Chapter 10

In this chapter it is good to remember that a picture tells a thousand words. It is therefore good to draw some kind of Brownian Motion to get inspiration from.

10.1 In these exercises, the variance parameter of the Brownian Motion is 1.

$$B(s) + B(t) = 2B(s) + [B(t) - B(s)] \sim 2N_1 + N_2,$$

where N_1 and N_2 are independent $N_1 \sim \mathcal{N}(0, s)$ and $N_2 \sim \mathcal{N}(0, t-s)$. So by properties of the normal distribution

$$B(s) + B(t) \sim \mathcal{N}(2 \times 0 + 0, 2^2 s + (t-s)) = \mathcal{N}(0, 3s+t).$$

10.2 For $t \geq 0$ set $B'(t) = B(t_1+t) - A$ and set $B' = B - A$ and $s' = s - t_1$ and $t_2' = t_2 - t_1$. $B'(\cdot)$ then defines a standard Brownian motion. We first compute the distribution of $B'(s')$ conditioned on $B'(t_2') = B'$. Or for $N_1 \sim \mathcal{N}(0, s')$ and $N_2 \sim \mathcal{N}(0, t_2' - s')$ and N_1 and N_2 independent we want to compute the conditional density

$$\begin{aligned}
f_{N_1|N_1+N_2}(x|B') &= \frac{f_{N_1,N_2}(x, B'-x)}{f_{N_1+N_2}(B')} \\
&= \frac{\frac{1}{\sqrt{2\pi s'}} \exp\left[-\frac{x^2}{2s'}\right] \frac{1}{\sqrt{2\pi(t_2'-s')}} \exp\left[-\frac{(B'-x)^2}{2(t_2'-s')}\right]}{\frac{1}{\sqrt{2\pi t_2'}} \exp\left[\frac{B'^2}{2t_2'}\right]} \\
&= \sqrt{\frac{t_2'}{2\pi s'(t_2'-s')}} \exp\left(-\frac{x^2(t_2'-s')t_2' + (B'-x)^2 s' t_2' - B'^2 s'(t_2'-s')}{2s'(t_2'-s')t_2'}\right) \\
&= \frac{1}{\sqrt{2\pi \frac{s'(t_2'-s')}{t_2'}}} \exp\left(-\frac{x^2(t_2')^2 - 2B'xs't_2' + B'^2 s'^2}{2s'(t_2'-s')t_2'}\right) \\
&= \frac{1}{\sqrt{2\pi \frac{s'(t_2'-s')}{t_2'}}} \exp\left(-\frac{(x - \frac{B's'}{t_2'})^2}{2\frac{s'(t_2'-s')}{t_2'}}\right)
\end{aligned}$$

which is the density function of a normal distribution with mean $B's'/t_2'$ (which lies on the straight line from the origin to (t_2', B')) and variance $\frac{s'(t_2'-s')}{t_2'}$. To obtain the density of $B(s)$ conditioned on $B(t_1) = A$ and $B(t_2) = B$, we note that $B(s) = B'(s-t_1)+A = B'(s')+A$.

10.3

$$\begin{aligned}\mathbb{E}[B(t_1)B(t_2)B(t_3)] &= \mathbb{E}[\mathbb{E}[B(t_1)B(t_2)B(t_3)|B(t_1)]] \\ &= \mathbb{E}[B(t_1)\mathbb{E}[B(t_2)B(t_3)|B(t_1)]].\end{aligned}$$

We now consider the inner expectation $\mathbb{E}[B(t_2)B(t_3)|B(t_1)]$:

$$\begin{aligned}\mathbb{E}[B(t_2)B(t_3)|B(t_1)] &= \mathbb{E}[\mathbb{E}[B(t_2)B(t_3)|B(t_1), B(t_2)]|B(t_1)] \\ &= \mathbb{E}[(B(t_2))^2|B(t_1)].\end{aligned}$$

So, using that $(B(t_2))^2 = (B(t_2) - B(t_1))^2 + 2(B(t_2) - B(t_1))B(t_1) + B(t_1)^2$,

$$\begin{aligned}&\mathbb{E}[B(t_1)B(t_2)B(t_3)] \\ &= \mathbb{E}[B(t_1)\mathbb{E}[(B(t_2))^2|B(t_1)]] \\ &= \mathbb{E}[B(t_1)]Var(B(t_2)|B(t_1)) + 2\mathbb{E}[B(t_1)^2\mathbb{E}[B(t_2) - B(t_1)|B(t_1)] + \mathbb{E}[B(t_1)[(B(t_1))^2]] \\ &= \mathbb{E}[B(t_1)](t_2 - t_1) + 0 + \mathbb{E}[B(t_1)(B(t_1))^2] \\ &= 0 + 0 + \mathbb{E}[(B(t_1))^3] \\ &= 0,\end{aligned}$$

where we have used that the odd moments of the normal distribution are 0.

10.4 Using equation (10.6) on page 611, we obtain that

$$\mathbb{P}(T_a < \infty) = \lim_{z\to 0} \frac{2}{\sqrt{2\pi}} \int_z^\infty e^{-y^2/2} dy = \frac{2}{\sqrt{2\pi}} \int_0^\infty e^{-y^2/2} dy = 1,$$

because it is twice the integral over the density of a standard normal distribution over the positve half-line.

$$\begin{aligned}\mathbb{E}[T_a] = \int_0^\infty \mathbb{P}(T_a > t)dt &= \int_0^\infty 1 - \mathbb{P}(T_a \le t)dt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty \int_0^{a/\sqrt{t}} e^{-y^2/2} dydt \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty \int_0^{a^2/y^2} e^{-y^2/2} dtdy \\ &= \frac{2}{\sqrt{2\pi}} \int_0^\infty (a^2/y^2) e^{-y^2/2} dy \\ &\ge \frac{2}{\sqrt{2\pi}} \int_0^1 (a^2/y^2) e^{-y^2/2} dy \\ &\ge \frac{2}{\sqrt{2\pi}} \int_0^1 (a^2/y^2) e^{-1/2} dy = \infty\end{aligned}$$

10.6 Translated to a Brownian Motion problem we want to know

$$\mathbb{P}(T_{b+c} > t|B(0) = b) = \mathbb{P}(T_c > t|B(0) = 0) = 1 - \mathbb{P}(T_c \le t)$$

and we can use equation (10.6) on page 611.

10.7

$$
\begin{aligned}
&\mathbb{P}(\max_{t_1\le s\le t_2} B(s) > x)\\
&= \int_{-\infty}^{x} \mathbb{P}(\max_{t_1\le s\le t_2} B(s) > x|B(t_1)=y) f_{B(t_1)}(y)dy + \int_{x}^{\infty} \mathbb{P}(\max_{t_1\le s\le t_2} B(s) > x|B(t_1)=y) f_{B(t_1)}(y)dy\\
&= \int_{-\infty}^{x} \mathbb{P}(\max_{t_1\le s\le t_2} B(s) > x|B(t_1)=y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)}dy + \int_{x}^{\infty} 1\times \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)}dy\\
&= \int_{-\infty}^{x} \mathbb{P}(\max_{0\le s'\le t_2-t_1} B(s') > x-y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)}dy + \int_{x}^{\infty} 1\times \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)}dy.
\end{aligned}
$$

Using the display on page 612, the first integral becomes

$$
\begin{aligned}
&\int_{-\infty}^{x} \mathbb{P}(\max_{0\le s'\le t_2-t_1} B(s') > x-y) \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)}dy\\
&= \int_{-\infty}^{x} \frac{2}{\sqrt{2\pi}} \int_{(x-y)/\sqrt{t_2-t_1}}^{\infty} e^{-z^2/2}dz \frac{1}{\sqrt{2\pi t_1}} e^{-y^2/(2t_1)}dy\\
&= \frac{1}{\pi\sqrt{t_1}} \int_{-\infty}^{x} \int_{(x-y)/\sqrt{t_2-t_1}}^{\infty} e^{-z^2/2} e^{-y^2/(2t_1)}dzdy
\end{aligned}
$$

and I do not think it gets nicer than this.

10.9 Let $\{X(t), t\ge 0\} = \{\sigma B(t) + \mu t, t \ge 0\}$. So,

$$
\begin{aligned}
f_{X(s),X(t)}(x,y) &= f_{\sigma B(s),\sigma B(t)}(x-\mu s, y-\mu t)\\
&= \frac{1}{\sigma^2} f_{B(s),B(t)}\left(\frac{x-\mu s}{\sigma}, \frac{y-\mu t}{\sigma}\right)\\
&= \frac{1}{\sigma^2} f_{B(s),B(t)-B(s)}\left(\frac{x-\mu s}{\sigma}, \frac{y-x-\mu(t-s)}{\sigma}\right)\\
&= \frac{1}{\sigma^2} f_{B(s)}\left(\frac{x-\mu s}{\sigma}\right) f_{B(t)-B(s)}\left(\frac{y-x-\mu(t-s)}{\sigma}\right)\\
&= \frac{1}{\sqrt{2\pi\sigma^2 s}} e^{-\frac{(x-\mu s)^2}{2s\sigma^2}} \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} e^{-\frac{(y-x-\mu(t-s))^2}{2(t-s)\sigma^2}}
\end{aligned}
$$

and it doesn't get much nicer than this.

10.10 Let $\{X(t), t \geq 0\} = \{\sigma B(t) + \mu t, t \geq 0\}$. If $s < t$ then,

$$\begin{aligned}
f_{X(t)|X(s)}(x|c) &= \frac{1}{\sigma} f_{B(t)|B(s)}\left(\frac{x-\mu t}{\sigma} | \frac{c-\mu s}{\sigma}\right) \\
&= \frac{1}{\sigma} \frac{f_{B(s),B(t)}\left(\frac{c-\mu s}{\sigma}, \frac{x-\mu t}{\sigma}\right)}{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right)} \\
&= \frac{1}{\sigma} \frac{f_{B(s),B(t)-B(s)}\left(\frac{c-\mu s}{\sigma}, \frac{x-c-\mu(t-s)}{\sigma}\right)}{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right)} \\
&= \frac{1}{\sigma} \frac{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right) f_{B(t)-B(s)}\left(\frac{x-c-\mu(t-s)}{\sigma}\right)}{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right)} \\
&= \frac{1}{\sigma} f_{B(t)-B(s)}\left(\frac{x-c-\mu(t-s)}{\sigma}\right) \\
&= \frac{1}{\sqrt{2\pi\sigma^2(t-s)}} \exp\left[-\frac{(x-c-\mu(t-s))^2}{2\sigma^2(t-s)}\right]
\end{aligned}$$

So conditioned on $X(s) = c$, $X(t)$ is normal distributed with expectation $c + \mu(t-s)$ and variance $\sigma^2(t-s)$.

Similarly if $s > t$, then

$$\begin{aligned}
f_{X(t)|X(s)}(x|c) &= \frac{1}{\sigma} f_{B(t)|B(s)}\left(\frac{x-\mu t}{\sigma} | \frac{c-\mu s}{\sigma}\right) \\
&= \frac{1}{\sigma} \frac{f_{B(t),B(s)}\left(\frac{x-\mu t}{\sigma}, \frac{c-\mu s}{\sigma}\right)}{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right)} \\
&= \frac{1}{\sigma} \frac{f_{B(t),B(s)-B(t)}\left(\frac{x-\mu t}{\sigma}, \frac{c-x-\mu(s-t)}{\sigma}\right)}{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right)} \\
&= \frac{1}{\sigma} \frac{f_{B(t)}\left(\frac{x-\mu t}{\sigma}\right) f_{B(s)-B(t)}\left(\frac{c-x-\mu(s-t)}{\sigma}\right)}{f_{B(s)}\left(\frac{c-\mu s}{\sigma}\right)} \\
&= \frac{1}{\sigma} \frac{\frac{1}{\sqrt{2\pi t}} \exp\left[-\frac{1}{2t}\left(\frac{x-\mu t}{\sigma}\right)^2\right] \frac{1}{\sqrt{2\pi(s-t)}} \exp\left[-\frac{1}{2(s-t)}\left(\frac{c-x-\mu(s-t)}{\sigma}\right)^2\right]}{\frac{1}{\sqrt{2\pi s}} \exp\left[-\frac{1}{2s}\left(\frac{c-\mu s}{\sigma}\right)^2\right]} \\
&= \frac{1}{\sqrt{2\pi \frac{t(s-t)}{s} \sigma^2}} \exp\left[\frac{s(s-t)(x-\mu t)^2 + st(c-x-\mu(s-t))^2 - t(s-t)(c-\mu s)^2}{2t(s-t)s\sigma^2}\right] \\
&= \frac{1}{\sqrt{2\pi \frac{t(s-t)}{s} \sigma^2}} \exp\left[\frac{[(x-c\frac{t}{s})^2}{2\frac{t(s-t)}{s}\sigma^2}\right].
\end{aligned}$$

This is the density of a Normal random variable with expectation ct/s and variance $\sigma^2 t(s-t)/s$, which is independent of μ.

10.11 If we consider the logarithm of the described process then with probability $\frac{1}{2}(1+\frac{\mu}{\sigma}\sqrt{h})$ the new process increases $\sigma\sqrt{h}$ per h time units and with probabilty $\frac{1}{2}(1-\frac{\mu}{\sigma}\sqrt{h})$ it decreases $\sigma\sqrt{h}$. The expected step is μh and the variance is $\sigma^2 h$. The new process has independent increments and by the CLT the increment after t/h steps is normal distributed with mean μt and variance $\sigma^2 t$.

10.32 A Brownian motion is a Gaussian process with expectation function $\mathbb{E}[B(t)] = 0$ and Covariance function $Cov(B(s), B(t)) = s$. The Brownian bridge is a Gaussian process with expectation function $\mathbb{E}[Z(t)] = 0]$ and $Cov(Z(s), Z(t)) = s(1-t)$ for $0 < s, t < 1$ (see page 630). For all sequences of times $t_1, t_2, \cdots, t_n$. $(Y(t_1), Y(t_2), \cdots, Y(t_n))$ is a multivariate normal because Z is a Gaussian process and the Ys are just constants times Z's evaluated at different times. Further $\mathbb{E}[Y(t)] = (t+1)[E][Z(t/(t+1))] = 0$, because the expectation function of a Brownian Motion is 0. To go to the covariance, for $s \leq t$ we have

$$Cov[Y(s), Y(t)] = Cov\left[(s+1)Z\left(\frac{s}{s+1}\right), (t+1)Z\left(\frac{t}{t+1}\right)\right]$$
$$= (s+1)(t+1)Cov\left[Z\left(\frac{s}{s+1}\right), Z\left(\frac{t}{t+1}\right)\right] = (s+1)(t+1)\frac{s}{s+1}\left(1-\frac{t}{t+1}\right) = s.$$

So the expectation function and covariance function of the Gaussian process $\{Y(t), t \geq 0\}$ are those of a Brownian motion and therefore the process is a Brownian Motion.

10.33 First assume $s > 1$, then

$$Cov(X(t), X(t+s)) = Cov(N(t+1) - N(t), N(t+s+1) - N(t+s)) = 0,$$

because it is the covariance between the number of arrivals in the non-overlapping intervals $(t, t+1)$ and $(t+s, t+s+1)$.

If $s \in [0, 1]$, then

$$\begin{aligned}
&Cov(X(t), X(t+s)) \\
=&Cov(N(t+1) - N(t), N(t+s+1) - N(t+s)) \\
=&Cov([N(t+1) - N(t+s)] + [N(t+s) - N(t)], [N(t+s+1) - N(t+1)] + [N(t+1) - N(t+s)]) \\
=&Cov([N(t+1) - N(t+s)], [N(t+s+1) - N(t+1)]) \\
&+ Cov([N(t+1) - N(t+s)], [N(t+1) - N(t+s)]) \\
&+ Cov([N(t+s) - N(t)], [N(t+s+1) - N(t+1)]) \\
&+ Cov([N(t+s) - N(t)], [N(t+1) - N(t+s)]) \\
=&0 + Var(N(t+1) - N(t+s)) + 0 + 0 \\
=&\lambda((t+1) - (t+s)) = \lambda(1-s)
\end{aligned}$$

Where we have used that covariance of the number of arrivals in non-overlapping interval is 0, while the variance of the number of arrivals in an interval of length x is the variance of Poisson distributed random variable with mean λx.

10.34 In a Poisson process the waiting times between arrivals are independent and identically exponentially distributed and the exponential distribution is memoryless. In particular $\{N(t); t \geq 0\}$ is distributed as $\{N(t+s) - N(s); t \geq 0\}$ and therefore $\{Y(t); t \geq 0\} = \{\min\{u \geq 0; N(t+u) - N(t) \geq 1\}, t \geq 0\}$ is distributed as $\{Y(t+s); t \geq 0\}$ for all $s > 0$.

$$Cov(Y(t), Y(t+s)) = \mathbb{E}[Y(t)Y(t+s)] - \mathbb{E}[Y(t)]\mathbb{E}[Y(t+s)] = \mathbb{E}[Y(t)Y(t+s)] - 1/\lambda^2$$

$Y(t)$ has distribution function $f(u) = \lambda e^{-\lambda u}$. So,

$$\mathbb{E}[Y(t)Y(t+s)] = \mathbb{E}[\mathbb{E}[Y(t)Y(t+s)|Y(t)]] = \int_0^\infty \lambda e^{-\lambda u}\mathbb{E}[Y(t)Y(t+s)|Y(t) = u]du$$
$$= \int_0^s \lambda e^{-\lambda u}\mathbb{E}[uY(t+s)|Y(t) = u]du + \int_s^\infty \lambda e^{-\lambda u}\mathbb{E}[uY(t+s)|Y(t) = u]du$$

Now, if $Y(t) > s$ then $Y(t+s) = Y(t) - s$, while if $Y(t) < s$ then $Y(t+s)$ is exponentially distributed with parameter λ and further independent of $Y(t)$. So,

$$\mathbb{E}[Y(t)Y(t+s)] = \int_0^s \lambda e^{-\lambda u}u\frac{1}{\lambda}du + \int_s^\infty \lambda e^{-\lambda u}u(u-s)du$$
$$= \int_0^s e^{-\lambda u}\frac{1}{\lambda}du - [e^{-\lambda u}\frac{u}{\lambda}]_{u=0}^s + \int_0^\infty \lambda e^{-\lambda(u'+s)}(u'^2 + u's)du'$$
$$= \frac{1}{\lambda^2}(1 - e^{-\lambda s}) - \frac{s}{\lambda}e^{-\lambda s} + e^{-\lambda s}(Var(Y(t) + \mathbb{E}[Y(t)]^2) + s\mathbb{E}[Y(t)])$$
$$= \frac{1}{\lambda^2}(1 - e^{-\lambda s}) - \frac{s}{\lambda}e^{-\lambda s} + e^{-\lambda s}\left(\frac{2}{\lambda^2} + \frac{s}{\lambda}\right) = \frac{1}{\lambda^2}(1 + e^{-\lambda s})$$

and
$$Cov(Y(t), Y(t+s)) = \mathbb{E}[Y(t)Y(t+s)] - \frac{1}{\lambda^2} = e^{-\lambda s}\frac{1}{\lambda^2}$$

10.35 a)

$$Var(X(t+s) - X(t)) = Cov(X(t+s) - X(t), X(t+s) - X(t))$$
$$= Cov(X(t+s), X(t+s)) - 2Cov(X(t), X(t+s) + Cov(X(t), X(t)) = R_X(0) - 2R_X(s) + R_X(0)$$

as desired.

b) $\mathbb{E}[Y(t)] = \mathbb{E}[X(t+1)] - \mathbb{E}[X(t)] = 0$, because $X(\cdot)$ is stationary, while

$$\begin{aligned}
&Cov(Y(t), Y(t+s)) \\
=&Cov(X(t+1) - X(t), X(t+s+1) - X(t+s)) \\
=&Cov(X(t+1), X(t+s+1)) - Cov(X(t), X(t+s+1)) \\
&- Cov(X(t+1), X(t+s)) + Cov(X(t), X(t+s)) \\
=&2R_X(s) - R_X(s+1) - R_X(s-1),
\end{aligned}$$

which is independent of t.