Brownian motion

1. Brownian motion
2. Variations on Brownian motion
3. Gaussian and stationary processes

Index

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2. Variations on Brownian motion
3. Gaussian and stationary processes

Symmetric random walk

Consider a **symmetric random walk** $\{X_n, n \in \mathbb{N}\}$, which at each time is equally likely to go one step up or down, i.e., the X_n's are i.i.d. with

$$\mathbb{P}(X_n = 1) = \mathbb{P}(X_n = -1) = \frac{1}{2}.$$

This is a Markov chain with $P_{i,i+1} = \frac{1}{2} = P_{i,i-1}$ for $i = 0, \pm 1, \pm 2, \ldots$.

- $S_n = \sum_{i=1}^{n} X_i$.
- $\mathbb{E}[S_n] = 0$ and $\mathrm{Var}(S_n) = n$.
- By the CLT $\frac{1}{\sqrt{n}} S_n$ converges in distribution to a $\mathcal{N}(0, 1)$.

Scaling limit of a random walk

Let's **speed up this process** by taking smaller and smaller steps in smaller and smaller time intervals. If at each Δt time unit we take a step of size Δx up or down with equal probabilities, then the position at time t is

$$X(t) = \Delta x \left(X_1 + \cdots + X_{\lfloor t/\Delta t \rfloor} \right),$$

where the X_i's are i.i.d. with $\mathbb{P}(X_i = 1) = \mathbb{P}(X_i = -1) = \frac{1}{2}$.

- $\mathbb{E}[X(t)] = 0$ and $\mathrm{Var}(X(t)) = (\Delta x)^2 \left\lfloor \frac{t}{\Delta t} \right\rfloor$.

Go to the limit: let $\Delta x = \sigma\sqrt{\Delta t}$ for $\sigma > 0$, and let $\Delta t \to 0$.

- $\mathbb{E}[X(t)] = 0$ and $\mathrm{Var}(X(t)) \to \sigma^2 t$.
- By the CLT $X(t)$ converges in distribution to a $\mathcal{N}(0, \sigma^2 t)$.

Since the random walk has independent and stationary increments, we expect the limiting process to have them as well.

Brownian motion

A stochastic process $\{X(t), t \geq 0\}$ is said to be a **Brownian motion** (BM) if

(i) $X(0) = 0$;

(ii) $\{X(t), t \geq 0\}$ has independent and stationary increments;

(iii) for every $t > 0$, $X(t) \sim \mathcal{N}(0, \sigma^2 t)$.

- If $X(0) = x$, then $\{X(t) - x, t \geq 0\}$ is a BM.
- If $\sigma = 1$, **standard Brownian motion** (SBM) $B(t) \sim \mathcal{N}(0, t)$. Any BM can be converted to the standard process by letting $B(t) = X(t)/\sigma$. From now on, we assume $\sigma = 1$.
- By symmetry, $\{X(t), t \geq 0\}$ is distributed as $\{-X(t), t \geq 0\}$.

History

- 1827: the English botanist Robert **Brown** studied the motion of a small particle that is totally immersed in a liquid or gas.
- 1905: **Einstein** explained the process by assuming that the immersed particle was continually being subjected to bombardment by the molecules of the surrounding medium.
- 1918: **Wiener** gave the precise mathematical definition (also called Wiener process)

Continuous but not differentiable

Consider the SBM $\{X(t), t \geq 0\}$.

- $X(t)$ is a **continuous function** of t.
  Intuition. We must show that $\lim_{h \to 0}(X(t+h) - X(t)) = 0$ a.s.. Note that the r.v. $X(t+h) - X(t) \sim \mathcal{N}(0, h)$ has mean 0 and variance h, and so it would seem to converge to a r.v. with mean 0 and variance 0 as $h \to 0$. □
- $X(t)$ is **nowhere differentiable**.
  Intuition. Note that $\frac{X(t+h)-X(t)}{h} \sim h^{-1}\mathcal{N}(0, h) \sim \mathcal{N}(0, h^{-1})$ has mean 0 and variance $1/h$, which converges to ∞ if $h \to 0$. Hence, it is not differentiable. □

Brownian bridge

For $0 < s < t$, we are interested in the conditional distribution of $X(s)$ given that $X(t) = B$. The conditional density is

$$
\begin{aligned}
f_{X(s)|X(t)=B}(x) &= \frac{f_{X(s),X(t)}(x,B)}{f_{X(t)}(B)} = \frac{f_{X(s)}(x) f_{X(t-s)}(B-x)}{f_{X(t)}(B)} \\
&= \frac{\frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} \frac{1}{\sqrt{2\pi(t-s)}} e^{-\frac{(B-x)^2}{2(t-s)}}}{\frac{1}{\sqrt{2\pi t}} e^{-\frac{B^2}{2t}}} = \frac{e^{-\left(\frac{x^2}{2s} + \frac{(B-x)^2}{2(t-s)} - \frac{B^2}{2t}\right)}}{\sqrt{2\pi \frac{s(t-s)}{t}}} \\
&= \frac{e^{-\frac{t(t-s)x^2 + st(B^2 - 2Bx + x^2) - s(t-s)B^2}{2st(t-s)}}}{\sqrt{2\pi \frac{s(t-s)}{t}}} = \frac{e^{-\frac{(x - Bs/t)^2}{2s(t-s)/t}}}{\sqrt{2\pi \frac{s(t-s)}{t}}},
\end{aligned}
$$

hence it is normal with

$$\mathbb{E}[X(s) \,|\, X(t) = B] = \frac{s}{t} B \qquad \text{and} \qquad \mathrm{Var}(X(s) \,|\, X(t) = B) = \frac{s}{t}(t-s).$$

Example

Example 10.1: Bicycle race.

Hitting times

Let $T_a = \inf\{t \geq 0 : X(t) \geq a\}$ be the **hitting time** of barrier a. For $a > 0$, since

$$\begin{aligned}
&\mathbb{P}(X(t) \geq a) \\
&= \mathbb{P}(X(t) \geq a | T_a \leq t)\mathbb{P}(T_a \leq t) + \mathbb{P}(X(t) \geq a | T_a > t)\mathbb{P}(T_a > t) \\
&= \frac{1}{2}\mathbb{P}(T_a \leq t) + 0\mathbb{P}(T_a > t) \\
&= \frac{1}{2}\mathbb{P}(T_a \leq t),
\end{aligned}$$

we have that (using $y = x/\sqrt{t}$)

$$\mathbb{P}(T_a \leq t) = 2\mathbb{P}(X(t) \geq a) = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t}\, dx = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^\infty e^{-y^2/2}\, dy.$$

By symmetry, for $a < 0$ the distribution of T_a is the same as that of T_{-a}, hence we obtain $\mathbb{P}(T_a \leq t) = \frac{2}{\sqrt{2\pi}} \int_{|a|/\sqrt{t}}^\infty e^{-y^2/2}\, dy$. By computing the density $f_{T_a}(t)$, we get $\mathbb{E}(T_a) = \infty$.

The maximum of Brownian motion

Let $M(t) = \max_{0 \leq s \leq t} X(s)$ be the **maximum of Brownian motion**. For $a > 0$,

$$\mathbb{P}(M(t) \geq a) = \mathbb{P}(T_a \leq t) = 2\mathbb{P}(X(t) \geq a) = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} e^{-y^2/2}\, dy.$$

Moreover, by symmetry,

$$2\mathbb{P}(X(t) \geq a) = \mathbb{P}(X(t) \geq a) + \mathbb{P}(X(t) \leq -a) = \mathbb{P}(|X(t)| \geq a).$$

Hence $M(t)$ has the same distribution of $|X(t)|$.

The reflection principle

Theorem (The reflection principle)

Let $\{X(t), t \geq 0\}$ is a SBM and T a stopping time. The process $\{X_T(t), t \geq 0\}$ defined as

$$X_T(t) = \begin{cases} X(t), & 0 \leq t \leq T, \\ 2X(T) - X(t), & t > T, \end{cases}$$

is also a SBM.

Example: exam 2021.

Index

1. Brownian motion
2. Variations on Brownian motion
3. Gaussian and stationary processes

Variations on Brownian motion

- If $\{B(t), t \geq 0\}$ is a SBM and $\mu \in \mathbb{R}$, then the process $\{X(t) = B(t) + \mu t, t \geq 0\}$ is a **Brownian motion with drift** μ. Since $\frac{B(t)}{t} \sim \mathcal{N}(0, 1/t)$, and its variance converges to 0, we have that $\frac{B(t)}{t} \to 0$ in probability and a.s.. Hence, as $t \to \infty$,

$$\frac{X(t)}{t} = \frac{B(t)}{t} + \mu \to \mu \qquad \text{a.s..}$$

- If $\{Y(t), t \geq 0\}$ is a BM with drift μ and variance parameter σ^2, the the process $\{X(t), t \geq 0\}$, defined by $X(t) = e^{Y(t)}$ is a **geometric Brownian motion**.

Geometric Brownian motion

Given the history of the process up to time $s < t$, the expected value of the process at time t is

$$\begin{aligned}\mathbb{E}[X(t) \,|\, X(u), 0 \le u \le s] &= \mathbb{E}[e^{Y(t)} \,|\, Y(u), 0 \le u \le s] \\ &= \mathbb{E}[e^{Y(t)-Y(s)+Y(s)} \,|\, Y(u), 0 \le u \le s] \\ &= e^{Y(s)}\mathbb{E}[e^{Y(t)-Y(s)} \,|\, Y(u), 0 \le u \le s) \\ &= X(s)\mathbb{E}[e^{Y(t)-Y(s)}]\end{aligned}$$

and, since $Y(t) - Y(s) \sim \mathcal{N}(\mu(t-s), \sigma^2(t-s))$,

$$\mathbb{E}[e^{Y(t)-Y(s)}] = e^{\mathbb{E}[Y(t)-Y(s)]+\mathrm{Var}(Y(t)-Y(s))/2} = e^{\mu(t-s)+\sigma^2(t-s)/2},$$

hence,

$$\mathbb{E}[X(t) \,|\, X(u), 0 \le u \le s] = X(s)e^{(t-s)(\mu+\sigma^2/2)}.$$

Similarly, we can compute

$$\mathrm{Var}(X(t) \,|\, X(u), 0 \le u \le s) = (X(s))^2 e^{2\mu(t-s)}(e^{2\sigma^2(t-s)} - e^{\sigma^2(t-s)}).$$

Application

Geometric BM is useful in the **modeling of stock prices** over time when the percentage changes are i.i.d..

- Suppose that $X(n)$ is the price of some stock at time n and $Y(n) = X(n)/X(n-1), n \geq 1$ are i.i.d..
- We have that $X(n) = Y(n)X(n-1) = Y(n)Y(n-1)\dots Y(1)X(0)$.
- Then $\log(X(n)) = \sum_{i=1}^{n} \log(Y(i)) + \log(X(0))$ and, since the $\log(Y(i))$'s are i.i.d., $\{\log(X(n)), n \geq 0\}$ is approximately (for large n) a BM with a drift, so $\{X(n), n \geq 0\}$ is approximately a geometric BM.

The maximum of Brownian motion with drift

Let $\{X(t), t \geq 0\}$ be a BM with drift μ and variance parameter σ^2, and define

$$M(t) = \max_{0 \leq s \leq t} X(s)$$

to be the **maximal value of the process** up to time t. We are interested in the distribution of $M(t)$.

- For $y > x$, we have that $\mathbb{P}(M(t) \geq y \,|\, X(t) = x) = e^{-\frac{2y(y-x)}{t\sigma^2}}$, $y \geq 0$. The proof uses the fact that the conditional distribution of $X(s), 0 \leq s \leq t$ given $X(t)$ does not depend on μ.
- Then we get $\mathbb{P}(M(t) \geq y) = e^{2y\mu/\sigma^2}\bar{\phi}(\frac{y+\mu t}{\sigma\sqrt{t}}) + \bar{\phi}(\frac{y-\mu t}{\sigma\sqrt{t}})$, with $\bar{\phi}(x) = 1 - \phi(x) = \mathbb{P}(Z > x)$ for $Z \sim \mathcal{N}(0,1)$.
- For $y > 0$, recall that $M(t) \geq y$ iff $T_y \leq t$.

Index

1. Brownian motion
2. Variations on Brownian motion
3. Gaussian and stationary processes

Gaussian processes

A stochastic process $\{X(t), t \geq 0\}$ is called a **Gaussian process** (or normal process) if $X(t_1), \ldots, X(t_n)$ have a multivariate normal distribution for all $t_1, \ldots, t_n$, $n \geq 1$.

- Let $Z_1, \cdots, Z_m$ be i.i.d. $\mathcal{N}(0,1)$ and let

$$X_i = \sum a_{ij} Z_j + \mu_i$$

  for some constants $a_{ij}, 1 \leq i \leq n, 1 \leq j \leq m$ and $\mu_i, 1 \leq i \leq n$. Then the r.v.'s $X_1, \cdots, X_n$ have a multivariate normal distribution.
- A multivariate normal distribution is completely determined by the marginal mean values and the covariance matrix.

Brownian motion is Gaussian

- If $\{X(t), t \geq 0\}$ is a BM, since each $X(t_1), \ldots, X(t_n)$ can be expressed as a linear combination of the independent normal variables $X(t_1), X(t_2) - X(t_1), \ldots, X(t_n) - X(t_{n-1})$ it follows that **BM is a Gaussian process**.
- Hence SBM can also be defined as a Gaussian process with $\mathbb{E}[X(t)] = 0$ and $\mathrm{Cov}(X(s), X(t)) = \min\{s, t\}$. Indeed, for $s \leq t$

$$\begin{aligned}\mathrm{Cov}(X(s), X(t)) =& \mathrm{Cov}(X(s), X(s) + X(t) - X(s)) \\ =& \mathrm{Cov}(X(s), X(s)) + \mathrm{Cov}(X(s), X(t) - X(s)) \\ =& \mathrm{Cov}(X(s), X(s)) = \mathrm{Var}(X(s)) = s.\end{aligned}$$

The Brownian bridge is Gaussian

- Let $\{X(t), 0 \le t \le 1 \,|\, X(1) = 0\}$ be a standard Brownian bridge (SBB). Since in general $X(t) \,|\, X(1) = B \sim \mathcal{N}(tB, t(1-t))$, we have that $X(t) \,|\, X(1) = 0 \sim \mathcal{N}(0, t(1-t))$.
- The conditional distribution is a multivariate normal, hence the standard **Brownian bridge is a Gaussian process** with mean value 0 and covariance $\mathrm{Cov}(X(s), X(t)) = s(1-t)$, $s \le t$. Indeed, since $\mathbb{E}[X(u) \,|\, X(1) = 0] = 0$ for all $u < 1$,

$$
\begin{aligned}
&\mathrm{Cov}(X(s), X(t) \,|\, X(1) = 0) \\
&= \mathbb{E}[X(s)X(t) \,|\, X(1) = 0] \\
&= \mathbb{E}\big[\mathbb{E}[X(s)X(t) \,|\, X(t), X(1) = 0] \,|\, X(1) = 0\big] \\
&= \mathbb{E}\big[X(t)\mathbb{E}[X(s) \,|\, X(t)] \,|\, X(1) = 0\big] = \mathbb{E}[X(t)\frac{s}{t}X(t) \,|\, X(1) = 0] \\
&= \frac{s}{t}\mathbb{E}[(X(t))^2 \,|\, X(1) = 0] = \frac{s}{t}t(1-t) = s(1-t).
\end{aligned}
$$

- Alternative definition: if $\{X(t), t \ge 0\}$ is a SBM, then $\{Z(t) = X(t) - tX(1), 0 \le t \le 1\}$ is a SBB.

Integrated Brownian motion

If $\{X(t), t \geq 0\}$ is a BM, the process $\{Z(t) = \int_0^t X(s)\,ds, t \geq 0\}$ is called **integrated Brownian motion**.

- It can be used to model the price of a commodity throughout time. Let $Z(t)$ be the price at time t and assume the rate of change $X(t) = \frac{d}{dt}Z(t)$ follows a BM. Then $Z(t) = Z(0) + \int_0^t X(s)\,ds$.
- Since BM is a Gaussian process, also **the integrated BM is a Gaussian process**. When $\{X(t), t \geq 0\}$ is a SBM, we get $\mathbb{E}[Z(t)] = \mathbb{E}\big[\int_0^t X(s)\,ds\big] = \int_0^t \mathbb{E}[X(s)]\,ds = 0$ and, for $s \leq t$,

$$\begin{aligned}\mathrm{Cov}(Z(s), Z(t)) = \mathbb{E}[Z(s)Z(t)] = \mathbb{E}\left[\int_0^s \int_0^t X(u)X(v)\,dudv\right] \\ = \int_0^s \int_0^t \mathbb{E}[X(u)X(v)]\,dudv = \int_0^s \int_0^t \min\{u, v\}\,dudv \\ = \int_0^s \left(\int_0^u v\,dv + \int_u^t u\,dv\right)du = s^2\left(\frac{t}{2} - \frac{s}{6}\right).\end{aligned}$$

Stationary processes

A stochastic process $\{X(t), t \geq 0\}$ is a **stationary process** if for all $n \geq 1$ and $s > 0$ the random vectors $(X(t_1), \dots, X(t_n))$ and $(X(s+t_1), \dots, X(s+t_n))$ have the same joint distribution.

Examples:

- An ergodic continuous-time Markov chain $\{X(t), t \geq 0\}$ when $\mathbb{P}(X(0) = j) = P_j$ for each state j, i.e., when the initial state is chosen according to the limiting probabilities.
- The process $\{X(t) = N(t+L) - N(t), t \geq 0\}$ with $L > 0$ constant and $\{N(t), t \geq 0\}$ Poisson Process with rate λ. It follows from the stationary and independent increment assumption of Poisson processes.
- BM is not a stationary process.

Example 10.5: the random telegraph signal process.

Weakly stationary processes

A stochastic process $\{X(t), t \geq 0\}$ is a **weakly stationary process** if for all $s, t > 0$, $\mathbb{E}[X(t)] = c$ constant and $\mathrm{Cov}(X(t), X(t+s)) = R(s)$ does not depend on t.

For Gaussian processes, weakly stationarity implies stationarity, because multivariate normal distributions are determined by their mean values and the covariance matrix. Note that BM is not weakly stationary.

Example 10.6: the Ornstein-Uhlenbeck process.

Example 10.8: the random telegraph signal process.

Exercises

Session 11. Chapter 10: 1-4, 6.

Session 12. Chapter 10: 7, 9, 10, 32, 35. Suggested extra: 11, 33, 34.

Note that answers might contain some integrals that are very hard to evaluate. It is okay to provide answers as not evaluated integrals.