Solutions Stochastic Processes and Simulation II, August 21, 2018

Problem 1: Poisson Processes

Let $\{N(t), t \geq 0\}$ be a homogeneous Poisson Process on $(0, \infty)$ with rate λ. Let $\{S_i, i = 1, 2, \cdots\}$ be the points of the Poisson Process, such that $0 < S_1 < S_2 < S_3 < \cdots$. Define $S_0 = 0$.

a) Provide the definition of a Poisson Process. (4p)

Solution: A non-decreasing non-negative integer valued process $\{N(t); t \geq 0\}$ is a Poisson Process with rate $\lambda > 0$ if

- $N(0) = 0$
- for all $0 \leq t_1 < t_2 < t_3 < t_4 < \infty$, $N(t_4) - N(t_3)$ is independent of $N(t_2) - N(t_1)$
- for $0 \leq s \leq t < \infty$, $N(t) - N(s)$ is Poisson distributed with parameter $\lambda(t - s)$

b) Let n be a strictly positive integer. Suppose that we know that $N(1) = n$. What is the distribution of S_1? That is, compute $\mathbb{P}(S_1 \leq t | N(1) = n)$ for $t \in [0, 1]$? (4p)

Solution: First note that

$$\mathbb{P}(S_1 \leq t | N(1) = n) = 1 - \mathbb{P}(S_1 > t | N(1) = n) = 1 - \mathbb{P}(N(t) = 0 | N(1) = n).$$

By the order statistic property the probability that all n points in $[0, 1]$ are in $[t, 1]$ is $(1 - t)^n$. So, $\mathbb{P}(S_1 \leq t | N(1) = n) = 1 - (1 - t)^n$.

Define independently of $\{N(t), t \geq 0\}$ a second homogeneous Poisson process $\{X(t), t \geq 0\}$ on $(0, \infty)$ with rate β.

c) What is the distribution of $X(S_1)$? (4p)

Solution You can see $\{N(t) + X(t), t \geq 0\}$ as a single Poisson Process with rate $\lambda + \beta$ and the points of the process belong independently to the N process with probability $\lambda/(\lambda + \beta)$ and the the X process otherwise. Set $p = \lambda/(\lambda + \beta)$. Then for $k \in \{0, 1, \cdots\}$, $\mathbb{P}(X(S_1) = k)$ is the probability that the first k points of the combined Poisson process are all belonging to X and the $k + 1$-st to N, which is $(1 - p)^k p$.

It is also possible to note that the first point of N arrives after an exponential time with density $\lambda e^{-\lambda t}$, and we note that $X(t)$ is Poisson distributed with expectation βt. Therefore, for $k \in \{0, 1, \cdots\}$

$$\mathbb{P}(X(S_1) = k) = \int_0^\infty \lambda e^{-\lambda t} \frac{(\beta t)^k}{k!} e^{-\beta t} dt = \lambda \beta^k \int_0^\infty \frac{t^k}{k!} e^{-(\lambda+\beta)t} dt,$$

which can be computed using (by repeated partial integration)

$$\int_0^\infty \frac{t^k}{k!} e^{-(\lambda+\beta)t} dt = \frac{1}{\lambda + \beta} \int_0^\infty \frac{t^{k-1}}{(k-1)!} e^{-(\lambda+\beta)t} dt = \cdots = \frac{1}{(\lambda + \beta)^k} \int_0^\infty e^{-(\lambda+\beta)t} dt = \frac{1}{(\lambda + \beta)^{k+1}},$$

which gives that $\mathbb{P}(X(S_1) = k) = \frac{\lambda \beta^k}{(\lambda+\beta)^{k+1}} = (1 - p)^k p$.

Problem 2: Renewal Theory

A factory has two machines. Each machine can be either broken or working. If both machines are working one is "producing", while the other is "on stand-by". If only one machine is working, that machine is "producing", while the other one is "in repair". If both machines are broken, then one machine is "in repair", while the other ones is "waiting to go in repair". So, the pair of machines can be in three states:

A One machine "producing", the other "on stand-by".
B One machine "producing", the other "in repair".
C One machine "in repair", the other waiting to go "in repair".

Assume that a "producing" machine breaks down after an exponentially distributed time with expectation $1/\lambda$, which is independent of everything else in the process. If just before that moment the other machine was "on stand-by", it becomes "producing" immediately and the machine which broke down immediately gets "in repair". If at the moment of break down of one machine, the other machine is "in repair", then the newly broken down machine has to wait for its repair. to start again until the first machine is fully repaired. Then it gets "in repair" itself. The time needed for repair for a machine is not random and equal to exactly one time unit.

Assume that at time $S_0 = 0$ one machine just became "producing", while the other just gets "in repair" (So, the pair just enters state B at time 0). Let

$$S_k := \min\{t > S_{k-1} : \text{A machine starts producing}\} \qquad \text{for } k \in \{1, 2, \cdots\}.$$

be the k-th time one of the machines just becomes "producing" (and by the definition of the model, the other just gets "in repair"). That is S_k is the k-th time strictly after time 0, that the pair of machines enters state B. For $k \in \{1, 2, \cdots\}$, define $X_k = S_k - S_{k-1}$.

a) Argue that for $k \in \{1, 2, \cdots\}$, the random variable X_k satisfies
$\mathbb{P}(X_k \le t) = 0$ for $t < 1$ and $\mathbb{P}(X_k \le t) = 1 - e^{-\lambda t}$ for $t \ge 1$.
Furthermore, show that $\mathbb{E}[X_k] = 1 + e^{-\lambda}/\lambda$. (4p)

Solution: Let X be distributed as X_k for $k = 1, 2, \cdots$. In order to return to state B both the working machine has to break down and the repair of the other machine has to finish. Let T be the time until break down. If $T < 1$, the repair of the second machine is still going on and the time until re-entering B is exactly 1 time unit (time until repair), while if $T > 1$, the repair of the second machine is already finished. So, the time until re-entering B is equal to T. That is, $X = \max(T, 1)$ Since T is exponentially distributed with parameter λ we obtain that $\mathbb{P}(X_1 \le t) = 0$ for $t \le 1$ and $\mathbb{P}(X_1 \le t) = \mathbb{P}(T \le t) = \int_0^t \lambda e^{-\lambda s} ds = 1 - e^{-\lambda t}$ for $t \ge 1$.

A straightforward computation now gives

$$\begin{aligned}\mathbb{E}[X_1] = \mathbb{E}[\max(1,T)] &= \int_0^1 \lambda e^{-\lambda s} ds + \int_1^\infty s\lambda e^{-\lambda s} ds \\ &= (1 - e^{-\lambda}) + \int_0^\infty (s'+1)\lambda e^{-\lambda(s'+1)} ds' = (1 - e^{-\lambda}) + e^{-\lambda} \int_0^\infty (s'+1)\lambda e^{-\lambda s'} ds' \\ &= (1 - e^{-\lambda}) + e^{-\lambda}\mathbb{E}[T+1] = (1 - e^{-\lambda}) + e^{-\lambda} + e^{-\lambda}/\lambda\end{aligned}$$

b) Compute the long run fraction of time that none of the machines is working (that is the fraction of time the pair of machines is in state C). (4p)

Solution: Use renewal reward theory. Where the reward is the time spend in state C between two "renewals". By a) the expected duration of a cycle is $1 + e^{-\lambda}/\lambda$, while the expected time in state C is $1 - T$ if $T < 1$ and 0 otherwise. This boils down to that the expected time spend in state C during a cycle is

$$\int_0^1 \lambda e^{-\lambda t}(1-t)dt = (1-e^{-\lambda}) - \int_0^1 \lambda t e^{-\lambda t}dt = (1-e^{-\lambda}) + \int_1^\infty \lambda t e^{-\lambda t}dt - \int_0^\infty \lambda t e^{-\lambda t}dt$$

We notice that $(1-e^{-\lambda}) + \int_1^\infty \lambda t e^{-\lambda t}dt$ is the answer to part a) and $\int_0^\infty \lambda t e^{-\lambda t}dt = \mathbb{E}[T] = 1/\lambda$. So, the expected time spend in state C during a cycle is $1 + e^{-\lambda}/\lambda - 1/\lambda$, and the asymptotic fraction of time spent in state C is given by

$$\frac{1 + e^{-\lambda}/\lambda - 1/\lambda}{1 + e^{-\lambda}/\lambda} = \frac{\lambda + e^{-\lambda} - 1}{\lambda + e^{-\lambda}}.$$

Let $N(t)$ be the number of "renewals" up to time t. That is $N(t) = n$ if and only if $S_n \leq t$ and $S_{n+1} > t$.

c) For $t \to \infty$, compute $\mathbb{E}[t - S_{N(t)}]$. That is, compute the expected time since the last renewal at time t, in the limit as $t \to \infty$. (4p)

Solution: Use the arguments of page 448 of the book: Assume that you obtain reward at rate $t - S_{N(t)}$ at time t. And let X be distributed as a cycle length X_1. Then the expected reward during a cycle is $\mathbb{E}[\int_0^X tdt] = \mathbb{E}[X^2]/2$. We can compute $\mathbb{E}[X^2]$ using that $X = \max(1, T)$ and we obtain

$$\begin{aligned}
\mathbb{E}[X^2] &= \int_0^1 \lambda e^{-\lambda t}dt + \int_1^\infty \lambda t^2 e^{-\lambda t}dt = (1-e^{-\lambda}) + e^{-\lambda}\int_0^\infty \lambda(t'+1)^2 e^{-\lambda t'}dt' \\
&= (1-e^{-\lambda}) + e^{-\lambda}\int_0^\infty \lambda e^{-\lambda t'}dt' + 2e^{-\lambda}\int_0^\infty \lambda t' e^{-\lambda t'}dt' + e^{-\lambda}\int_0^\infty \lambda (t')^2 e^{-\lambda t'}dt' \\
&= (1-e^{-\lambda}) + e^{-\lambda}(1 + 2\mathbb{E}[T] + \mathbb{E}[T^2]) = 1 + \frac{2e^{-\lambda}}{\lambda} + \frac{2e^{-\lambda}}{\lambda^2}.
\end{aligned}$$

So, $\mathbb{E}[t - S_{N(t)}]$ converges to

$$\frac{1 + \frac{2e^{-\lambda}}{\lambda} + \frac{2e^{-\lambda}}{\lambda^2}}{1 + e^{-\lambda}/\lambda} = \frac{\lambda^2 + \lambda 2e^{-\lambda} + 2e^{-\lambda}}{\lambda(1+e^{-\lambda})}$$

Problem 3: Queueing Theory

Consider an $M/M/1$ queue with impatient customers. In this model customers arrive according to a Poisson Process with rate λ. Customers have independent workloads which are exponentially distributed with expectation $1/\mu$. In addition, customers which are in the queue, but not in service might become impatient and leave at rate μ independently of other customers. That is, customers leave because they are impatient after an exponentially distributed time with expectation $1/\mu$, unless they already started service by that time. Note that this expectation is equal to the expected workload of a customer.

a) For $k \in \{0, 1, 2, \cdots\}$, let P_k be the probability that there are k customers in the system (in service and in the queue) if the number of customers in the systems starts in the stationary distribution. Show that

$$P_k = \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu}$$

for $k \in \{0, 1, 2, \cdots\}$. (4p)

Solution: Each customer, whether in service or not, leaves at rate μ. New customers arrive at rate λ. So the rate of going from $k$to $k+1$ customers is λ for $k = 0, 1, \cdots$ and the rate of going from k to $k-1$ customers is μk for $k = 1, 2, \cdots$. Writing the balance equations (the rate of leaving a state should be equal to entering a state, then gives that $\lambda P_0 = \mu P_1$ and $(\lambda + \mu k)P_k = \lambda P_{k-1} + \mu(k+1)P_{k+1}$. It is easyly checked that $P_k = \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu}$ for $k \in \{0, 1, 2, \cdots\}$ satisfies this equation, and that $\sum_{k=0}^{\infty} \frac{(\lambda/\mu)^k}{k!} e^{-\lambda/\mu} = 1$.

b) In the long run, what fraction of arriving customers will enter service before they loose their patience? (4p)

Hint: Compute the long run number of arriving customers per time unit and the long run number of served customers leaving the system per time unit.

Solution: Customers enter at rate λ and served customes leave at rate μ as long as there are customers in the queue. So, in the long run the number of served customers leaving per time unit is $\mu(1 - P_0)$. So the long run fraction of customers that will enter service is $\frac{\mu(1-P_0)}{\lambda} = \frac{(1-e^{-\lambda/\mu})}{\lambda/\mu}$.

c) What is the expected time a customer is in the system and what is the expected time a customer is in the queue? (4p)

Solution: Since customers leave at rate μ, independently of whether they are in service or not, the expected time they are in the system is $1/\mu$. The expected time an arriving customer is in service is $1/\mu$ times the probability that a customer gets into service. This latter probability is computed in part *b*). So, the expected time an arriving customer spends in service is $\frac{(1-e^{-\lambda/\mu})}{\lambda}$, and the expected time a customer spends in the queue is $\frac{1}{\mu} - \frac{(1-e^{-\lambda/\mu})}{\lambda}$.

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion. Let $\alpha > 0$ be a strictly positive constant and let $\{V(t), t \geq 0\}$ be an Ornstein Uhlenbeck Process, defined through $V(t) = e^{-\alpha t/2} B(e^{\alpha t})$ for $t \geq 0$.

a) Compute $\mathbb{E}[V(t)]$ for $t \geq 0$. (2p)

Solution: $\mathbb{E}[V(t)] = \mathbb{E}[e^{-\alpha t/2} B(e^{\alpha t})] = e^{-\alpha t/2}\mathbb{E}[B(e^{\alpha t})] = 0$ by the definition of a standard Brownian Motion.

b) Compute the covariance $Cov[V(t), V(t+s)]$ for $t > 0$ and $s > 0$. (4p)

Solution: $Cov[V(t), V(t+s)] = \mathbb{E}[V(t)V(t+s)] - \mathbb{E}[V(t)]\mathbb{E}[V(t+s)] = \mathbb{E}[V(t)V(t+s)]$, where we have used part a) for the final equality. Further

$$\begin{aligned}\mathbb{E}[V(t)V(t+s)] &= e^{-\alpha(t+(t+s))/2}\mathbb{E}[B(e^{\alpha t})B(e^{\alpha(t+s)})] \\ &= e^{-\alpha(t+s/2)}\mathbb{E}[B(e^{\alpha t})([B(e^{\alpha(t+s)}) - B(e^{\alpha t})] + B(e^{\alpha t}))] \\ &= e^{-\alpha(t+s/2)}\left(\mathbb{E}[B(e^{\alpha t})]\mathbb{E}[B(e^{\alpha(t+s)}) - B(e^{\alpha t})] + \mathbb{E}[(B(e^{\alpha t}))^2]\right).\end{aligned}$$

Here we have used the independent increment property of the Brownian motion in the last equation. Now note that $\mathbb{E}[B(e^{\alpha(t+s)}) - B(e^{\alpha t})] = 0$ and $\mathbb{E}[(B(e^{\alpha t}))^2] = Var(B(e^{\alpha t})) = e^{\alpha t}$, which gives that $\mathbb{E}[V(t)V(t+s)] = e^{-\alpha s/2}$.

c) Provide the distribution of $V(1)$. (2p)

Solution: By the definition of Brownian motion $V(1) = e^{-\alpha/2}B(e^{\alpha})$ has a Normal distribution with expectation $e^{-\alpha/2} \times 0 = 0$ and Variance $(e^{-\alpha/2})^2 \times e^{\alpha} = 1$.

d) Let $x > 0$ and $t > 1$, compute $\mathbb{P}\left(\min_{1 \leq s \leq t} V(s) > 0 | V(1) = x\right)$. (4p)

Solution: Note that $V(s)$ has the same sign as $B(e^{\alpha s})$. So, we are interested in

$$\mathbb{P}\left(\min_{1\leq s\leq t} B(e^{\alpha s}) > 0 | V(1) = x\right) = \mathbb{P}\left(\min_{1\leq s\leq t} B(e^{\alpha s}) > 0 | B(e^{\alpha}) = e^{\alpha/2}x\right).$$

By symmetry and translation invariance of the Brownian Motion the Right Hand Side equals

$$\mathbb{P}\left(\max_{1\leq s\leq t} B(e^{\alpha s}) < e^{\alpha/2}x | B(e^{\alpha}) = 0\right) = \mathbb{P}\left(\max_{0\leq s\leq e^{\alpha t}-e^{\alpha}} B(s) < e^{\alpha/2}x | B(0) = 0\right).$$

Now using the cheat-cheet and noting that $\mathbb{P}\left(\max_{0\leq s\leq e^{\alpha t}-e^{\alpha}} B(s) < e^{\alpha/2}x | B(0) = 0\right) = \mathbb{P}\left(T_{e^{\alpha/2}x} > e^{\alpha t} - e^{\alpha}\right)$, we obtain that $\mathbb{P}\left(\min_{1\leq s\leq t} V(s) > 0 | V(1) = x\right) = 1 - 2\mathbb{P}(B(e^{\alpha t} - e^{\alpha}) > e^{\alpha/2}x)$.

Problem 5: Simulation

Consider $\{N(t); t \geq 0\}$, a homogeneous Poisson Process on $[0, \infty)$ with rate λ. Note that the number of points in this Poisson process on the interval $[0, t]$ is denoted by $N(t)$. For $k \in \{1, 2, \cdots\}$, let U_k be independent and identically distributed random variables which are uniformly distributed on the interval $[0, 1]$. And assume that U_k is easy to simulate.

a) Show in detail that the position of the first point of the Poisson process $\{N(t); t \geq 0\}$ can be obtained through simulating U_1 and then computing $-\log[U_1]/\lambda$. (4p)

Solution: The first point of Poisson process has density $\lambda e^{-\lambda t}$ for $t \geq 0$. So the probability that this point is less than t is $1 - e^{-\lambda t}$. Similarly

$$\mathbb{P}(-\log[U_1]/\lambda \leq t) = \mathbb{P}(\log[U_1] \geq -\lambda t) = \mathbb{P}(U_1 \geq e^{-\lambda t}) = 1 - e^{-\lambda t},$$

where we have used that U_1 is uniform on (0,1).

b) Provide a method to simulate $N(T)$ for given $T > 0$, where you generate the uniformly distributed random variable U_2 and no other random variables. (4p)

Solution: We know that $N(T)$ is Poisson distributed with parameter λT. $N(T)$ is distributed as $\min\{K \geq 0; \sum_{k=0}^{K} \frac{(\lambda T)^k}{k!} e^{-\lambda T} > U_2\}$. That is, given U_2, $N(T)$ is given by the minimal K for which $\sum_{k=0}^{K} \frac{(\lambda T)^k}{k!} e^{-\lambda T} > U_2$. This can be seen by observing that $\mathbb{P}(N(T) \leq n) = \sum_{k=0}^{n} \frac{(\lambda T)^k}{k!} e^{-\lambda T}$ and

$$\mathbb{P}(\min\{K \geq 0; \sum_{k=0}^{K} \frac{(\lambda T)^k}{k!} e^{-\lambda T} > U_2\} \leq n) = \mathbb{P}(\sum_{k=0}^{n} \frac{(\lambda T)^k}{k!} e^{-\lambda T} > U_2) = \sum_{k=0}^{n} \frac{(\lambda T)^k}{k!} e^{-\lambda T},$$

Since U_2 is uniformly distributed

c) Let S_n be the location of the n-th point of the process $\{N(t); t \geq 0\}$. That is, $S_n = \min\{t \geq 0; N(t) = n\}$. Describe how you would use the rejection method to simulate S_n, using the density function $g(x) = (\lambda/n)e^{-(\lambda/n)x}$ as "trial density". (4p)

Solution: The position of the n-th point has Gamma distribution with parameters λ and n. That is, the density of the n-th point is $f(x) = \frac{\lambda^n}{(n-1)!} x^{n-1} e^{-\lambda x}$ (see cheat-sheet). This gives that $f(x)/g(x) = \frac{n\lambda^{n-1} x^{n-1}}{(n-1)!} e^{-\lambda(1-1/n)x}$. The derivative of this expression with respect to x is given by

$$\frac{n\lambda^{n-1} x^{n-2}}{(n-2)!} e^{-\lambda(1-1/n)x} - \frac{\lambda^n x^{n-1}}{(n-2)!} e^{-\lambda(1-1/n)x} = (n - \lambda x) \frac{\lambda^{n-1} x^{n-2}}{(n-2)!} e^{-\lambda(1-1/n)x},$$

which has a 0 at $x = n/\lambda$. Note that $f(0)/g(0) = 0$ and $f(x)/g(x) \to 0$ as $n \to \infty$, while $f(n/\lambda)/g(n/\lambda) = \frac{n^n}{(n-1)!} e^{-(n-1)}$. So $f(x)/g(x)$ takes its maximum in n/λ and this maximum is $M = \frac{n^n}{(n-1)!} e^{-(n-1)}$. Now simulate a value y from the distribution $g(\dot{)}$ (perhaps using part a)) and simulate a random variable U_3. If $U_3 \leq f(y)/(Mg(y))$ then keep y, otherwise repeat the procedure.