injectivity.

Similarly, ~~Ag~~ $A \models \varphi_{surj}$ iff f_1^A is surjective, and $A \models \varphi_{invol}$ iff f_1^A is an involution (i.e. $f(f(a)) = a$, for all $a \in |A|$).

By soundness, if a theory has a model, it is consistent.
We therefore have:

(a) $\{\varphi_{inj}; \neg\varphi_{surj}\}$ is consistent:

$\langle \mathbb{N}; (-)^2; \rangle$ is a model of it,
since squaring (on $\mathbb{N}$) is injective but not surjective.

(b) $\{\neg\varphi_{inj}, \varphi_{surj}\}$ is also consistent:

$\langle \mathbb{N}; \lfloor \frac{\cdot}{2} \rfloor ; \rangle$ is a model of it (where $\lfloor \frac{n}{2} \rfloor$ is the largest integer m s.t. $m \leq \frac{n}{2}$), since $\lfloor \frac{\cdot}{2} \rfloor$ is surj. on $\mathbb{N}$ but not inj.

(c) $\{\varphi_{invol}, \neg\varphi_{surj}\}$ is inconsistent:

$$\dfrac{\neg\varphi_{surj} \qquad \dfrac{\dfrac{\dfrac{[(x_0)]^1 \quad \forall x_0\, f_1(f_1(x_0)) \doteq x_0}{(x_0) \quad f_1(f_1(x_0)) \doteq x_0}\ \forall E}{(x_0)\ \exists x_1\ f_1(x_1) \doteq x_0}\ \exists I \quad (x_1 := f_1(x_0))}{\forall x_0\, \exists x_1\, (f_1(x_1) \doteq x_0)}\ \forall I_1}{\bot}\ \to E$$

(d) $\{\varphi_{invol}, \varphi_{inj}\}$ is again consistent,
since e.g. $\langle \mathbb{Z}; -; \rangle$ is a model of it (where $(-)$ denotes unary negation, i.e. $x \mapsto -x$).