STOCKHOLMS UNIVERSITET
MATEMATISKA INSTITUTIONEN
Avd. Matematisk statistik

MT5012
TENTAMEN
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# First exam: Stochastic Processes and Simulation II

## May 30th 2023, kl. 14–19

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Grading: The exam consists of five exercises, each divided into three questions. Each question is worth a maximum of 4 points, for a total of 12 points per exercise. Exercise 2 contains a bonus question which is worth 2 extra points, for a total of 14 points. The maximum score for the exam is then 62 points.
In order to pass the exam, you have to score at least 3 points in each exercise and 30 points in total.

| Grade | A | B | C | D | E |
|---|---|---|---|---|---|
| Needed points | 54 | 48 | 42 | 36 | 30 |

Partial answers might be worth points, while answers "out of the blue" will not be rewarded. The questions differ in difficulty.

## Exercise 1: Poisson processes

**(i)** Give the definition of a compound Poisson process $X(t)$ and write an expression for its mean $\mathbb{E}[X(t)]$ and its variance $\mathrm{Var}(X(t))$.

Due to the current climate crisis, lots of families are migrating from climate-affected countries seeking better living conditions. Data show that most of the climate refugees migrate from Afghanistan, India and Pakistan. Assume that families are migrating from Afghanistan according to a Poisson process with a rate of 7 per day. Suppose also that the size of a family is $1, 2, 3, 4$, respectively with probability $1/8, 3/8, 3/8, 1/8$, independently for each family and from the migration process.

**(ii)** What is the expected value and variance of the number of refugees migrating from Afghanistan in a week?

**(iii)** Using the central limit theorem, find the approximate probability that at least 1800 refugees migrate from Afghanistan in the next 100 days. The answer can be given in terms of $\phi(x) = \mathbb{P}(Z \leqslant x)$, where $Z \sim \mathcal{N}(0, 1)$ is a standard normal random variable.

## Exercise 2: Renewal theory

**(i)** Let $\{N(t), t \geqslant 0\}$ be a renewal process with i.i.d. interarrival times $X_n$, $n \geqslant 1$, and let $\mu = \mathbb{E}[X_n]$. Prove that the rate of the renewal process $\frac{N(t)}{t}$ converges to $\frac{1}{\mu}$ almost surely as $t \to \infty$.

Suppose that at one of the border control stations between Afghanistan and Iran, all the refugees that arrive from Afghanistan receive help to reach the nearest city in Iran thanks to a bus service provided by some volunteers authorized by the governments. Assume that refugees arrive according to a Poisson process with rate $\lambda$. Assume that, as soon as there are $N$ refugees at the station, a bus picks them all up and departs. The bus service association incurs a cost at a rate of $nc$ per unit time whenever there are $n$ refugees waiting at the station.

**(ii)** Describe the problem in terms of a renewal reward process. State and use the renewal reward theorem to compute the long-run average cost.

Consider now a slightly different model and assume that, as soon as there are $N$ refugees at the station, a bus is called and takes $T$ units of time to arrive. Again, when it arrives all the refugees are picked up.

**(iii)** What is the expected cost in a cycle?

**(iv)** Bonus (2 points): What is the long-run average cost?

## Exercise 3: Queueing theory

When the refugees arrive at the border control station they are sent to a registration desk to fill in some documents. Suppose that there is only one active desk, that the refugees arrive at a Poisson rate $\lambda$ independently of each other, and that the time it takes for each registration is exponentially distributed with mean $1/\mu$, independently of everything else.

**(i)** Specify what type of queueing model best describes the registration process. Write down the balance equations and show how they can be solved to compute the limiting probability $P_0$ that there are no refugees at the border control station. What condition must $\lambda$ and $\mu$ satisfy in order for the limiting probabilities to exist?

**(ii)** What is the average number of refugees at the station? What is the average number of refugees in the queue at the station?

**(iii)** Assume now that the refugees arrive on average every 10 minutes according to a Poisson process, and that the time $S$ it takes for each registration is not anymore exponentially distributed, but $\mathbb{E}[S] = \mathrm{Var}(S) = 5$ minutes. What is the average number of refugees at the station? What is the average number of refugees in the queue at the station?

## Exercise 4: Simulation

**(i)** Describe and prove the inverse transformation method to simulate a random variable with distribution function $F$.

**(ii)** How can we simulate a Poisson random variable with mean $\lambda$ starting from independent uniform random variables $U_1, U_2, \ldots$? Describe the method and argue that it gives the desired distribution.

Suppose that the refugees arrive at the border control station according to a nonhomogeneous Poisson process $\{N(t), t \geqslant 0\}$ with rate $\lambda(t) = 3t^2$.

**(iii)** Explain in detail how we can simulate the first $T$ time units of the refugees' arrival process by first simulating the random variable $N(T)$ and then simulating $N(T)$ random variables representing the arrival times.

## Exercise 5: Brownian motion

**(i)** Give the definition of a Brownian motion $\{X(t), t \geqslant 0\}$ with drift coefficient $\mu$ and variance parameter $\sigma^2$.

Data from this year show that, due to emissions of greenhouse gases from human activities, the global temperature has increased by 1.1 degrees Celsius compared with pre-industrial levels in the period 1850-1900. Recent studies predicted that it will be 2 degrees Celsius warmer than pre-industrial times by the year 2050.

**(ii)** Assume that the global temperature evolves according to a Brownian motion with drift and that it will most likely reach the 2 degrees Celsius threshold exactly in 2050. What is the value of the drift coefficient $\mu$ in the unit Celsius/years?

Assume that we will be able to find solutions to the climate crisis and completely cancel the effect of the drift in the year 2035, so that the global temperature will evolve according to a Brownian motion with drift $\mu$ until 2035 and then without drift until 2050.

**(iii)** If in 2035 the global temperature will be exactly at its mean value and if $\sigma = 1$, what is the probability that it will still reach the 2 degrees Celsius threshold by 2050?