Some observations on Fatou sets of rational functions

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Abstract

This article is about repeated iterations of rational functions in the complex plane, Fatou sets and Julia sets. While exploring some properties of the Fatou set by computer simulations, I encountered an interesting pattern. After generating more data, I became even more convinced that there was a correlation between the Fatou set of a rational function, and its derivatives on the iterated functions. In conclusion; my research strongly suggests that in most cases, the points in the Fatou set that does not converge to infinity under iteration lie in the limit of the points where the absolute value of the derivative of the iterated function is at most one as the number of iterations grows.
Contents

Abstract 1

Table of contents 3

Introduction 5
  Definition of the Fatou and Julia sets 5
  Definition of periodic points 6
  Definition of Fatou Component 6

Main hypothesis 6

Method towards a proof of the main hypothesis 7
  Lemma 1: No wandering domain theorem 7
  Lemma 2: Fixed points in Fatou components 7
  The Classification theorem 8
  Lemma 3: Convergence of the derivatives 8

Theorem: Convergence in components 9

Working with the computer 12
  Algorithm 12
  Some images and examples 13

Practical uses 15

References 16
Introduction

A rational function \( f(z) \) is a map from \( \mathbb{C} \rightarrow \mathbb{C} \), where \( \mathbb{C} \) is the extended complex plane, and \( f(z) = \frac{P(z)}{Q(z)} \) where \( P, Q \) are polynomials. We may assume that common zeros have been cancelled out, i.e. \( P \) and \( Q \) are coprime. The degree of a rational function, \( \text{deg}(f) \) is defined as \( \max(\text{deg}(P), \text{deg}(Q)) \).

Definition

Define \( f_k \) as \( f \circ f \circ ... \circ f \), \( f_0(z) = z \), and let \( z_k = f_k(z_0) \).

Obviously, \( z_n \) will for some \( f \) and \( z_0 \) diverge to infinity as \( n \) grows, \( (f(z) = z^2 + 1, z_0 = 1 \) will do the job), but the sequence behaves quite differently with different choices of \( z_0 \).

Definition of the Fatou and Julia sets

A sequence of functions \( \{f_n\} \) is said to be equicontinuous in a set \( X \) if there for every \( \epsilon > 0 \) and every \( z \in X \) exists a \( \delta > 0 \) such that for \( \zeta : |\zeta - z| < \delta \) we have that \( |f_n(\zeta) - f_n(z)| < \epsilon \) for all \( n \).

The Fatou set, \( \mathcal{F}(f) \), is defined to be the maximum open subset in \( \mathbb{C} \) where the family of functions \( f_1, f_2, f_3, ... \) is equicontinuous. This means that \( f_n \) will preserve the proximity of points, i.e. two points near each other will behave quite similar when iterated under \( f \).

The Julia set \( \mathcal{J}(f) \) is defined as the complement to \( \mathcal{F}(f) \). This means that the Julia set is closed and compact by definition.

Definitions

A set \( D \) is forward invariant if \( f(D) = D \), and backward invariant if \( f^{-1}(D) = D \). A set is completely invariant if it is both forward and backward invariant.

It is clear that \( \mathcal{F}(f) = \mathcal{F}(f_k) \) and \( \mathcal{J}(f) = \mathcal{J}(f_k) \) and one can show \(^1\) that both these sets are completely invariant under \( f \).

\(^1\)p 54 in Alan F. Beardon, Iterations of Rational Functions
Definition of periodic points
A point $z_0$ is called a periodic point of $f$ with period $p$ if $z_0 = f_p(z_0)$ and $z_0 \neq f_k(z_0)$ for $k < p$.

Furthermore, a point $z_0$ is
a) attracting if $|f'(z_0)| < 1$

b) indifferent if $|f'(z_0)| = 1$
c) repelling if $|f'(z_0)| > 1$

A point $z_0$ is called preperiodic if $f_m(z_0)$ is periodic for some $m$.

It is easy to show that if $z_0$ is periodic, then $f(z_0)$ is also a periodic point of the same type. For rational maps, it is shown that all attracting periodic points lies in the Fatou set, and the repelling ones in the Julia set.

Definition of Fatou Component
A Fatou component is a maximum subset in the Fatou set such that there exists a path between any two points in the subset.

A Fatou component $\Omega$ is called a limited component of the Fatou set $\mathcal{F}(f)$ if the point at infinity is not a point in $f_n(\Omega)$ for any $n$. The union of all the limited components is denoted by $\mathcal{L}(f)$.

Main hypothesis
Define $A_k = \{z_0 : |f'_k(z_0)| \leq 1\}$, i.e. where the absolute value of the derivative of the iterated function in $z_0$ is less than or equal to one, and define $A_\infty$ as $\{z_0 : \limsup_{k \to \infty} |f'_k(z_0)| \leq 1\}$.

The hypothesis is that if $\text{deg}(f) \geq 2$, and if $f$ is not an analytic conjugate to a Euclidean rotation of the unit disc or some annulus onto itself, then

$$\mathcal{L}(f) \subset A_\infty$$

Computer simulations point towards this statement and an abundance of data strengthens the hypothesis. The following lemmas and sketches will show that the hypothesis seems intuitively true for some cases.

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2 p 104,109 in Alan F. Beardon, Iterations of Rational Functions
Method towards a proof of the main hypothesis

Definitions

A Fatou component $\Omega$ of $f$ is:

a) periodic if $f_k(\Omega) = \Omega$ for some positive integer $k$;
b) eventually periodic if $f_m(\Omega)$ is periodic for some positive integer $m$;
c) wandering if $\Omega, f(\Omega), f_2(\Omega), \ldots$ are all pairwise disjoint.

Lemma 1: No wandering domain theorem

If $\Omega$ is a Fatou component of a rational map $f$, where $\deg(f) \geq 2$, then it is either periodic or eventually periodic.

The lemma follows directly from The No wandering domain theorem (Sullivan’s Theorem) \(^3\) but as the proof is very complicated it has been excluded from this article.

Lemma 2: Fixed points in Fatou components

Let $\Omega$ be a forward invariant Fatou component of the rational function $f$ such that it contains an attracting fixed point $\alpha \neq \infty$. Then $\lim_{n \to \infty} f_n(z) = \alpha$ for all $z \in \Omega$ and $|f'(\alpha)| < 1$.

Proof:

Because there is an attracting fixed point $\alpha$ in $\Omega$, $|f'(\alpha)| < 1$ and there exists an $r$ such that $r < 1$ and a disc $D$ centred in $\alpha$ where we have that $|f(z) - \alpha| = |f(z) - f(\alpha)| < r|z - \alpha|$. This shows that $f_n$ maps $D$ into itself, and Vitali’s Theorem \(^4\) gives the limit $f_n \to \alpha$ in $\Omega$.

\(^3\) p 176 in Alan F. Beardon, Iterations of Rational Functions, p 69 in Lennart Carleson, Theodore W Gamelin, Complex Dynamics

\(^4\) p 56 in Alan F. Beardon, Iterations of Rational Functions
The Classification theorem

One can show that a forward invariant component $F_0$ of $F(f)$ where $f$ is a rational function is exactly one of the following:

(a) an attracting component if it contains an attracting fixed point of $f$;
(b) a parabolic domain if there exists a fixed point $\zeta$ on the boundary of $F_0$ such that $f'(\zeta)$ is a root of unity and if $f_n \to \zeta$ on $F_0$.
(c) a Siegel disc if $f : F_0 \to F_0$ is analytically conjugate to a Euclidean rotation of the unit disc onto itself;
(c) a Herman ring if $f : F_0 \to F_0$ is analytically conjugate to a Euclidean rotation of some annulus onto itself.

Having this in our minds, we have come to a critical step in the proof, and the following lemma is crucial; the lemma is certainly true if $f$ is a polynomial and the Fatou component has an attracting fixed point, but it might be possible to extend the theorem to include the parabolic domains.

The reason why this lemma feels intuitively true is that the chain rule gives that $f_k'(z_0) = f'(z_0) \cdot f'(z_1) \cdots f'(z_{k-1})$. Furthermore since each $z_k$ lie closer and closer to the fixed point $\alpha$, the derivative $f'(z_k)$ must be close to $f'(\alpha)$ which is less than one if $\alpha \in \Omega$.

Lemma 3: Convergence of the derivatives

If $\Omega$ is a forward invariant Fatou component of $f$ which contains an attracting fixed point $\alpha \neq \infty$, where $f$ is holomorphic in $\Omega$, then

$$\lim_{n \to \infty} |f'(z_n)| = |f'(\alpha)| < 1 \quad \text{for all} \quad z_0 \in \Omega$$

and

$$\lim_{n \to \infty} f'_n(z_0) = 0 \quad \text{for all} \quad z_0 \in \Omega$$

Proof:

Let $\zeta \in \Omega$, and choose $\epsilon > 0$. Then there exists an $r > 0$ such that $|\zeta - \alpha| < r$ implies that $|f'(\zeta) - f'(\alpha)| < \epsilon$, since $f$ is holomorphic in $\Omega$.

But Lemma 2 gives that for each $r > 0$, there exists $N$, such that $n \geq N$ implies that $|f_n(z_0) - \alpha| < r$, since $f_n(z_0) \to \alpha$ as $n$ grows.

Hence for each $\epsilon > 0$, there exists $N$ such that $n \geq N \Rightarrow |f'(f_n(z_0)) - f'(\alpha)| < \epsilon$, and this gives that $|f'(f_n(z_0))| = |f'(z_n)|$ converges to $|f'(\alpha)| <$
1, and thus the first limit is proved.

Using the chain rule on $f'_n(z_0)$, we get $f'(z_{n-1}) \cdot f'(z_{n-2}) \cdots f'(z_1) \cdot f'(z_0)$ and because of the limit we just proved only a finite number of factors have an absolute value that is greater than or equal to 1, and hence the second limit is proved.

**Theorem: Convergence in components**

Let $f$ be a rational function where $\deg(f) \geq 2$ and let $\Omega$ be a Fatou component in $\mathcal{L}(f)$ such that $f_m(\Omega)$ contains an attracting periodic point for some $m$. Then $\Omega \subset A_\infty$.

**Proof:**

Define $\Omega_k$ as $f_k(\Omega)$, and $\Omega_0 = \Omega$. By Lemma 1, there exists a number $M$ such that $\Omega_{m_1}$ is periodic under $f$ with period $n$ if $m_1 \geq M$. We also know that there exists an $m_2$ such that $f_{m_2}(\Omega)$ contains an attracting periodic point. Let $m = \max(m_1, m_2)$. Then $\Omega_m$ is periodic and contains an attracting periodic point with period $n$.

This gives us that $f_n(\Omega_m) = \Omega_m$ and it must then contain an attracting fixed point $\alpha_0$ of $f_n$.

For each point $z \in \Omega$, $|f_k(z)| < \infty$ for all $k$ since $\Omega \subset \mathcal{L}(f)$. $\Omega_m$ is the image of $\Omega$ under $f_m$ so each $z \in \Omega_m$ is finite. The attracting fixed point in $\Omega_m$ must therefore also be finite and $\Omega_m$ is therefore also free from poles. We can thus be sure that $f_n$ is holomorphic in $\Omega_m$.

The conditions in Lemma 3 are now satisfied because $\Omega_m$ is forward invariant under $f_n$. Hence $\lim_{k \to \infty} f'_{nk} = 0$ in $\Omega_m$. But if $\Omega_m$ is forward invariant for $f_n$, then are $\Omega_{m+1}, \Omega_{m+2}, \ldots$ forward invariant as well, so in all these components, Lemma 2 and 3 are true, since they all have finite fixed points of $f_n$. Hence $\lim_{k \to \infty} f'_{nk} = 0$ in $\Omega_{m+i}$. 

9
For $k$ large enough, we can write $k = nq + r$ where $m \leq r < m + n$. Then for an arbitrary point $z_0 \in \Omega$ we have by the chain rule that $f'_k(z_0) = f'_{nq+r}(z_0) = f'_r(z_0) \cdot f'_{nq}(z_r)$.

As $k \to \infty$, we have that $q \to \infty$ which implies that $f'_{nq}(z_r) \to 0$ since $z_r$ lies in some $\Omega_{m+i}$ and we have already established that this limit is 0 in $\Omega_{m+i}$.

This shows that if $z_0 \in \Omega$, $|f'_k(z_0)| \to 0 \leq 1$, and the proof is complete.

**Corollary:**

Let $f$ be a rational function where $\text{deg}(f) \geq 2$. If $\mathcal{F}(f)$ is free from parabolic domains, Siegel discs and Herman rings, then $\mathcal{L}(f) \subset A_\infty$.

The Classification Theorem gives any forward invariant component $\Omega \in \mathcal{L}(f)$ must be attracting. No Wandering Domain Theorem says that $\Omega$ is eventually periodic and hence there exists $m$ and $n$ such that $f_m(\Omega)$ is forward invariant under $f_n$. We conclude that $f_m(\Omega)$ contains an attracting periodic point, and we apply the theorem about convergence in components. This implies that $\Omega \subset A_\infty$.

**Remarks:**

This result does not state how fast $A_n$ converges to a set that contains $\mathcal{L}(f)$, although I will briefly touch that subject later on. Since the Fatou set can have only 0, 1, 2 or infinitely many components, the hypothesis is only proved when every limited component converges to a cycle of components.

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5p 94 in Alan F. Beardon, Iterations of Rational Functions
that contains attracting periodic points and \( f \) is holomorphic. One such example is page 13 in reference [1].

Figure 1: \( f(z) = z^2 - 1 \): from left to right: The Fatou set (black), the Fatou set and \(|f'_5(z_0)| \leq 1\) (white), the Fatou set and \(|f'_{10}(z_0)| \leq 1\) (white).

Another example where one can easily would be \( f(z) = z^2 - 1/2 \) which has only two Fatou components; one limited, and one unbounded. The limited component contains an attracting fixpoint \( z = \frac{1-\sqrt{3}}{2} \).

Figure 2: \( f(z) = z^2 - 1/2 \): from left to right: The Fatou set (black), the Fatou set and \(|f'_{10}(z_0)| \leq 1\) (white).
Working with the computer

The algorithm I use to find the limited components of the Fatou set is the usual; iterate a maximum of 400 iterations and if \( |f_{400}(z)| < 100 \), then \( z \) is most probably in \( \mathcal{L}(f) \), (or in the Julia set, but this set has almost always zero area, and can be omitted). The polynomials I have examined have zeros \( a + ib : a, b \in [-1, 1] \) in the numerator and the denominator to make sure that the limited Fatou set is roughly in the center of my image. By simply calculating the pixels I compute the different areas (letting the resolution be about 800x800 pixels).

The pseudo-code for my algorithm is the following:

\[
\begin{align*}
\text{MAX\_ITERATIONS} & := 400 \\
\text{FOR each point } z & \text{ DO} \\
& \quad i := 0 \\
& \quad \text{WHILE abs}(z)<100 \text{ AND } i<\text{MAX\_ITERATIONS} \text{ DO} \\
& \quad \quad z := f(z) \\
& \quad \quad i := i+1 \\
& \quad \text{IF } i = 400 \text{ THEN point is in the limited components} \\
& \quad \text{ELSE point is not in the limited components}
\end{align*}
\]

Remark
This algorithm clearly gives an overestimation of the Fatou set; pixels that diverge towards infinity really slowly might be included. No pixels that should be in the Fatou set are excluded assuming all points \( z \) in the Fatou set fulfils \( |z| < 100 \).
Some images and examples

The leftmost image below shows the Fatou set (greyscale) and Julia set (black), and thereafter the sets $A_1, A_4, A_9$ (white). As we can see, $A_k$ converges rapidly towards $\mathcal{L}(f)$, the difference between the area of $\mathcal{L}(f)$ and $A_9$ is only 3%.

![Fatou set and Julia set](image)

Figure 3: $f(z) = (z - 0.052 + 0.391i) \cdot (z - 0.314 - 0.332i)$

The images in Figure 4 show that even when $\mathcal{L}(f)$ is infinitely disconnected (right), $A_k$ (left) will eventually converge to $\mathcal{L}(f)$.

In the example, the difference between $\mathcal{L}(f)$ and $A_{300}$ is less than 0.2%.

![Infinitely disconnected](image)

Figure 4: $f(z) = (z - 0.146 + 0.612i) \cdot (z - 0.325 - 0.993i) \cdot (z + 0.482 + 0.82i)/(z - 0.913 + 0.02i)$
The following image is the first function I found which had a really slow convergence; the difference between $A_{15}$ and $\mathcal{L}(f)$ is usually only a few percents. (The picture shows $A_1$ to $A_{40}$, then $\mathcal{L}(f)$, and the last square is $A_{400}$)

Figure 5: $f(z) = (z - 0.507 + 0.055i) \cdot (z - 0.932 - 0.665i) \cdot (z + 0.378 - 0.707i)$
Practical uses

What new methods can be developed using the above results? It is only possible to approximate the Fatou set because of the infinite number of iterations needed, so instead of using the regular algorithm (which I described about approximating the Fatou set), it is sometimes more accurate to compute the derivatives. Computer simulations indicate that if the limited Fatou components of a rational map have an area close to zero and is infinitely disconnected, it is more accurate to compute the derivative instead of the absolute value of $f_n(z_0)$.

The following image shows the different results: The leftmost frame is the Fatou set, computed with 400 iterations. The second one is the same algorithm using 10 iterations. The last frame is $A_{10}$ (white). $A_{10}$ is an underestimate which happened in most cases, but the interesting part is that $A_{10}$ is about 79% of the Fatou set, while the one with 10 iterations is an overestimation of about 33%.

*Future work might show that an interpolation between the regular algorithm and using the derivatives gives the most accurate approximation of the Fatou set.*

Figure 6: $f(z) = (z + 0.566 + 0.795i) \cdot (z + 0.185 + 0.101i) \cdot (z + 0.753 + 0.714i)$
References

[1] Alan F. Bearden *Iterations of Rational Functions*

[2] Lennart Carleson, Theodore W. Gamelin *Complex Dynamics*