Solutions Stochastic Processes and Simulation II, August 21, 2019

Problem 1: Poisson Processes

Let $\{N_R(t), t \geq 0\}$ be a homogeneous Poisson Process of red points on $(0, \infty)$ with rate $\lambda_R > 0$ and let $\{N_B(t), t \geq 0\}$ be a homogeneous Poisson Process of blue points on $(0, \infty)$ with rate $\lambda_B > 0$. The two Poisson processes are independent of each other. Let

$$\{N(t), t \geq 0\} = \{N_R(t) + N_B(t), t \geq 0\}$$

be the point process which contains the points of both $\{N_R(t), t \geq 0\}$ and $\{N_B(t), t \geq 0\}$.

For $i \in \mathbb{N}_0$ define

$$S_i = \min\{t \geq 0; N(t) \geq i\},$$

That is, $S_0 = 0$ and the points of $\{N(t), t \geq 0\}$ are denoted by $\{S_i, i \in \mathbb{N}\}$, satisfying

$$0 = S_0 < S_1 < S_2 < S_3 < \cdots .$$

a) Provide the distribution of S_1 and the probability that the point at S_1 is red? That is, Compute

$$\mathbb{P}(S_1 \leq t) \qquad \text{for } t \geq 0$$

and

$$\mathbb{P}\left(\min\{t \geq 0; N_R(t) = 1\} < \min\{t \geq 0; N_B(t) = 1\}\right).$$

Solution: $\mathbb{P}(S_1 \leq t) = 1 - \mathbb{P}(N_R(t) + N_B(t) = 0) = 1 - \mathbb{P}(N_R(t) = 0, N_B(t) = 0)$, but $N_R(t)$ and $N_B(t)$ are independent. So, $\mathbb{P}(S_1 \leq t) = 1 - \mathbb{P}(N_R(t) = 0)\mathbb{P}(N_B(t) = 0)$, which by the definition of a Poisson process equals $1 - e^{-\lambda_R t} e^{-\lambda_B t}$.

Similarly, let $f_R(t) = 1 - e^{-\lambda_R t}$ be the density of an exponential distribution with expectation $1/\lambda_R$, which is also the density of the first red point by (one of) the definitions of a Poisson process. We then obtain that $\mathbb{P}\left(\min\{t \geq 0; N_R(t) = 1\} < \min\{t \geq 0; N_B(t) = 1\}\right)$ equals $\int_0^\infty f_R(t)\mathbb{P}(N_B(t) = 0)dt = \int_0^\infty 1 - e^{-\lambda_R t} e^{-\lambda_B t} dt = \frac{1}{\lambda_B} - \frac{1}{\lambda_B + \lambda_R} = \frac{\lambda_R}{\lambda_B + \lambda_R}$.

The answer also directly follows by using that if one label the points of a Poisson process independently (in this case red and blue) then for each label, the points of that label form a Poisson process itself and the Poisson processes of different labels are independent. The density of the Poisson process with label R is the density of the original Poisson process times p_R the probability of assigning label R to a point. In our case we can see that if we have a (not labelled) Poisson process with rate $\lambda_R + \lambda_B$ and then make points red with probability $p_R = \frac{\lambda_R}{\lambda_B + \lambda_R}$ and blue otherwise, we obtain a red Poisson process with rate λ_R and an independent blue Poisson process with intensity λ_B. By the definition of a Poisson process it then follows that S_1 is exponentially distributed with rate $\lambda_B + \lambda_R$ and the first point is red with probability p_R.

Let $T > 0$ be a constant.

b) Assume that $N(T) = n$, where $n \in \mathbb{N}$. What is the distribution of $N_R(T)$? That is, provide

$$\mathbb{P}(N_R(T) = k | N_R(T) + N_B(T) = n) \qquad \text{for } n \in \mathbb{N}.$$

Solution: Using the independent labeling argument of part a), we obtain that each of the n points is independently red with probability $p_R = \frac{\lambda_R}{\lambda_B + \lambda_R}$ and therefore $N_R(T)$ conditioned on $N(T) = n$ is binomially distributed with parameters n and p_R. That is,

$$\mathbb{P}(N_R(T) = k | N_R(T) + N_B(T) = n) = \binom{n}{k} (p_R)^k (1 - p_R)^{n-k}$$

for $0 \leq k \leq n$, while the probability is 0 otherwise.

c) Compute $\mathbb{E}\left[\sum_{i=1}^{N(T)} S_i\right]$. Recall $\sum_{i=1}^{0} S_i = 0$ by definition.

Solution:

Use the order statistic property we obtain

$$\mathbb{E}\left[\sum_{i=1}^{N(T)} S_i\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N(T)} S_i | N(T)\right]\right] = \mathbb{E}\left[\mathbb{E}\left[\sum_{i=1}^{N(T)} X_i | N(T)\right]\right],$$

where the X_i are independent and identically distributed random variables which are uniform on $(0, T)$ (and therefore have expectation $T/2$). So,

$$\mathbb{E}\left[\sum_{i=1}^{N(T)} S_i\right] = \mathbb{E}\left[N(T) \times T/2\right] = (\lambda_R + \lambda_B) T^2/2.$$

Problem 2: Renewal Theory

Let $\{N(t), t \geq 0\}$ be a renewal process, with interarrival distribution function $F(t)$ and density function $\frac{d}{dt}F(t) = f(t)$. Define $m(t) = \mathbb{E}[N(t)]$ for $t \geq 0$.

a) Justify the renewal equation. That is, show that for $t \geq 0$,

$$m(t) = F(t) + \int_0^t m(t-x)f(x)dx.$$

Solution: Condition on the first arrival (say S_1) then observe

$$\mathbb{E}[N(t)] = \mathbb{E}[\mathbb{E}[N(t)|S_1]] = \int_0^t \mathbb{E}[N(t)|S_1 = x]\mathbb{P}(S_1 \in dx) + \int_t^\infty \mathbb{E}[N(t)|S_1 = x]\mathbb{P}(S_1 \in dx).$$

Observe that for $x > t$, we have that $\mathbb{E}[N(t)|S_1 = x] = 0$, because the first arrival is after time t. Furthermore for $x < t$ we have that $\mathbb{E}[N(t)|S_1 = x] = 1 + \mathbb{E}[N(t-s)]$ because the system "renews" at the time of the first arrival. So

$$\mathbb{E}[N(t)] = \int_0^t (1 + \mathbb{E}[N(t-x)])f(x)dx = F(t) + \int_0^t m(t-x)f(x)dx$$

as desired.

b) Assume that the interarrival times are uniformly distributed on $(0,1)$, i.e.

$$f(t) = \begin{cases} 1 & \text{for } t \in (0,1) \\ 0 & \text{otherwise} \end{cases}.$$

Show that for $t \in [0,2]$, $m(t)$ is given by

$$m(t) = \begin{cases} e^t - 1 & \text{for } t \in (0,1] \\ e^t - 1 - (t-1)e^{t-1} & \text{for } t \in (1,2] \end{cases}.$$

Solution: For $t \in (0,1]$ fill in the proposed solution in the renewal equation. The Left Hand Side is $e^t - 1$, while the Right Hand Side is

$$t + \int_0^t (e^{t-x} - 1) \times 1 dx = e^t - 1,$$

as desired. For $t \in (0,2]$, the Right Hand Side of the renewal equation becomes

$$1 + \int_0^1 m(t-x) \times 1 dx = 1 + \int_{t-1}^t m(x)dx = 1 + \int_{t-1}^1 m(x)dx + \int_1^t m(x)dx.$$

Filling in the suggestion for $m(t)$ then gives that this equals

$$1 + \int_{t-1}^1 (e^x - 1)dx + \int_1^t e^x - 1 - (x-1)e^{x-1}dx = 1 + \int_{t-1}^t (e^x - 1)dx - \int_1^t (x-1)e^{x-1}dx.$$

The first integral is equal to

$$e^t - e^{t-1} - 1,$$

while the second integral is (by partial integration)

$$\int_0^{t-1} xe^x dx = xe^x|_{x=0}^{t-1} - \int_0^{t-1} e^x dx = (t-1)e^{t-1} - (e^{t-1} - 1) = (t-2)e^{t-1} + 1.$$

So the Right Hand Side of the renewal equation is given by

$$1 + (e^t - e^{t-1} - 1) - ((t-2)e^{t-1} + 1) = e^t - 1 - (t-1)e^{t-1},$$

as desired.

c) Let $U_1, U_2, \cdots$ be independent random variables all uniformly distributed on $(0,1)$. Define

$$N = \min\{n \in \mathbb{N}; \sum_{i=1}^n U_i \geq 2\}.$$

Compute $\mathbb{E}[N]$.

Solution: by definition $N = N(2) + 1$ so $E[N] = m(2) + 1 = e^2 - e$.

Problem 3: Queueing Theory

Consider an $M/G/1$ queue in which customers arrive according to a Poisson Process with rate λ, and customers have independent workloads which are distributed as the random variable S. Let $m_1 = \mathbb{E}[S] < \infty$ be the expected time a customer needs service and $m_2 = \mathbb{E}[S^2] < \infty$ be the second moment of this workload.

a) Provide a necessary and sufficient condition on λ for the queue length not to go to infinity?

Solution: It is enough that (in expectation) less than 1 customers arriving during a service. So we require $\lambda\mathbb{E}[S] < 1$.

For part b) and c) assume that the condition of part a) is satisfied. For $n \in \mathbb{N}$, let Y_n be the number of new customers arriving during the service period of the n-th customer and let X_n be the number of customers that the n-th departing customer leaves behind in the system.

b) Argue that for $n \in \mathbb{N}$

$$X_{n+1} = X_n - 1 + Y_{n+1} + \mathbb{1}(X_n = 0), \tag{1}$$

where $\mathbb{1}(X_n = 0) = \begin{cases} 1 & \text{if } X_n = 0 \\ 0 & \text{otherwise} \end{cases}$.

Solution: If the n-th departing customer leaves 0 customers behind, then the first customer to arrive after the departure of the n-th departing customer is the $n+1$-st customer to depart and only the Y_{n+1} customers will be there when the $n+1$-st customer departs. So if $X_n = 0$, then $X_{n+1} = Y_{n+1}$, which is consisten with (**??**).

If $X_n \neq 0$, then the first customer in the queue at the departure of n-th departing customer is the $n+1$-st customer to depart. All $X_n - 1$ other customers will still be in the queue at the $n+1$-st departure and in addition the Y_{n+1} customers which arrive between the n-th and $n+1$-st departure will add to X_{n+1}. So in this case $X_{n+1} = (X_n - 1) + Y_n$, as desired.

c) Compute the long run fraction of customers that depart without leaving anybody in the system and compute the expected number of customers that the n-th departing customers leaves behind in the system for $n \to \infty$.

Hint: As intermediate steps, compute

$$\lim_{n\to\infty} \mathbb{P}(X_n = 0) = \lim_{n\to\infty} \mathbb{E}[\mathbb{1}(X_n = 0)]$$

by taking the expectations on both sides of equation (**??**) and compute $\lim_{n\to\infty} \mathbb{E}[X_n]$ by taking expectations of the squares of both sides of equation (**??**).

Solution: Follow the hint

$$\lim_{n\to\infty} \mathbb{E}[X_{n+1}] = \lim_{n\to\infty} \mathbb{E}[X_n] - 1 + \lim_{n\to\infty} \mathbb{E}[Y_{n+1}] + \lim_{n\to\infty} \mathbb{E}[\mathbb{1}(X_n = 0)].$$

Now observe that

$$\lim_{n\to\infty} \mathbb{E}[X_{n+1}] = \lim_{n\to\infty} \mathbb{E}[X_n] \qquad \text{and} \qquad \mathbb{E}[Y_{n+1}] = \lambda m_1 \quad \text{for all } n.$$

So, we obtain $\lim_{n\to\infty} \mathbb{E}[\mathbb{1}(X_n = 0)] = 1 - \lambda m_1$. This implies that the long run fraction of customers that depart without leaving anybody in the system is $1 - \lambda m_1$

Taking the square of equation (**??**). we obtain

$$\begin{aligned}\mathbb{E}[X_{n+1}])^2 &= \mathbb{E}[(X_n)^2] + 1 + \mathbb{E}[(Y_{n+1})^2] + \mathbb{E}[(\mathbb{1}(X_n = 0))^2] \\ &- 2\mathbb{E}[X_n] + 2\mathbb{E}[X_n Y_n] + 2\mathbb{E}[X_n \mathbb{1}(X_n = 0)] - 2\mathbb{E}[Y_n] - 2\mathbb{E}[\mathbb{1}(X_n = 0)] + 2\mathbb{E}[Y_n \mathbb{1}(X_n = 0)].\end{aligned}$$

Now observe that $\mathbb{E}[(\mathbb{1}(X_n = 0))^2] = \mathbb{E}[\mathbb{1}(X_n = 0))]$ and that Y_n is independent of X_n. Also, $\mathbb{E}[Y_n] = \lambda m_1$ and for S_{n+1} the workload of the $n+1$-st customer,

$$\mathbb{E}[(Y_n)^2] = \mathbb{E}[\mathbb{E}[(Y_n)^2 | S_n]] = \mathbb{E}[\lambda S_n + (\lambda S_n)^2)] = \lambda m_1 + \lambda^2 m_2.$$

Furthermore, $\mathbb{E}[X_n \mathbb{1}(X_n = 0)] = 0$ and we obtain

$$\begin{aligned}\mathbb{E}[X_{n+1}])^2 &= \mathbb{E}[(X_n)^2] + 1 + \lambda m_1 + \lambda^2 m_2 + \mathbb{E}[\mathbb{1}(X_n = 0)] \\ &- 2\mathbb{E}[X_n] + 2\mathbb{E}[X_n]\lambda m_1 - 2\lambda m_1 - 2\mathbb{E}[\mathbb{1}(X_n = 0)] + 2\lambda m_1 \mathbb{E}[\mathbb{1}(X_n = 0)].\end{aligned}$$

taking limits, substracting $\lim_{n\to\infty} \mathbb{E}[(X_n)^2]$ from both sides and filling in

$$\lim_{n\to\infty} \mathbb{E}[\mathbb{1}(X_n = 0)] = 1 - \lambda m_1,$$

we then obtain

$$\begin{aligned}0 &= 1 - \lambda m_1 + \lambda^2 m_2 - (1 - \lambda m_1) - 2(1 - \lambda m_1) \lim_{n\to\infty} \mathbb{E}[X_n] + 2\lambda m_1 (1 - \lambda m_1) \\ &= \lambda^2 m_2 + 2\lambda m_1 (1 - \lambda m_1) - 2(1 - \lambda m_1) \lim_{n\to\infty} \mathbb{E}[X_n].\end{aligned}$$

So,

$$\lim_{n\to\infty} \mathbb{E}[X_n] = \frac{\lambda^2 m_2}{2(1 - \lambda m_1)} + \lambda m_1.$$

Problem 4: Brownian Motion and Stationary Processes

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and for $t > 0$, let

$$M(t) := \max_{0 \leq s \leq t} B(s),$$

be the maximum of the Brownian motion up to time t. Here we assume that $B(0) = 0$ and that the variance parameter $\sigma^2 = 1$ is part of the definition of a standard Brownian motion.

Let μ be a constant and define the Brownian motion with drift process

$$\{B_\mu(t), t \geq 0\} = \{B(t) + \mu t, t \geq 0\}.$$

a) Provide the distribution of $B_\mu(t)$ for $t > 0$.

Solution: $B_\mu(t) = B(t) + \mu t$ We know from the definition of a Brownian motion that $B(t)$ is Normal distributed with mean 0 and variance t. Therefore, $B_\mu(t) = B(t) + \mu t$ is Normal distributed with mean μt and variance t.

b) For $0 < s < t$ and constants a and b compute

$$\mathbb{P}(B_\mu(s) \leq a | B_\mu(t) = b),$$

and show that this expression does not depend on μ.

Solution:

$$\mathbb{P}(B_\mu(s) \leq a | B_\mu(t) = b) = \mathbb{P}(B(s) + \mu s \leq a | B(t) + \mu t = b) = \mathbb{P}(B(s) \leq a - \mu s | B(t) = b - \mu t).$$

Using what we know about the Brownian Bridge (see cheat sheet) we obtain that conditioned on $B(t) = b - \mu t$, $B(s)$ is Normal distributed with mean $(b - \mu t)(s/t)$ and variance $s(t - s)/t$. So,

$$\begin{aligned}
\mathbb{P}(B_\mu(s) \leq a | B_\mu(t) = b) &= \mathbb{P}(B(s) \leq a - \mu s | B(t) = b - \mu t) \\
&= \int_{-\infty}^{a - \mu s} \frac{1}{\sqrt{2\pi s(t-s)/t}} \exp\left[-\frac{(x - (b - \mu t)(s/t))^2}{2s(t-s)/t}\right] dx.
\end{aligned}$$

Making the change of variables $y = x + \mu s$, we then obtain

$$\mathbb{P}(B_\mu(s) \leq a | B_\mu(t) = b) = \int_{-\infty}^{a} \frac{1}{\sqrt{2\pi s(t-s)/t}} \exp\left[-\frac{(y - bs/t)^2}{2s(t-s)/t}\right] dx.$$

Which is to say that conditioned on $B_\mu(t) = b$, $B_\mu(s)$ is Normal distributed with mean bs/t and variance $s(t - s)/t$ (which is independent of μ).

c) Consider again the Brownian Motion without drift. For a constant $a > 0$ and given time $t > 0$, compute

$$\mathbb{P}(M(t) < a|B(t) < 0).$$

Solution: Use the reflection principe and let T_a be the first hitting time of a.

$$\begin{aligned}
\mathbb{P}(M(t) < a|B(t) < 0) &= 1 - \mathbb{P}(M(t) > a|B(t) < 0) \\
&= 1 - \frac{\mathbb{P}(T_a < t, B(t) < 0)}{\mathbb{P}(B(t) < 0)} \\
&= 1 - \frac{\mathbb{P}(T_a < t, B(t) < 0)}{1/2} \\
&= 1 - 2\mathbb{P}(T_a < t, B(t) - B(T_a) < -a) \\
&= 1 - 2\mathbb{P}(T_a < t, B(t) - B(T_a) > a) \\
&= 1 - 2\mathbb{P}(T_a < t, B(t) > 2a) \\
&= 1 - 2\mathbb{P}(B(t) > 2a)
\end{aligned}$$

Problem 5: Simulation

Let $\{N(t), t \geq 0\}$ be an nonhomogeneous Poisson Process with strictly positive and finite intensity function $\lambda(t)$, $t \geq 0$. Define

$$m(t) = \int_0^t \lambda(s)ds$$

and assume that $m(\infty) = \infty$.

Because the intensity function is strictly positive and finite, $m(t)$ is continuous and strictly increasing to ∞ in t. This implies that the inverse function $m^{-1}(\cdot)$ satisfying

$$m(m^{-1}(t)) = m^{-1}(m(t)) = t \qquad \text{for all } t \geq 0,$$

is well defined.

Define

$$X_i := \min\{t \geq 0; N(t) \geq i\} \qquad \text{for } i \in \mathbb{N}.$$

That is, X_i is the position of the i-th point of the nonhomogeneous Poisson Process. For completeness define $X_0 = 0$.

a) Provide the definition of an nonhomenous Poisson Process.

Solution: A non-decreasing non-negative integer valued process $\{N(t); t \geq 0\}$ is a nonhomogeneous Poisson Process with intensity function $\lambda(t) \in [0, \infty)$, $t \geq 0$ if

- $N(0) = 0$
- The process has independent increments
- $\mathbb{P}(N(t+h) - N(t) = 1) = \lambda(t)h + o(h)$
- $\mathbb{P}(N(t+h) - N(t) > 1) = o(h)$

The mean value function is defined by $m(t) = \int_0^t \lambda(s)ds$

b) Show that $m(X_1)$ is exponentially distributed with expectation 1. That is, show that

$$\mathbb{P}(m(X_1) > t) = e^{-t} \qquad \text{for } t \geq 0.$$

Solution: Note that $\mathbb{P}(m(X_1) > t) = \mathbb{P}(X_1 > m^{-1}(t))$, which is the probability that the nonhomogeneous Poisson process contains no points up to $m^{-1}(t)$, which by the definition of a nonhomogeneous Poisson process is equal to

$$e^{-\int_0^{m^{-1}(t)} \lambda(s)ds} = e^{-m(m^{-1}(t))} = e^{-t}$$

as desired.

c) Show that for $i \in \mathbb{N}_0$ the random variables $m(X_{i+1}) - m(X_i)$ are independent and identically distributed all exponentially distributed with expectation 1.

Remark: The above result implies that we may simulate $\{N(t), t \geq 0\}$ by first simulating the points $Y_1, Y_2, \cdots$ of a homogeneous Poisson Process with intensity 1 and then set $X_i = m^{-1}(Y_i)$ for $i \in \mathbb{N}$.

Solution: Note that if we know $m(X_i)$ then we know X_i and vice-versa, because m is a one-to-one function. So, for every $x > 0$ and $i \in \mathbb{N}$ we have

$$\mathbb{P}(m(X_{i+1}) - m(X_i) > x | m(X_j) - m(X_{j-1}), j = 1, \cdots, i) = \mathbb{P}(m(X_{i+1}) - m(X_i) > x | X_1, \cdots, X_i).$$

By the definition of a nonhomogeneous Poisson Process and because $X_j < X_i$ for $j < i$, we obtain that this is equal to $\mathbb{P}(m(X_{i+1}) - m(X_i) > x | X_i)$.

We now show that the above probability is exponentially distributed with expectation 1 and independent of X_i.

$$\mathbb{P}(m(X_{i+1}) - m(X_i) > x | X_i) = \mathbb{P}(m(X_{i+1}) > x + m(X_i) | X_i) = \mathbb{P}(X_{i+1} > m^{-1}(x + m(X_i)) | X_i),$$

which is the probability that there is no point between X_i and $m^{-1}(x + m(X_i))$. which is by the definition of a non-homogeneous Poisson Process given by

$$\exp\left[-\int_{X_i}^{m^{-1}(x+m(X_i))} \lambda(t) dt\right] = \exp\left[-\{m\left(m^{-1}\left(x + m(X_i)\right)\right) - m(X_i)\}\right] = e^{-\{x+m(X_i)-m(X_i)\}} = e^{-x},$$

as desired.