Solutions Stochastic Processes and Simulation II, June 1, 2020

Problem 1: Poisson Processes

To describe the spread of an infectious disease, such as Covid-19, mathematical modellers often use SEIR models. In those models people can be Susceptible, Exposed (i.e. infected, but not yet infectious), Infectious or Removed (which might mean recovered and eternally immune or dead). At time $t = 0$ there is one infectious person, patient 0, who just turned Infectious (so just before $t = 0$ patient 0 was exposed). All other individuals in the population are susceptible at time $t = 0$. If an infectious person makes an "infectious contact" (more about this below) with a susceptible person, the susceptible one immediately becomes exposed and stays so for a random time, which is distributed as X: the exposed period. After the exposed period the infected person becomes infectious and stays so for a random time distributed as Y: the infectious period. After the infectious period the infected person is removed forever. The durations of the exposed and infectious periods of different people are independent.

An infectious individual makes "infectious contacts" during his or her infectious period according to a homogeneous Poisson Process with rate λ.

Set

$$\mathbb{E}[X] = \mu_X, \quad Var(X) = \sigma_X^2, \quad \mathbb{E}[Y] = \mu_Y \quad \text{and} \quad Var(Y) = \sigma_Y^2.$$

Let Y_0 be the random infectious period of patient 0 (which is distributed as Y) and let for given Y_0, the process $\{N_0(t); t \in [0, Y_0]\}$ be the Poisson process describing the "infectious contacts" made by patient 0.

a) Provide the mean and variance of the random number of "infectious contacts" made by patient 0. That is, provide the mean and variance of the random variable $N_0(Y_0)$. (6pt)

Hint: first compute the mean and variance of $N_0(Y_0)$, conditioned on Y_0 and use this to compute the unconditional mean and variance.

Solution: We follow the hint and we use that N_0 is a Poisson process on the interval $[0, Y_0]$. So conditioned on Y_0 we know that $N_0(Y_0)$ is Poisson distributed with expectation and variance both equal to λY_0. Using telescoping expectations:

$$\mathbb{E}[N_0(Y_0)] = \mathbb{E}[\mathbb{E}(N_0(Y_0)|Y_0)] = \mathbb{E}[\lambda Y_0] = \lambda\mu_Y$$

While

$$\begin{aligned}
Var[N_0(Y_0)] &= \mathbb{E}[(N_0(Y_0))^2] - (\mathbb{E}[N_0(Y_0)])^2 \\
&= \mathbb{E}[(N_0(Y_0))^2] - (\lambda\mu_Y)^2 \\
&= \mathbb{E}[\mathbb{E}[(N_0(Y_0))^2|Y_0]] - (\lambda\mu_Y)^2 \\
&= \mathbb{E}[Var(N_0(Y_0)|Y_0) + (\mathbb{E}[(N_0(Y_0))|Y_0])^2] - (\lambda\mu_Y)^2 \\
&= \mathbb{E}[\lambda Y_0 + (\lambda Y_0)^2] - (\lambda\mu_Y)^2 \\
&= \lambda\mu_Y + \lambda^2(\sigma_Y^2 + (\mu_Y)^2) - (\lambda\mu_Y)^2 \\
&= \lambda\mu_Y + \lambda^2\sigma_Y^2.
\end{aligned}$$

For part b) assume that $\mathbb{P}(Y=1)=1$, (so also $\mathbb{P}(Y_0=1)=1$) and that $\mathbb{P}(X \leq 1)=1$ and assume that all infectious contacts made by patient 0 are with susceptible individuals (and thus lead to infection).

b) Provide the distribution of the number of people infected by patient 0 which are still not infectious at time 1 (i.e. who are still exposed at time 1). (6pt)

Solution: Denote the number we are interested in by the random variable K and the total number of people infected by patient 0 by the random variable L. We first condition on L:

$$\mathbb{P}(K=k)=\sum_{\ell=0}^{\infty}\mathbb{P}(K=k|L=\ell)\mathbb{P}(L=\ell)=\sum_{\ell=k}^{\infty}\mathbb{P}(K=k|L=\ell)\frac{\lambda^{\ell}}{\ell!}e^{-\lambda},$$

where we use that $K \leq L$. The next step is to compute $\mathbb{P}(K=k|L=\ell)$. To use this we use the order statistic property and notice that the ℓ times of infections in the interval (0,1) are independent and uniformly distributed points on that interval. The probability that a person infected at time $x \in (0,1)$ is still exposed at time 1 is given by $\mathbb{P}(X>1-x)$. So the probability that a person who is infected at a uniform time in $(0,1)$ is still exposed at time 1 is given by

$$\int_0^1 \mathbb{P}(X>1-x)dx=\int_0^1 \mathbb{P}(X>x)dx=\mathbb{E}[X]=\mu_X.$$

So conditioned on $L=\ell$ the random variable K is binomially distributed with parameters ℓ and μ_X and therefore,

$$\begin{aligned}
\mathbb{P}(K=k) &= \sum_{\ell=k}^{\infty}\binom{\ell}{k}(\mu_X)^k(1-\mu_X)^{\ell-k}\frac{\lambda^{\ell}}{\ell!}e^{-\lambda} \\
&= \sum_{\ell=k}^{\infty}\frac{\ell!}{k!(\ell-k)!}(\lambda\mu_X)^k[\lambda(1-\mu_X)]^{\ell-k}\frac{1}{\ell!}e^{-\lambda} \\
&= \sum_{\ell=k}^{\infty}\frac{(\lambda\mu_X)^k}{k!}\frac{(\lambda(1-\mu_X))^{\ell-k}}{(\ell-k)!}e^{-\lambda} \\
&= \sum_{\ell=k}^{\infty}\frac{(\lambda\mu_X)^k}{k!}\frac{(\lambda(1-\mu_X))^{\ell-k}}{(\ell-k)!}e^{-\lambda\mu_x}e^{-\lambda(1-\mu_x)} \\
&= \frac{(\lambda\mu_X)^k}{k!}e^{-\lambda\mu_X}\left(\sum_{\ell=k}^{\infty}\frac{(\lambda(1-\mu_X))^{\ell-k}}{(\ell-k)!}e^{-\lambda(1-\mu_x)}\right) \\
&= \frac{(\lambda\mu_X)^k}{k!}e^{-\lambda\mu_X}.
\end{aligned}$$

Problem 2: Renewal Theory

Consider the following epidemic model describing the spread of an antibiotic resistant pathogen in an infinitely large intensive care unit (ICU). At time $t = 0$ there are no infectious people in the ICU. Throughout the process, infectious people are admitted at the ICU according to a homogeneous Poisson process with strictly positive rate α. As long as the number of infected people is less than the strictly positive integer N, each infectious person infects other people in the ICU independently at strictly positive constant rate λ. That is the total rate of going from k to $k+1$ infectious people is $\alpha + k \times \lambda$ ($k \in \{0, 1, 2, \cdots, N-1\}$), where α is the rate of importation of infectious people and $k \times \lambda$ the contribution of "within-ICU" transmissions. Infectious people stay infectious until there is a "clearance". As soon as there are N infectious people in the ward "clearance" takes place and the entire ICU is cleared of the pathogen and the number of infectious people goes instanteneously back to 0.

a) Argue that the expected time between two subsequent clearances of the ICU is

$$\sum_{k=0}^{N-1} \frac{1}{\alpha + k\lambda}.$$

(4pt)

Solution: for $k \in \{0, 1, \cdots N-1\}$ let T_k be the time to go from k to $k+1$ patients in the ICU. We know that T_k is exponenententially distributed with expectation $1/(\alpha + k\lambda)$ and that the time from a clearance until the ICU first has again N infectious people (the next clearance) is $\sum_{k=0}^{N-1} T_k$. The expectation of this random variable is given by

$$\mathbb{E}[\sum_{k=0}^{N-1} T_k] = \sum_{k=0}^{N-1} \mathbb{E}[T_k] = \sum_{k=0}^{N-1} \frac{1}{\alpha + k\lambda},$$

as desired.

b) What is the (almost sure) long-run average of the number of infectious people at the ICU? Note that if ugly sums appear in your answer, you do not have to evaluate them. (4pt)

Solution: Use Renewal reward theory and find the sum of the durations of times patients are infectious at the ward. A cycle is the period between "clearances". for $k \in \{0, 1, \cdots N-1\}$ there are k infectious patients in the system for duration T_k. So the cummulative time of infectious patients in the ward during a cycle is

$$\sum_{k=0}^{N-1} \frac{k}{\alpha + \lambda k}.$$

From part a) we know that the expected duration of a cycle is

$$\sum_{k=0}^{N-1} \frac{1}{\alpha + k\lambda}.$$

By renewal reward theory we then know that the long run average number of infectious patients in the ICU is

$$\sum_{k=0}^{N-1} \frac{k}{\alpha + \lambda k} / \sum_{k=0}^{N-1} \frac{1}{\alpha + k\lambda}.$$

c) In the long run, what is (almost surely) the fraction among all infectious people that have been at the ICU, that has been infected outside the ICU (those are the people who were infected when admitted at the ICU)? That is, if $N_A(t)$ is the number of people that are admitted while infectious up to time t and $N_T(t)$ is the total number of people that has been infectious at the ICU, then provide the almost sure limit of $N_A(t)/N_T(t)$ as $t \to \infty$. (4pt)

Solution: Let $A(t)$ be the number of patients who are admitted while infectious into the ICU up to time t. And let $B(t)$ be the total number of patients who have been infectious at the ICU (either through infection or through being admitted infectious).

We know that $A(t)/t \to \alpha$ almost surely as $t \to \infty$, because infectious patients are admitted at constant rate α no matter the state of the "system". We also know by renewal reward theory that $B(t)/t$ converges almost surely to the expected number of infected patients per cycle (i.e. N) divided by the expected duration of a cycle. So

$$\frac{A(t)}{B(t)} = \frac{A(t)/t}{B(t)/t}$$

converges almost surely to

$$\alpha/[N/\sum_{k=0}^{N-1} \frac{1}{\alpha + k\lambda}] = \frac{\alpha}{N} \sum_{k=0}^{N-1} \frac{1}{\alpha + k\lambda}.$$

Problem 3: Queueing Theory Consider the following queueing system with a fast server (Alice) and a slow server (Barbara). If a customer is served by Alice, the time needed for the service is exponentially distributed with parameter μ_A (i.e. with expectation $1/\mu_A$), while if a customer is serverd by Barbara, then the time needed for the service is exponentially distributed with parameter μ_B.

Customers arrive at the system according to a homogeneous Poisson Process with rate λ. If upon arrival of a new customer both Alice and Barbara are idle, the new customer will be served by Alice.

a) Provide a necessary and sufficient relationship between λ, μ_A and μ_B for the queue length not to go to infinity? (2pt)

Solution: For all but possibly a finite number of customers in the queue the arrival rate should be less than the departure rate. That is $\mu_A + \mu_B > \lambda$. Assume for the remainder of the problem that the condition of part a) is satisfied.

b) Provide an appropriate state space $\mathcal{S}$ in order to describe the queueing process as a Continuous Time Markov Chain. This Markov chain should contain (among other things) how many customers are in the system. (2pt)

Solution: A state space could have the following states.
State 0: 0 customers in the system,
State 1A: 1 customer in the system served by Alice
State 1B: 1 customer in the system served by Barbara
For $k \in \{2, 3, \cdots\}$, State k: k customers in the system.

c) Let $\{P_s\}_{s\in\mathcal{S}}$ be the stationary distribution of the Queueing system described in this problem. Provide the "balance equations" characterizing this stationary distribution. That is, provide relations beween the P_s $(s \in \mathcal{S})$, which are in theory enough to compute them all. Note you do not have to solve the balance equations. (4pt)

Solution: We get the following balance equations, where the left hand side is the rate of leaving a state, and the right hand side the rate of entering it.
(0) $\lambda P_0 = \mu_A P_{1A} + \mu_B P_{1B}$
(1A) $(\lambda + \mu_A)P_{1A} = \lambda P_0 + \mu_B P_2$
(1B) $(\lambda + \mu_B)P_{1B} = \mu_A P_2$
(2) $(\lambda + \mu_A + \mu_B)P_2 = (\mu_A + \mu_B)P_3 + \lambda(P_{1A} + P_{1B})$
(k) $(\lambda + \mu_A + \mu_B)P_k = (\mu_A + \mu_B)P_{k+1} + \lambda P_{k-1}$ for $k \in \{3, \cdots\}$

d) What is the long run fraction of customers that is served by Alice? Express your answer in terms of $\{P_s\}_{s\in\mathcal{S}}$. (4pt)

Solution: If a customer arrives when the system is in state 0 or 1B then he or she will be served by Alice. While if a customer arrives while the system is in state k for $k \geq 2$ then he or she is served by Alice if, when the customer gets to the first place in the queue, Alice is the first to finish her job which by the Markov property of the system occurs with probability $\mu_A/(\mu_A + \mu_B)$. Using the PASTA principle the probability that an arriving customer is served by Alice is given by

$$P_0 + P_{1B} + \frac{\mu_A}{\mu_A + \mu_B} \sum_{k=2}^{\infty} P_k.$$

An alternative is to assume that there is a clock which rings according to a Poisson Process with intensity μ_A. If Alice is working just before the clock rings she finishes her job at the ringing. This gives that Alices finishes jobs at rate $\mu_A(1 - P_0 - P_{1B})$, while (because the system does not explode jobs finish at total rate λ). So the fraction served by Alice is $\frac{\mu_A}{\lambda}(1 - P_0 - P_{1B})$. One can deduce from the balance equations that those numbers are indeed the same.

Problem 4: Brownian Motion and Stationary Processes

Let $\{X_1(t); t \geq 0\}$ and $\{X_2(t); t \geq 0\}$ be independent standard Brownian motions satisfying $X_1(0) = X_2(0) = 0$ and both having variance parameter 1. Consider the 2 dimension Brownian motion

$$\{\mathbf{X}(t); t \geq 0\} = \{(1 + X_1(t), 1 + X_2(t)); t \geq 0\}.$$

Note that this process starts in $(1, 1)$, i.e. $\mathbf{X}(0) = (1, 1)$.

Define for $i \in \{1, 2\}$ the random time $T_i = \inf\{t \geq 0; 1 + X_i(t) \leq 0\}$.

a) Show that the density functions of T_1 and T_2 are given by

$$f_{T_1}(t) = f_{T_2}(t) = \frac{1}{\sqrt{2\pi t^3}} e^{-1/(2t)} \quad \text{for } t \geq 0.$$

(3pt)

Solution: By definition $f_{T_1}(t) = f_{T_2}(t)$, while by symmetry arguments, T_1 is the first time a standard Brownian motion hits 1. Taking the derrivative of (10.6) on page 611 of the book. We obtain that

$$f_{T_1}(t) = \frac{d}{dt} \frac{2}{\sqrt{2\pi}} \int_{1/\sqrt{t}}^{\infty} e^{-y^2/2} dy = -\left(\frac{d}{dt} t^{-1/2}\right) \frac{2}{\sqrt{2\pi}} \left[e^{-y^2/2}\right]_{y=t^{-1/2}} = \frac{1}{\sqrt{2\pi t^3}} e^{1/(2t)}$$

Let T be the exit time of the positive quadrant. i.e.

$$T = \inf\{t \geq 0; \min(1 + X_1(t), 1 + X_2(t)) \leq 0\}.$$

b) Provide the distribution function of T, i.e. compute $\mathbb{P}(T \leq t)$ for $t \geq 0$. (3pt)

Solution: Define T_1 and T_2 as above. Then $T = \min(T_1, T_2)$ and

$$\mathbb{P}(T > t) = \mathbb{P}(T_1 > t)\mathbb{P}(T_2 > t).$$

By (10.7) on page 611 of the course book, this probability is equal to

$$\left(1 - \frac{2}{\sqrt{2\pi}} \int_{1/\sqrt{t}}^{\infty} e^{-y^2/2} dy\right)^2 = \left(\frac{2}{\sqrt{2\pi}} \int_0^{1/\sqrt{t}} e^{-y^2/2} dy\right)^2 = \frac{2}{\pi} \left(\int_0^{1/\sqrt{t}} e^{-y^2/2} dy\right)^2$$

c) Compute $\mathbb{P}(1+X_1(T)>x)$ for $x>0$. (6pt)

Hint: Use that

$$\mathbb{P}(1+X_1(T)>x)=\int_0^\infty f_{T_2}(t_2)\left(\int_0^\infty \mathbb{P}(1+X_1(T)>x|T_1=t_1,T_2=t_2)f_{T_1}(t_1)dt_1\right)dt_2.$$

Further note that $1+X_1(T)>x>0$ implies $T=T_2<T_1$. One can then proceed to first show that

$$\mathbb{P}(1+X_1(T)>x)=\int_0^\infty f_{T_2}(t_2)\mathbb{P}(1+X_1(t_2)>x,T_1>t_2)dt_2$$

and then use this for further computations.

Solution: Start with

$$\mathbb{P}(1+X_1(T)>x)=\int_0^\infty f_{T_2}(t_2)\left(\int_0^\infty \mathbb{P}(1+X_1(T)>x|T_1=t_1,T_2=t_2)f_{T_1}(t_1)dt_1\right)dt_2.$$

Then using the hint that $T=T_2<T_1$ we obtain that

$$\mathbb{P}(1+X_1(T)>x)=\int_0^\infty f_{T_2}(t_2)\left(\int_{t_2}^\infty \mathbb{P}(1+X_1(t_2)>x|T_1=t_1,T_2=t_2)f_{T_1}(t_1)dt_1\right)dt_2.$$

We then note that $X_1(t_2)$ is independent of the event $T_2=t_2$. So, the above is equal to

$$\int_0^\infty f_{T_2}(t_2)\left(\int_{t_2}^\infty \mathbb{P}(1+X_1(t_2)>x|T_1=t_1)f_{T_1}(t_1)dt_1\right)dt_2.$$

Observe that

$$\int_{t_2}^\infty \mathbb{P}(1+X_1(t_2)>x|T_1=t_1)f_{T_1}(t_1)dt_1=\mathbb{P}(1+X_1(t_2)>x,T_1>t_2),$$

which is equal to

$$\mathbb{P}(1+X_1(t_2)>x)-\mathbb{P}(1+X_1(t_2)>x,T_1\le t_2)=\mathbb{P}(X_1(t_2)>x-1)-\mathbb{P}(X_1(t_2)>x+1),$$

where we have used the reflection principle. This is equal to $\mathbb{P}(x-1<X_1(t_2)<x+1)$. So,

$$\begin{aligned}\mathbb{P}(1+X_1(T)>x) &= \int_0^\infty \frac{1}{\sqrt{2\pi(t_2)^3}}e^{-1/(2t_2)}\int_{x-1}^{x+1}\frac{1}{\sqrt{2\pi t_2}}e^{-y^2/(2t_2)}dydt_2\\ &= \int_{x-1}^{x+1}\int_0^\infty \frac{1}{2\pi t^2}e^{-(y^2+1)/(2t_2)}dt_2dy\end{aligned}$$

Integrating with respect to t_2 (noting that $\frac{d}{dt}e^{-(y^2+1)/(2t)}=\frac{(y^2+1)}{2(t)^2}e^{-(y^2+1)/(2t)}$) we obtain.

$$\mathbb{P}(1+X_1(T)>x)=\int_{x-1}^{x+1}\frac{1}{\pi(y^2+1)}dy.$$

Problem 5: Simulation

Let $\{Z(t), t \geq 0\}$ be a linear birth and death process. That is, $\{Z(t), t \geq 0\}$ is a continuous time Markov process on state space $\{0, 1, \cdots\}$ where the rate of going from k to $k+1$ is $\lambda \times k$ and the rate of going from k to $k-1$ is $\mu \times k$. Assume $Z(0) = 1$.

a) Argue that one can simulate this process by using two sequences of independent and identically distributed uniform random variables on $(0,1)$, say $U_1, U_2, \cdots$ and $V_1, V_2, \cdots$ as follows.

- Define the time of the first event $S_1 = -\frac{\log(U_1)}{\lambda+\mu}$. Let $Z(t) = 1$ for $t \in [0, S_1)$ and set $Z(S_1) = 2$ if $V_1 \leq \frac{\lambda}{\lambda+\mu}$ and $Z(S_1) = 0$ otherwise.
- For $k, n \in \{1, 2, \cdots\}$, if $Z(S_n) = k$, set $S_{n+1} = S_n + -\frac{\log(U_{n+1})}{(\lambda+\mu)k}$. Let $Z(t) = k$ for $t \in [S_n, S_{n+1})$ and $Z(S_{n+1}) = k+1$ if $V_{n+1} \leq \frac{\lambda}{\lambda+\mu}$ and $Z(S_{n+1}) = k-1$ otherwise.
- If $Z(t) = 0$ then $Z(s) = 0$ for all $s > t$. (4pt)

Solution: From the theory of Markov processes we know that we can model the process by having an exponential distribution of staying in state k with expectation $1/[k(\lambda+\mu)]$. So the time of stay in state k is by the inverse distribution method also distributed as $-\log[U]/[k(\lambda+\mu)]$.

The probability of going up is then $\frac{\lambda}{\lambda+\mu}$ and this event is independent of everything else in the process. Note that the way we use V gives the same probability of increasing by 1. When the state is 0 the rates of leaving are 0 and no further events will occur.

b) Argue that one can simulate the number of customers in the system in the first busy period of an M/M/1 queueing process $\{Q(t); t \geq 0\}$ with arrival rate λ and departure rate μ, by defining $\tau(t) = \int_0^t Z(s)ds$ and defining $\{Q(\tau(t)); t \geq 0\} = \{Z(t); t \geq 0\}$. That is, the queueing process and the the linear birth and death process are the same apart from a random time change. (4pt)

Solution: Many arguments are possible. E.g. Let $\{Z(t); t \geq 0\}$ be as in exercise a. By the definition of τ and $S_1, S_2, \cdots$, from part a we know that $\tau(S_1) = S_1$, and if $Z(S_k) > 0$, $\tau(S_{k+1}) - \tau(S_k) = Z(S_k)(S_{k+1} - S_k)$, which is exponentially distributed with parameter $\lambda + \mu$. So if we define $\{Q(\tau(t)); t \geq 0\}$ as $\{Z(t); t \geq 0\}$ then $\{Q(\tau(t)); t \geq 0\}$ leaves its current state if the argument $\tau(t)$ has made an exponential$(\lambda + \mu)$ increase (which will only be after a finite time if $Z(t) = Q(\tau(t)) > 0$), as desired for an $M/M/1$ queue (in the first busy period), while the probabilities of making up and down jumps are the same for the $M/M/1$ queue as for the linear birth and death process.

c) Argue that the expected number of births in a linear birth and death process (the expected number of "+1 jumps" in $\{Z(t), t \geq 0\}$) is equal to $\lambda\mathbb{E}$[duration of busy period in process $\{Q(t); t \geq 0\}$]. (4pt)

Solution: From part *b* we know that the number of "+1 jumps" in the busy period of an $M/M/1$ queue is the same as the number of "+1 jumps" in a linear birth and death process.

If the busy period is infinite, then infinitely many customers arrive in the busy period and the statement follows. If the expected busy period is finite, we know from Renewal reward theory that the expected number of arrivals during a cycle (idle period + busy period) divided by the expected duration of a cycle converges to the long run arrival rate, which is λ.

The number of arrivals in a cycle is 1 plus the number of arrivals during the busy period (just as we do not count the initial person in the birth and death process, we do not count the person who starts the busy period), while the duration of the busy period is the duration of a cycle - $1/\lambda$ (the duration of an idle period. Combining the above we obtain that

$$\lambda = \frac{1 + \mathbb{E}[[\text{number of arrivals during busy period}]}{\frac{1}{\lambda} + \mathbb{E}[\text{duration of busy period}]}.$$

Which implies

$$1 + \mathbb{E}[\text{number of arrivals during busy period}] = 1 + \lambda\mathbb{E}[\text{duration of busy period}].$$

as desired.