Some solutions, Stochastic processes II

Chapter 5

5.3 X is exponentially distributed and therefore memoryless. This implies that for all $t, s > 0$, we have

$$\mathbb{P}(X > t + s | X > t) = \mathbb{P}(X > s).$$

In particular, this implies that for any continuous function $f : \mathbb{R}_+ \to \mathbb{R}_+$ (or any other nice enough function), we have

$$\mathbb{E}[f(X)|X > t] = \mathbb{E}[f(X + t)].$$

So,

$$\mathbb{E}[X^2|X > 1] = \mathbb{E}[(X + 1)^2] = \mathbb{E}[X^2] + 2\mathbb{E}[X] + 1,$$

which is only equal to $\mathbb{E}[X^2]+1$ if $\mathbb{E}[X] = 0$ (which is never for an exponental distribution), and equal to

$$(1 + \mathbb{E}[X])^2 = (\mathbb{E}[X])^2 + 2\mathbb{E}[X] + 1$$

if $\mathbb{E}[X^2] = (\mathbb{E}[X])^2$ (i.e. if $Var(X) = 0$), which is also never. So only 3(a) is true.

5.8 Let X have density function $f_X(t) = \lambda e^{-\lambda t}$ and Y have density function $f_Y(t) = \mu e^{-\mu t}$, both for $t \geq 0$. Furthermore X and Y are independent.

We compute

$$\begin{aligned}\mathbb{P}(X > t|X \leq Y) &= \frac{\mathbb{P}(X > t, X \leq Y)}{\mathbb{P}(X \leq Y)} = \frac{\int_t^\infty \int_x^\infty f_Y(y)f_X(x)dydx}{\int_0^\infty \int_x^\infty f_Y(y)f_X(x)dydx} \\ &= \frac{\int_t^\infty \int_x^\infty \mu e^{-\mu y}\lambda e^{-\lambda x}dydx}{\int_0^\infty \int_0^\infty \mu e^{-\mu y}\lambda e^{-\lambda x}dydx} = \frac{\int_t^\infty \lambda e^{-\lambda x}e^{-\mu x}dx}{\int_0^\infty \lambda e^{-\lambda x}e^{-\mu x}dx} = \frac{\frac{\lambda}{\lambda+\mu}e^{-(\lambda+\mu)t}}{\frac{\lambda}{\lambda+\mu}} = e^{-(\lambda+\mu)t},\end{aligned}$$

which is equal to $\mathbb{P}(\min(X, Y) > t)$. (See page 288).

5.36 Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process with rate λ and let for $i = 1, 2, \cdots$ the random variables X_i be independent identically distributed exponential random variables with mean $1/\mu$, which are independent of $\{N(t), t \geq 0\}$. Define $S(t) = s\prod_{i=1}^{N(t)} X_i$. Then using the "telescoping property of expectations"

$$\mathbb{E}[S(t)] = \mathbb{E}[\prod_{i=1}^{N(t)} X_i] = \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)} X_i|N(t)]] = \mathbb{E}[\prod_{i=1}^{N(t)} \mathbb{E}[X_i|N(t)]] = \mathbb{E}[\prod_{i=1}^{N(t)}(1/\mu)] = \mathbb{E}[(1/\mu)^{N(t)}],$$

where we used the independence of the X_i's for the third identity. We may now use that $N(t)$ is Poisson distributed with expectation λt and thus that

$$\mathbb{E}[(1/\mu)^{N(t)}] = \sum_{k=0}^{\infty} \mathbb{P}(N(t) = k)(1/\mu)^k = \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t}(1/\mu)^k = e^{-\lambda(1-1/\mu)t}.$$

Similarly, using that $\mathbb{E}[(X_i)^2] = Var(X_i) + (\mathbb{E}[X_i])^2 = 1/\mu^2 + 1/\mu^2 = 2/\mu^2$, we obtain

$$\mathbb{E}[(S(t))^2] = \mathbb{E}[(\prod_{i=1}^{N(t)} X_i)^2] = \mathbb{E}[\mathbb{E}[\prod_{i=1}^{N(t)}(X_i)^2|N(t)]] = \mathbb{E}[\prod_{i=1}^{N(t)}(2/\mu^2)] = \mathbb{E}[(2/\mu^2)^{N(t)}] = e^{-\lambda(1-2/\mu^2)t}.$$

5.40 Use definition 5.2 (page 299) and see that
(1) $N_1(0) + N_2(0) = 0$,
(2) for all $0 \leq t_1 < t_2 < t_3 < t_4$. $N_1(t_2) - N_1(t_1)$, $N_1(t_4) - N_1(t_3)$, $N_2(t_2) - N_2(t_1)$ and $N_2(t_4) - N_2(t_3)$ are independent and therefore, $N_1(t_2) + N_2(t_2) - N_1(t_1) - N_2(t_1)$ and $N_1(t_4) + N_2(t_4) - N_1(t_3) - N_2(t_3)$ are independent, and we have independent increments,
(3)

$$\begin{aligned}
&\mathbb{P}(N_1(t+h) + N_2(t+h) - N_1(t) - N_2(t) = 1) \\
=&\mathbb{P}(N_1(t+h) - N_1(t) = 1, N_2(t+h) - N_2(t) = 0) \\
&+ \mathbb{P}(N_1(t+h) - N_1(t) = 0, N_2(t+h) - N_2(t) = 1) \\
=&(\lambda_1 h + o(h))(1 - \lambda_2 h + o(h)) + (1 - \lambda_1 h + o(h))(\lambda_2 h + o(h)) \\
=&(\lambda_1 + \lambda_2)h + o(h)
\end{aligned}$$

while (4)

$$\begin{aligned}
&\mathbb{P}(N_1(t+h) + N_2(t+h) - N_1(t) - N_2(t) \geq 2) \\
=&\mathbb{P}(N_1(t+h) - N_1(t) = N_2(t+h) - N_2(t) = 1) \\
&+ \mathbb{P}(N_1(t+h) - N_1(t) \geq 2 \cup N_2(t+h) - N_2(t) \geq 2) \\
\leq&\mathbb{P}(N_1(t+h) - N_1(t) = N_2(t+h) - N_2(t) = 1) \\
&+ \mathbb{P}(N_1(t+h) - N_1(t) \geq 2) + \mathbb{P}(N_2(t+h) - N_2(t) \geq 2) \\
=&(\lambda_1 h + o(h))(\lambda_2 h + o(h)) + o(h) + o(h) \\
=&o(h).
\end{aligned}$$

Where we have used that the product of two functions which are linear in h is $o(h)$. This finishes the proof.

5.45 Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process with rate λ, which is independent of T (≥ 0), which has mean μ and variance σ^2. Note that

$$Cov(T, N(T)) = \mathbb{E}[TN(T)] - \mathbb{E}[T]\mathbb{E}[N(T)].$$

Then observe

$$\mathbb{E}[TN(T)] = \mathbb{E}[\mathbb{E}[TN(T)]|T]] = \mathbb{E}[T \times \lambda T] = \lambda\mathbb{E}[T^2] = \lambda(Var(T)+(\mathbb{E}[T])^2) = \lambda(\sigma^2+\mu^2)$$

and

$$\mathbb{E}[N(T)] = \mathbb{E}[\mathbb{E}[N(T)]|T]] = \mathbb{E}[\lambda T] = \lambda\mathbb{E}[T] = \lambda\mu.$$

So,

$$Cov(T, N(t)) = \lambda(\sigma^2 + \mu^2) - \mu \times \lambda\mu = \lambda\sigma^2.$$

The variance of $N(T)$ can be computed similarly: $Var(N(T)) = \mathbb{E}[(N(T))^2] - (\mathbb{E}[N(T)])^2$, where

$$\mathbb{E}[(N(T))^2] = \mathbb{E}[\mathbb{E}[(N(T))^2]|T] = \mathbb{E}[\lambda T + \lambda^2 T^2] = \lambda\mu + \lambda^2(\sigma^2 + \mu^2).$$

So,

$$Var(N(T)) = \lambda\mu + \lambda^2(\sigma^2 + \mu^2) - (\lambda\mu)^2 = \lambda\mu + \lambda^2\sigma^2.$$

5.49 A translation of this problem is to compute $\mathbb{P}(N(T)-N(s) = 1)$, where $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process with rate λ. Because if $N(T) - N(s) > 1$, then the first arrival after s is not the last one before T while if $N(T) - N(s) = 0$, the first arrival after s is after T. By definition we have

$$\mathbb{P}(N(T) - N(s) = 1) = \frac{(\lambda(T-s))^1}{1!} e^{-\lambda(T-s)} = \lambda(T-s)e^{-\lambda(T-s)}. \tag{1}$$

This value is maximized if the derivative with respect to s is 0 or if $s = 0$ or if $s = T$. if $s = T$, (1) is clearly 0, while if $s = 0$ this value is $\lambda T e^{-\lambda T}$, which is easily checked to be at most e^{-1}. The derivative of $\mathbb{P}(N(T)-N(s) = 1)$ is 0, if $\lambda(\lambda(T-s)-1)e^{-\lambda(T-s)} = 0$, which implies that $T-s = 1/\lambda$. Filling in this value for s in (1) gives $\mathbb{P}(N(T)-N(s) = 1) = e^{-1}$, which is the maximal probability of winning.

5.60 By the order statistic property we obtain (a) $(1/3)^2 = 1/9$, (b) $1 - (2/3)^2 = 5/9$.

5.46 Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process with rate λ and let for $i = 1, 2, \cdots$ the random variables X_i be independent identically distributed random variables with mean μ, which are independent of $\{N(t), t \geq 0\}$. Then,

$$\begin{aligned} Cov(N(t), \sum_{i=1}^{N(t)} X_i) &= \mathbb{E}[N(t) \sum_{i=1}^{N(t)} X_i] - \mathbb{E}[N(t)]\mathbb{E}[\sum_{i=1}^{N(t)} X_i] \\ &= \mathbb{E}[\mathbb{E}[N(t) \sum_{i=1}^{N(t)} X_i | N(t)]] - \mathbb{E}[N(t)]\mathbb{E}[\mathbb{E}[\sum_{i=1}^{N(t)} X_i | N(t)]] \\ &= \mathbb{E}[N(t)\mu N(t)] - \mathbb{E}[N(t)]\mathbb{E}[\mu N(t)] = \mu(\lambda t + \lambda^2 t^2) - \lambda t \times \mu \lambda t = \mu \lambda t. \end{aligned}$$

5.78 Consider an inhomogeneous process between time 0 and 9, where the time is the time (in hours) since 8AM. $\lambda(t) = 4$ for $t \in (0, 2]$; $\lambda(t) = 8$ for $t \in (2, 4]$; $\lambda(t) = 8 + (t-4)$ for $t \in (4, 6]$ and $\lambda(t) = 10 - 2(t-6)$ for $t \in (6, 9]$. From the theory on inhomogeneous Poisson processes we know that the total number of arrivals of this Poisson Process is Poisson distributed with expectation $\int_0^9 \lambda(t)dt = 8 + 16 + 18 + 21 = 63$.

5.81b Use part a), but define $G(x) = m(x)/m(t)$ for $x \leq t$ and $G(x) = 1$ for $x > t$. Assume that there are $N(t)$ workers injured before time t. Note that this number is Poisson distributed with expectation $m(t)$. By part (a), the times of injury are independent and have distribution $G(x)$ and density $g(x) = m'(x)/m(t)$ for $x \leq t$. So, the probability that a worker is still injured at time t is given by $\int_0^t g(x)(1-F(t-x))dx$. The expected number of workers injured at time t is then given by this probability times the expected number of workers injured before time t. This product is given by $\int_0^t m'(x)(1 - F(t-x))dx$.

5.95 Let $\{N(t), t \geq 0\}$ be a mixed Poisson Process with random rate L. We first want to compute $\mathbb{E}[L|N(t) = n]$. By the definition of conditional expectation we have

$$\mathbb{E}[L|N(t) = n] = \frac{\mathbb{E}[L\mathbb{1}(N(t) = n)]}{\mathbb{P}(N(t) = n)}.$$

The latter is equal to

$$\frac{\mathbb{E}[\mathbb{E}[L\mathbb{1}(N(t) = n)|L]]}{\mathbb{E}[\mathbb{P}(N(t) = n|L)]} = \frac{\mathbb{E}[L\frac{(Lt)^n}{n!}e^{-Lt}]}{\mathbb{E}[\frac{(Lt)^n}{n!}e^{-Lt}]} = \frac{\mathbb{E}[L^{n+1}e^{-Lt}]}{\mathbb{E}[L^n e^{-Lt}]}.$$

For $s > t$ we obtain

$$\begin{aligned} \mathbb{E}[N(s)|N(t) = n] &= \mathbb{E}[\mathbb{E}[N(s)|N(t) = n, L]|N(t) = n] \\ &= \mathbb{E}[n + \mathbb{E}[N(s) - N(t)|L]|N(t) = n] = n + \mathbb{E}[L(t-s)|N(t) = n] \\ &= n + (t-s)\mathbb{E}[L|N(t) = n]. \end{aligned}$$

For $s < t$ we obtain

$$\mathbb{E}[N(s)|N(t) = n] = \mathbb{E}[\mathbb{E}[N(s)|N(t) = n, L]|N(t) = n] = \mathbb{E}[ns/t|N(t) = n] = ns/t,$$

where we have used the order statistic property for computing $\mathbb{E}[N(s)|N(t) = n, L]$.