Exam Solutions. 05/01/2023

Exercise 1

(a) a_n is the number of partitions of n which have only 10, 20, 50, 100 and 500 as a summand. Thus we have that

$$f(x) = \frac{1}{1-x^{10}} \cdot \frac{1}{1-x^{20}} \cdot \frac{1}{1-x^{50}} \cdot \frac{1}{1-x^{100}} \cdot \frac{1}{1-x^{500}}$$

(b) We need to compute the coefficient of degree 100 of

$$(1+x^{10}+x^{20}+\cdots)(1+x^{20}+x^{40}+x^{60}+\cdots)\cdot$$
$$(1+x^{50}+x^{100}+\cdots)(1+x^{100}+\cdots)(1+x^{500}+\cdots)$$

this is the same as the coefficient of degree 10 of

$$(1+x+x^2+x^3+x^4+x^5+\cdots)(1+x^2+x^4+x^6+x^8+x^{10})$$
$$(1+x^5+x^{10})(1+x^{10})$$

this is $(I)+(II)+(III)+(IV)$ where

(I) is the coefficient of degree 10 of

$$(1+x+x^2+\cdots)(1+x^2+x^4+x^6+x^8+x^{10})$$

(II) is the coefficient of degree 5 of
$$(1+x+x^2+\cdots)(1+x^2+x^4\ldots)$$

(III) is the coefficient of degree 0 of
$$(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)$$
(IV) is the coefficient of deg 0 of $(1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^5\ldots$

$(III) = (IV) = 1$

(II): using that $\frac{1}{1-x} = 1+x+x^2+\cdots$ is the summation operator we have that

$$\frac{1}{1-x}(1+x^2+x^4+x^6+x^8+x^{10}) = 1+x+2x^2+2x^3+3x^4+3x^5+4x^6+4x^7+$$

$5x^8 + 5x^9 + 6x^{10}$

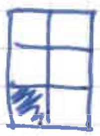
thus $(\mathrm{I}) = 6$

$(\mathrm{II}) = 3$

<u>Answer</u> $\boxed{a_{100} = 6+3+1+1 = 11}$

Exercise 2

(a) Up to swapping rows we can assume that C_2 is the chessboard

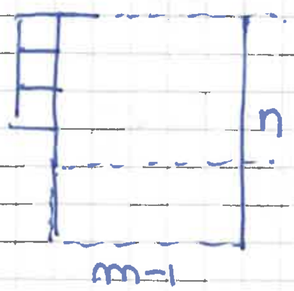


$$r(C_2, x) = r(\boxplus, x) + x\, r(\boxminus, x)$$

$$= (1 + 4x + 2x^2) + x(1 + 2x)$$

$$= \boxed{1 + 5x + 4x^2}$$

(b) Up to swapping rows we can assume that C_3 is



Let $C_{n,m}$ be the complete $n \times m$ chessboard

$$r(C_3^{n,m}, x) = r(C_2^{n,m}, x) + x\, r(C_{n-1,m-1}, x)$$

$$= r(C_1^{n,m}, x) + x\, r(C_{n-1,m-1}, x) + x\, r(C_{n-1,m-1}, x)$$

$$= r(C_{n,m-1}, x) + x\, r(C_{n-1,m-1}) + 2x\, r(C_{n-1,m-1})$$

$=$

$$= \sum_{k=0}^{m-1} \binom{n}{k}\binom{m-1}{k} k!\, x^k + 3\sum_{k=0}^{m-1}\binom{n-1}{k}\binom{m-1}{k} k!\, x^{k+1}$$

$$= \sum_{k=0}^{m-1}\left[\binom{n}{k}\binom{m-1}{k}k! + 3\binom{n-1}{k-1}\binom{m-1}{k-1}(k-1)!\right] x^k + 3\binom{n-1}{m-1}(m-1)!\,x^m$$

Exercise 3

We first find the general solution of the homogeneous problem

$$a_{n+2} + a_{n+1} + a_n = 0$$

the characteristic equation is

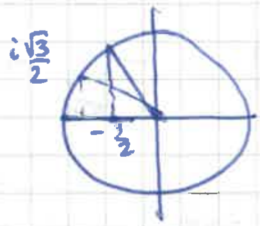
$$\lambda^2 + \lambda + 1 = 0$$

which has solution

$$\lambda_\pm = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$|\lambda_\pm| = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1 = \rho$$

$$\theta = \arccos\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$$



thus the general solution of the homogeneous problem is

$$a_n^{(h)} = A\cos\left(n\frac{2}{3}\pi\right) + B\sin\left(n\frac{2}{3}\pi\right) \quad +1$$

An educated guess for a particular solution of the non-homogeneous problem is

$$a_n^{(p)} = A_1 n + A_0 \quad +1$$

this is not a solution of the homogeneous problem, thus we can plug in and find coefficients

$$A_1(n+2) + A_0 + A_1(n+1) + A_0 + A_1(n) + A_0 = n$$

$$3A_1 n + 3A_1 + 3A_0 = n$$

$$3A_1 = 1 \qquad A_1 = \frac{1}{3}$$

$$3A_1 + 3A_0 = 0 \qquad A_0 = -\frac{1}{3}$$

$$a_n^{(p)} = \frac{1}{3}n - \frac{1}{3} \quad +1$$

The general solution of the problem is

$$a_n = A\cos\left(n\frac{2\pi}{3}\right) + B\sin\left(n\frac{2\pi}{3}\right) + \frac{1}{3}n - \frac{1}{3}$$

now we have to impose boundary conditions

$$0 = a_0 = A\cos(0) + B\sin(0) - \frac{1}{3} = A - \frac{1}{3}$$

$$1 = a_1 = A\cos\left(\frac{2\pi}{3}\right) + B\sin\left(\frac{2\pi}{3}\right) + \frac{1}{3} - \frac{1}{3}$$

$$= -\frac{1}{2}A + \frac{\sqrt{3}}{2}B$$

we get that $A = \frac{1}{3}$

$$1 = -\frac{1}{6} + \frac{\sqrt{3}}{2}B$$

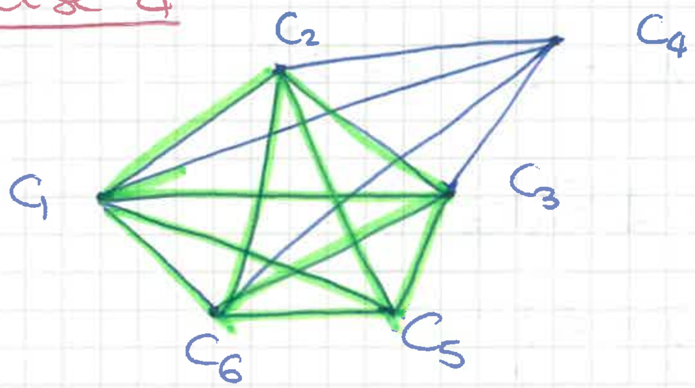
$$\frac{\sqrt{3}}{2}B = \frac{7}{6} \qquad B = \frac{7}{\cancel{6}_3} \cdot \frac{\cancel{2}}{\sqrt{3}} = \frac{7\sqrt{3}}{9}$$

Thus we get that

$$\boxed{a_n = \frac{1}{3}\cos\left(n\frac{2\pi}{3}\right) + \frac{7}{9}\sqrt{3}\sin\left(n\frac{2\pi}{3}\right) + \frac{1}{3}n - \frac{1}{3}}$$

Exercise 4

(a)



$\deg(C_i) = 5$ for $i = 1, 2, 3, 6$

$\deg(C_4) = \deg(C_5) = 4$

(b) There are vertices with odd degree so there is no Euler path

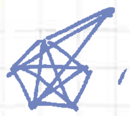
(c) G is not planar as it contains a subgraph (in green in the picture) isomorphic to K_5

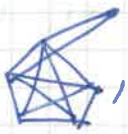
(d) The Chromatic number is 5: we need two colors to color the subgraph isomorphic to K_5 but we can color C_4 with the same color used for C_5

(e) We observe that, when we collapse any edge connecting C_4 to K_5 we get K_5

We use the formula

$$P(G,\lambda) = \underset{\text{Graph}-e}{P(G_e,\lambda)} - \underset{\text{Graph with e collapsed}}{P(G'_e,\lambda)}$$

$P(G,\lambda) = P($  $,\lambda) - P(K_5,\lambda)$

$= P($  $,\lambda) - P(K_5,\lambda) - P(K_5,\lambda)$

$= P($  $,\lambda) - P(K_5\lambda) - 2P(K_5\lambda) =$

$$= P(\;\;, \lambda) - P(K_5 \lambda) - 3P(K_5 \lambda)$$

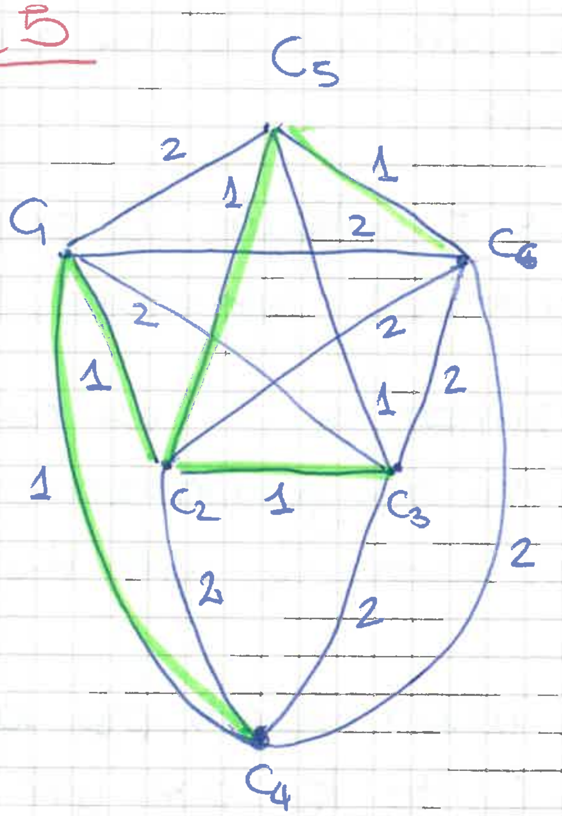
$$= P(K_1 \lambda) \cdot P(K_5 \lambda) - 4P(K_5 \lambda)$$

$$= P(K_5 \lambda)(P(K_1 \lambda) - 4)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)(\lambda-4)$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)^2$$

Exercise 5



We use Kruskal algorithm

$S = \{\{C_1, C_4\}\}$

$S = \{\{C_1 C_4\} \{C_1 C_2\}\}$

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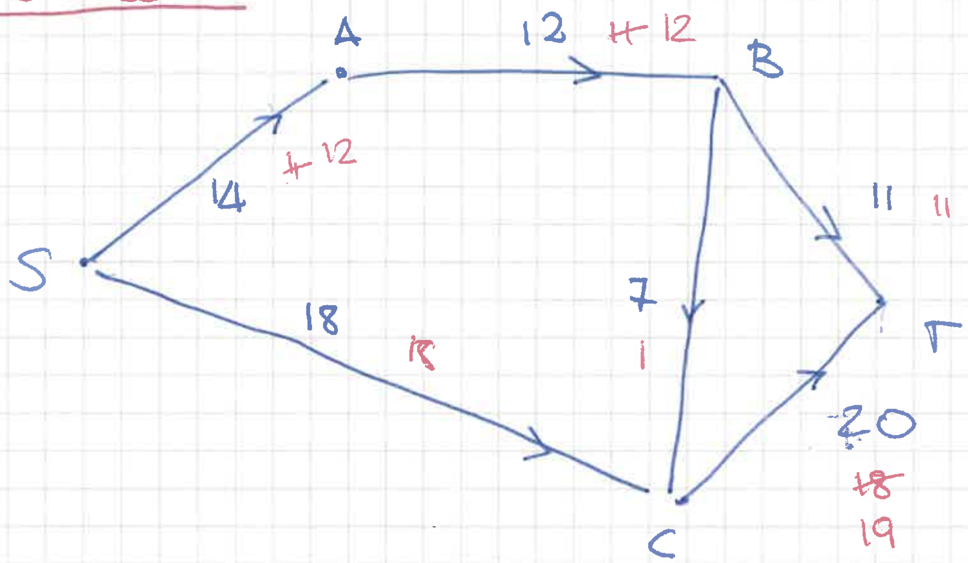
$S = \{\{C_1 C_4\} \{C_1 C_2\} \{C_2 C_5\} \{C_2 C_3\}\}$

$S = \{\{C_1 C_4\} \{C_1 C_2\} \{C_2 C_5\} \{C_2 C_3\}\}$

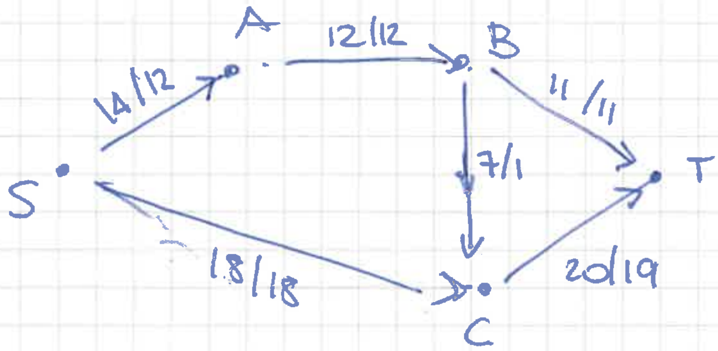
$S = \{\{C_1 C_4\} \{C_1 C_2\} \{C_2 C_5\} \{C_2 C_3\} \{C_5 C_6\}\}$

$\boxed{\text{The weigth is } 5}$

Exercise 6



by running the algorithm that we learned in class I find that a flow with maximal value is



$$Val(f) = 12 + 18 = 30$$

By using the method learned in class we find that $P = \{S, A\}$ $P^c = \{B, C, T\}$ is the cut associated to the flow.

$$c(P, P^c) = 12 + 18 = 30$$

As this is equal the value of the flow, at posteriori we can conclude that our answer is correct.