Home Work Problems 5

The problems are part of the examination of the course. This is the fifth and final set of problems. They are to be solved individually and are due by December 22 (2014). Please write clearly and present the solutions well.

1. With the help of Easton’s Theorem, and general facts about cardinal numbers, decide (with proof) which of the following statements are consistent with ZFC

   (a) \(2^{\aleph_{\alpha+1}} = \aleph_{\alpha+17}\), for all ordinals \(\alpha\),
   
   (b) \(2^{\aleph_0} = \aleph_{\omega}\),
   
   (c) \(2^{\aleph_0} = \aleph_{\omega_1}\).

   (5 p)

2. Generic sets do not belong to the ground model except in trivial cases. Let \((P, <)\) be a notion of forcing in \(M\) with the following "non-triviality property": For each \(p \in Q\), there are \(q \leq p\) and \(r \leq p\) such that \(q\) and \(r\) are incompatible. Prove that if \(G \subseteq P\) is generic over \(M\), then \(G \notin M\). (See Jech 2002, Exercise 14.6 for a hint.) Check that Cohen’s forcing notion (cf. Theorem 14.32, Jech) satisfies non-triviality. (5 p)

3. Chose one, and only one, of the following problems, and solve it.

   (alt. 1) Prove the Prime Ideal Theorem for Boolean algebras (Theorem 7.10, Jech) using Zorn’s lemma. (5 p)

   (alt. 2) Let \(P = (P, \leq)\) be a partial order. Define \(H(P)\) to be the set of cuts in \(P\). Prove that \((H(P), \subseteq)\) is a complete Heyting algebra (see definition in e.g. Bell 2005). Show that \(p \mapsto U_p = \{q \in P : q \leq p\}\) gives an order preserving embedding of \(P\) into \(H(P)\). (5 p)