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On the bunkbed conjecture

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Madeleine Leander

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MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 10691 STOCKHOLM

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Madeleine Leander

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Handledare: Svante Linusson

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ON THE BUNKBED CONJECTURE

MADELEINE LEANDER

Abstract. Let G be a finite graph and consider the bunkbed graph $\tilde{G} = G \times K_2$ where K_2 is the graph consisting of two vertices, $\{0, 1\}$ and one edge connecting them. On $\tilde{G}$ consider the percolation model with p the probability that an edge e exists, for all $e \in \tilde{G}$. All edges will exist or not independently of each other. We write $u \leftrightarrow v$ for the event "there is a path from u to v". The bunkbed conjecture states that for any bunkbed graph $\tilde{G} = G \times K_2$, corresponding to a finite graph G the following holds

$$P(u_0 \leftrightarrow v_0) \geq P(u_0 \leftrightarrow v_1),$$

for all $u, v \in V(G)$ and any probability p.

The bunkbed conjecture was first informally stated by P. W. Kasteleyn around 1985 and has influenced the research of mathematicians like van den Berg, Kahn, Häggström and Linusson since.

The main purpose of this thesis is to use combinatorial tools to work on the bunkbed conjecture. The bunkbed conjecture will be proven to be true for wheels and some small graphs.

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Finally I would like to give all my love to my family and to my friends.

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1. Introduction to the problem

Studying stochastic models on graphs is a big and important area in probability theory, see for example Aldous [1], Häggström [7], [6], [8] and Lyons [11]. Percolations on product graphs can be used in several different contexts. In theoretical physics one example is electrical networks, as described by Doyle and Snell in [3]. Another example is described in [5], where the graph corresponds to a quadratic or cubic lattice and is used for the Ising model.

The product graphs we will study here are bunkbed graphs. We can think of a bunkbed graph as two copies of a finite graph G such that every vertex in one copy is adjacent to the corresponding vertex in the other copy. We will call the copys the 0-layer and the 1-layer. A vertex $x \in G$ will have two copies in the bunkbed graph, we will call them x_0 and x_1 depending on the layer. We will write $\tilde{G}$ for the bunkbed graph $G \times K_2$ on G.

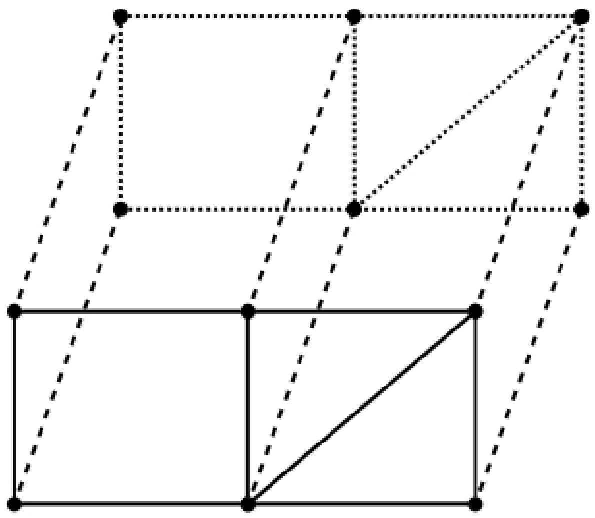

Figure 1.1. An example of a bunkbed graph. This bunkbed graph corresponds to the graph with solid edges in the figure.

One can ask if a node $u_0 \in \tilde{G}$ is "closer" to a vertex v_0 than it is to v_1. The answer to this question depends on the way closeness is defined. If we define a vertex to be closer if one can find a shorter path (a path with less edges) to that vertex the answer is of course yes.

One can on the other hand define closeness to be how likely a random walk starting at u_0 hits v_0 and v_1 respectively by time t. This was proven not to be the case (for discrete time) by Bollobas and Brightwell.

Another way of defining closeness is to compare the probability that there exists a path from u_0 to v_0 with the probability that there exists a path from u_0 to v_1, where every edge in the bunkbed graph exists with probability p

independent of each other. The existence of the edges will be i.i.d. The probability that there exists a path from a vertex u to a vertex v in the graph G will be written as $P_G(u \leftrightarrow v)$ or $P(u \leftrightarrow v)$ for short when it is clear which graph is regarded. We might say that u_0 is closer to v_0 than it is to v_1 if

$$P(u_0 \leftrightarrow v_0) \geq P(u_0 \leftrightarrow v_1).$$

The bunkbed conjecture states that this inequality is true.

Conjecture 1. ***The bunkbed conjecture (BBC).*** *Let G be a finite graph and let $\tilde{G} = G \times K_2$ be the corresponding bunkbed graph. Then for any two vertices $u, v \in V(G)$*

$$P(u_0 \leftrightarrow v_0) \geq P(u_0 \leftrightarrow v_1).$$

If we manage to prove things like the bunkbed conjecture we might get a better understanding of closeness between nodes in different stochastic models also for other graphs.

Product graphs are also interesting while studying random walks. In [2] Bollobás and Brightwell proved some results concerning this.

We shall begin in Section 2 and 3 by presenting some known results to initiate the reader to some recent results on the bunkbed conjecture due to Linusson. We will also introduce new models, operations and generalizations in those sections. In Section 3 we prove a generalization of the bunkbed conjecture for wheels and a subset of series-parallel graphs. In Section 3 we will also give some restrictions for minimal counterexamples to the BBC. The rest of the paper will be organized as follows. In Section 4, 5 and 6 we give some new generalizations of BBC. In Section 4 we will formulate a conjecture corresponding to the bunkbed conjecture where we allow paths to start and end in more than one vertex. In Section 5 we present the corresponding problem for acyclic directed graphs. In Section 6 we give a new way to attack the bunkbed conjecture. For the last section, Section 7, we will summary some results about series-parallel graphs we found while trying to prove the bunkbed conjecture for series-parallel graphs.

2. Models and generalizations

If nothing else is said all the graphs will be considered simple and undirected. For a graph G we will as usual define $V(G)$ to be the set of vertices in G and $E(G)$ the set of edges in G. If we have an edge between two vertices x and y we call it xy. We will write $G\backslash U$ for the graph obtained from G with the vertices in U removed. When we write $\tilde{G}$ for a graph, we will mean that $\tilde{G}$ is a bunkbed graph, $\tilde{G} = G \times K_2$.

For the first model we will condition on the vertical edges existing in $\tilde{G}$. Let G be a finite graph. Let further $T \subset V(G)$ be such that the vertical edges of $\tilde{G}$ exist exactly at the positions in T. Let the horizontal edges exist with probability p independently and identically-distributed, as usual. The vertices in T will be called **transcendental**. The following two models will be given the same names as in [10].

Model 1. $E_2^{p,T}$: *For a graph G, let $T \subset V(G)$ be the set of transendental nodes in G. Let p $(0 \leq p \leq 1)$ be the probability that an edge exists in the 0-layer or in the 1-layer of $\tilde{G} = G \times K_2$ independent of each other.*

With this model we can formulate the conjecture below that also seems likely to be true. The probability that there exists a path from u to v in G using the model BBC$E_2^{p,T}$ will be written $P_{E_2^{p,T}(G)}(u \leftrightarrow v)$ or $P_{E_2^{p,T}}(u \leftrightarrow v)$ for short. This will be written in a corresponding way for other models. Also if it is clear which model is used we may skip the model index.

Conjecture 2. *(BBC$E_2^{p,T}$). Let G be a finite graph with the corresponding bunkbed graph $\tilde{G} = G \times K_2$. Let further $T \subset V(G)$ be the set of transcendental nodes. Then for any $u, v \in V(G)$ and for any $0 \leq p \leq 1$ we have*

$$P_{E_2^{p,T}}(u_0 \leftrightarrow v_0) \geq P_{E_2^{p,T}}(u_0 \leftrightarrow v_1).$$

It is easy to realize that if BBC$E_2^{p,T}$ is true for all $T \subset V(G)$ then so is also the original bunkbed conjecture. Using a mirror argument Linusson also proved the following lemma that will be useful later. Recall first that a cutset $C \subset V(G)$ is such that $G\backslash C$ is disconnected.

Lemma 1. *If the set of transcendental nodes, $T \subset V(G)$, contains a cutset of G that separates u from v or $u \in T$ or $v \in T$, then*

$$P_{E_2^{p,T}}(u_0 \leftrightarrow v_0) = P_{E_2^{p,T}}(u_0 \leftrightarrow v_1),$$

and hence BBC$E_2^{p,T}$ is true.

Now, for the next model, fix any edge $e \in E(G)$. If $0 < p < 1$ we have four cases. If both e_0 and e_1 exists in $\tilde{G}$, it is equivalent to contract the edge e. On the other hand if neither e_0 nor e_1 exists in $\tilde{G}$ it is equivalent to remove the edge e. With equivalent we will mean that the probability that there exists a path between two vertices doesn't change. We end up with the model below, with the remaining two cases.

Model 2. E_3^T*: Let G be a graph and let the vertical edges exist exactly on positions in T for a subset T of $V(G)$. Every horisontal edge in $\tilde{G}$ exists either in the 0-layer or in the 1-layer with equal probability, and no edges exist in both layers.*

This model can be reformulated in the following way. *Let every edge in G be coloured either red or blue with equal probability and let a subset $T \subset V(G)$consist of transcendental nodes where a path may change colour.*

In the reformulation we will think of the red edges to exist in the 0-layer and the blue edges to exist in the 1-layer. Every path is allowed to change color at the transcendental nodes. We generalize the bunkbed conjecture to the one below.

Conjecture 3. ***(BBCE_3^T)*** *Let G be a finite graph in model E_3^T. Then for any $u, v \in V(G)$ we have*

$$P_{E_3^T}(u_0 \leftrightarrow v_0) \geq P_{E_3^T}(u_0 \leftrightarrow v_1),$$

where the zeroes means that we start/end in a red edge (or u/v transcendental) and v_1 means ending in a blue edge (or v transcendental).

In other words, the probability that there exists a path from u to v ending in a red edge (or v transcendental) is at least the same as the probability that there exists a path from u to v ending in a blue edge (or v transcendental). All paths will start with a red edge (if $u \notin T$). Sometimes we will call a path a red (blue) path when it starts at u in red and ends at v in red (blue).

Recall that a minor of G is a graph obtained by deleting or contracting edges in G. The conjecture was stated by Linusson in [10] were he also proved that if it is true for any minor of G and every set of transcendental nodes, $T \subset V(G)$, then so is also the bunkbed conjecture for G. A motivation for a model where one needs to regard all minors of a graph is that it is enough to show that no minimal counterexample exists, which was used to prove the bunkbed conjecture for outerplanar graphs in the same paper. By this model it will also be easier to use combinatorial tools.

By the same mirror argument that Linusson used to prove Lemma 1 we will also have the following lemma.

Lemma 2. *If $T \subset V(G)$ contains a cutset of G that separates u from v or $u \in T$ or $v \in T$ then*

$$P_{E_3^T}(u_0 \leftrightarrow v_0) = P_{E_3^T}(u_0 \leftrightarrow v_1),$$

and hence $BBC_{E_3^T}$ is true.

3. Classes of graphs

In this section we will prove the bunkbed conjecture E_3^T to be true for some classes of graphs, and we will also give some restrictions for minimal counterexamples. Here a graph G will be said to be a **minimal counterexample** if no graph obtained by deleting or contracting one or more edges of G, or removing one or more vertices of G is a counterexample. A graph G' obtained by deleting vertices or edges, or contracting edges of a graph G is said to be **equivalent** to G if no **connection probability**, a probability of the form $P(x \leftrightarrow y)$, changes. This will be more clear when we come to the operations in this section. We will start by presenting some useful operations that will reduce the possible number of minimal counterexamples. If nothing else is said, model E_3^T will be used in this section.

We will start by stating a result about outerplanar graphs by Linusson (Theorem 1).

First, a planar graph is said to be **outerplanar** if it does not have any K_4 or $K_{2,3}$ minor. Equivalently a graph is called outerplanar if it can be embedded in the plane such that the vertices lie on a fixed circle and the edges lie inside the disk of the circle and don't intersect. It is also possible to define outerplanar graphs as graphs containing a face that includes every vertex in the graph.

Theorem 1. *(Linusson) BBCE_3^T and BBC is true for outerplanar graphs.*

We will continue with a nice result for not-2-connected graphs.

Theorem 2. *A not-2-connected graph G can not be a minimal counterexample to BBCE_3^T.*

Proof. Assume that G is a minimal counterexample. First if G is not 1-connected the graph has at least two components. If u and v lie in different components we have

$$P_{E_3^T}(u_0 \leftrightarrow v_0) = P_{E_3^T}(u_0 \leftrightarrow v_1) = 0$$

and hence G can not be a minimal counterexample. Obviously G can not be a minimal counterexample if u and v lie in the same component. This since we can remove any edge and any vertex from the other component without altering the connection probability.

Now assume G is 1-connected and let x be a cut vertex, i.e. a vertex such that removing it splits the graph into two components. We start with the case where x is such that $G\backslash\{x\}$ has a component containing neither u nor v. Let G' be the graph obtained from G by removing all components of $G\backslash\{x\}$ that contain u or v. If $x \in T$ then G' does not influence the probabilities $P_{(G)}(u_0 \leftrightarrow v_0), P_G(u_0 \leftrightarrow v_1)$. Hence we can remove $G'\backslash\{x\}$ to obtain a minor of G, and so G can not be a minimal counterexample in this case. Now we condition on the color of the edges in G' and get the following two cases.

(1) There exists a path in G' starting at x with a red edge and ending at x with a blue edge.

For this case we can remove $G'\backslash\{x\}$ from G and let $x \in T$ to obtain a smaller graph without altering the connection probability we are considering.

(2) There is no path in G' that starts at x with a red edge and ends at x with a blue edge.

In this case G' can never be of any use for a path from u to v, so $G'\backslash\{x\}$ can be removed to obtain a smaller graph without changing the probabilities of going between u and v.

To complete the proof we need to show that G can not be a minimal counterexample when $G\backslash\{x\}$ consists of two components, one containing u and one containing v. In this case x separates u from v in G. If x is transcendental we are done by Lemma 2, and hence we may assume $x \notin T$.

We want to show that

$$P_{E_3^T(G)}(u_0 \leftrightarrow v_0) - P_{E_3^T(G)}(u_0 \leftrightarrow v_1) \geq 0.$$

Let G_1 be the subgraph of G obtained by deleting the component of $G\backslash\{x\}$ containing v. In the same way define G_2 to be the subgraph of G obtained by deleting the component of $G\backslash\{x\}$ containing u.

A path from u to v (which must pass through x) must go in and out from x with the same color.

We have

$$P_{E_3^T(G)}(u_0 \leftrightarrow v_0) - P_{E_3^T(G)}(u_0 \leftrightarrow v_1) =$$

$$\begin{aligned} &= P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) + P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1) \cdot P_{E_3^T(G_2)}(x_1 \leftrightarrow v_0) \\ &- P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1) - P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1) \cdot P_{E_3^T(G_2)}(x_1 \leftrightarrow v_1) \end{aligned}$$

Note that $P_{E_3^T(G)}(u_0 \leftrightarrow v_0)$ is not equal to

$$P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) + P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1) \cdot P_{E_3^T(G_2)}(x_1 \leftrightarrow v_0)$$

since in some cases we have both the event $u_0 \leftrightarrow x_0$ and $u_0 \leftrightarrow x_1$. But for these cases $P(u_0 \leftrightarrow v_0) = P(u_0 \leftrightarrow v_1)$ so we do not have to consider them for the difference.

We also have

$$P_{E_3^T}(x_0 \leftrightarrow v_1) = P_{E_3^T}(x_1 \leftrightarrow v_0)$$

and

$$P_{E_3^T}(x_0 \leftrightarrow v_0) = P_{E_3^T}(x_1 \leftrightarrow v_1)$$

by symmetry. By assumption we have

$$P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) - P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1) \geq 0$$

and

$$P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) - P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1) \geq 0$$

Hence their product is bigger or equal than zero.

The following calculation completes the proof.

$$\big(P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) - P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1)\big)\big(P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) - P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1)\big)$$

$$= P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) + P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1)$$
$$- P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1) - P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0)$$
$$= P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) + P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_1 \leftrightarrow v_0)$$
$$- P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) \cdot P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1) - P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1) \cdot P_{E_3^T(G_2)}(x_1 \leftrightarrow v_1)$$
$$= P_{E_3^T(G)}(u_0 \leftrightarrow v_0) - P_{E_3^T(G)}(u_0 \leftrightarrow v_1)$$

And hence a not-2-connected graph can not be a minimal counterexample to BBCE_3^T. □

To illustrate the last part of the proof, see for example the graph in Figure 3.1.

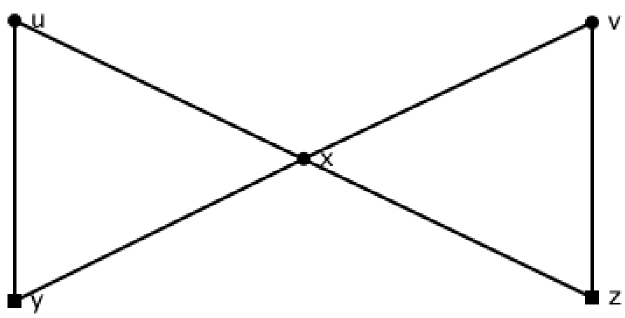

FIGURE 3.1. Here we have a 1-connected graph with transcendental nodes $\{y, z\}$, and a cutvertex x.

For this graph we have

$$\big(P_{E_3^T(G_1)}(u_0 \leftrightarrow x_0) - P_{E_3^T(G_1)}(u_0 \leftrightarrow x_1)\big)\big(P_{E_3^T(G_2)}(x_0 \leftrightarrow v_0) - P_{E_3^T(G_2)}(x_0 \leftrightarrow v_1)\big) =$$
$$= \left(\frac{5}{8} - \frac{2}{8}\right) \cdot \left(\frac{5}{8} - \frac{2}{8}\right) = \frac{3}{8} \cdot \frac{3}{8} = \frac{9}{64}$$

and

$$P_{E_3^T(G)}(u_0 \leftrightarrow v_0) - P_{E_3^T(G)}(u_0 \leftrightarrow v_1) = \frac{28}{64} - \frac{19}{64} = \frac{9}{64}.$$

Now we will define some operations and state and prove some lemmas that uses them.

T-operation: If two vertices $x, y \in T$ of G and $xy \in E(G)$ we can contract xy without changing the probability of any path to exist.

As a direct consequence of the T-operation we have the following lemma.

Lemma 3. *If G is a minimal counterexample to BBCE_3^T there can be no two transcendental vertices in G that are adjacent.*

Also we have the following two lemmas.

Lemma 4. *If G is a minimal counterexample to $BBCE_3^T$ there is no vertex x, different from u and v, in G with degree two such that $x \notin T$.*

Proof. Let x be a non-transcendental vertex of degree two in G different from u and v. Assume G is a minimal counterexample to BBCE_3^T. We have two cases. Either the two edges from x have different colors. Then vertex x and its two edges can never be used. Thus in this case we can remove x and its two edges without altering any probability. This gives a smaller graph, G_1 for which the statement is true by assumption. Hence G can not be a minimal counterexample in this case. Now for the other case, when the two edges from x have the same color. Then we can contract one of the edges to obtain a minor, G_2, of G without changing any probability. For G_2 the statement is true by assumption. Thus, G can not be a minimal counterexample to BBCE_3^T. □

In this proof we split the colorings of G into two cases. For both cases we found that the probabilities for paths is the same as for some smaller graphs, G_1 and G_2. We know the lemma to be true for these graphs by assumption and since G is just a linear combination of these two graphs the lemma must be true also for G. This idea will be used in other proofs without further comments.

Lemma 5. *If G is a minimal counterexample to $BBCE_3^T$ there can be no two vertices $x, y \in V(G)$ of degree two, different from u and v, such that xy exists.*

Proof. Let G be a graph with two vertices $x, y \in V(G)$ of degree two, different from u and v, such that xy exists. Assume G is a minimal counterexample to BBCE_3^T. Then both x and y then must be transcendental by Lemma 4 for the graph to be able to be a minimal counterexample. But we can not have any two transcendental adjacent vertices by Lemma 3. Hence G can not be a minimal counterexample to BBCE_3^T. □

Recall that the graph obtained by contracting the edge e in a graph G is denoted by G/e.

Δ-operation: Let $x, y, z \in V(G)$ and let $xy, xz, yz \in E(G)$. Assume that no other edges in $E(G)$ depend on the color of these edges. Form the following cases:

$G_1^\Delta = G/xy$
$G_2^\Delta = G/xz$
$G_3^\Delta = G/yz$
$G_4^\Delta = G$ with the edges xy, xz, yz having the same color.

If say xz and yz have the same color, but different from the color of xy, then all paths can walk between x and y with either of the colors and hence the case is equivalent to G_1^Δ. If xy and yz have the same color, but different from the color of xz we can walk between x and z with any of the two colors and hence we can contract the edge xz. This case is equivalent to G_2^{Δ} and is illustrated in Figure 3.2.

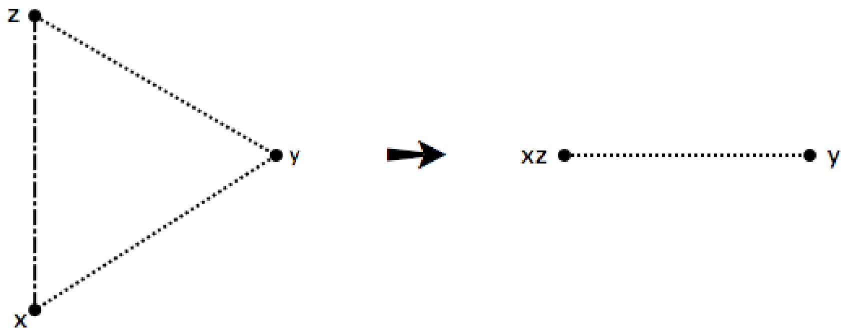

FIGURE 3.2. An illustration of the case G_2^{Δ}.

If xy and xz have the same color but different from the color of yz we can walk freely between y and z and hence we can contract the edge yz without altering any probability. This is the case G_3^{Δ}, and here G_3^{Δ} is said to be equivalent to G. For the last case all the edges in the triangle have the same color. We can not be sure that we keep the probabilities while contracting an edge since new paths can arise. See for example the case in Figure 3.3.

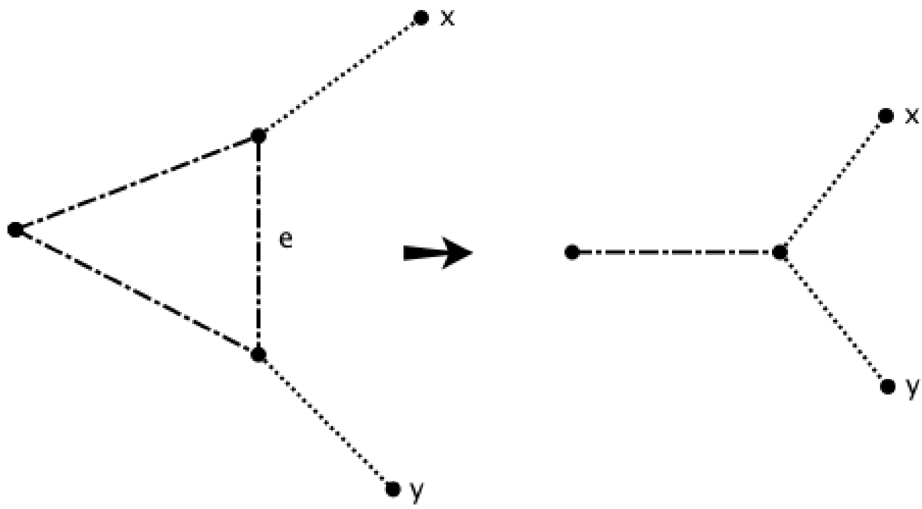

FIGURE 3.3. An illustration of why a graph obtained by contracting an edge in case G_4^{Δ} of the Δ-operation is not necessarily equivalent to G_4^{Δ}. In the figure we contract the edge e and a path between x and y arise.

It is easy to realize that if the BBC is true for all four cases it also holds for G. We have that G is equivalent to $G_i^{\Delta}, i = 1, 2, 3, 4$ with equal probability and

$$\begin{aligned} P_G(u_0 \leftrightarrow v_i) &= \frac{1}{4} P_{G_1^\Delta}(u_0 \leftrightarrow v_i) + \frac{1}{4} P_{G_2^\Delta}(u_0 \leftrightarrow v_i) \\ &+ \frac{1}{4} P_{G_3^\Delta}(u_0 \leftrightarrow v_i) + \frac{1}{4} P_{G_4^\Delta}(u_0 \leftrightarrow v_i) \end{aligned}$$

When we write "$P_{G_i^\Delta}(u_0 \leftrightarrow v_i)$" G_i^Δ implicitly tells us that model E_3^T is used, the same for the following two models.

Note that if the color of any other edges depended on the one of xy, xz or yz the graph obtained by contracting one of them would be a graph with some restrictions. For example say that in a graph G xy and yz have the same color, λ_1, and xz have the same color, λ_2, as another edge $e \in E(G)$. If $\lambda_1 \neq \lambda_2$ we may contract the edge xz without changing any probability. But the graph we obtain by doing this has the restriction that e and $y(xz)$ have different colors, and hence we can not discard G as a minimal counterexample. This will be more clear while observing the $Y-$operation.

Restricted Δ-operation Let $x, y, z \in V(G)$ such that $xy, xz, yz \in E(G)$. Assume that $xy \in U, |U| \geq 2$ for some connected subgraph U where the edges are required to have the same color. Assume also that the color of xz and yz doesn't depend on the color of any other edge. By arguments similar to them around the Δ-operation we get the following subgraphs:

$G_1^{R\Delta} = G/xz$
$G_2^{R\Delta} = G/yz$
$G_3^{R\Delta} = G$ but we require that xz and yz have the same color.

Again if BBCE_3^T is true for $G_i^{R\Delta}, i = 1, 2, 3$ then it is also true for G by reasoning analogues to the Δ case.

Y-operation Let $x \in V(G) \backslash T$ such that deg(x)=3 and $x \neq u, v$.. Let us call the neighbours of x for x_1, x_2, x_3. Assume that the color of no other edges is dependent of the colors of the edges xx_i. If say xx_1 and xx_2 have the same color but different from the color of xx_3 then xx_3 can not be used (since $x \notin T$) so we may remove it without altering any probability. We can also contract xx_1, and we end up with a graph, G_1^Y, equivalent to G without any restrictions. By the same reasoning we end up with the following four cases:

$G_1^Y = (G \backslash xx_1)/xx_2$
$G_2^Y = (G \backslash xx_2)/xx_3$
$G_3^Y = (G \backslash xx_3)/xx_1$
$G_4^Y = G$ with the edges xx_1, xx_2, xx_3 having the same color.

As for the other operations we see that if conjecture 3 is true for $G_i, i = 1, 2, 3, 4$ it is also true for G.

The operations will be used to exclude some graphs that can't be minimal counterexamples. When using them we need to be careful so that there are no restrictions on the minors G_i, such as edges having the same (or different) color. For example, it is easy to realize that contracting one of the edges xx_i in G_4^Y does not change any probability. We end up with a minor, $G_4^{Y'}$, for which we have a restriction of two edges having the same color. If we assume that G is a minimal counterexample to BBCE_3^T we have that $G_i^Y, i = 1, 2, 3$ are not counterexample. But we do not have the same for $G_4^{Y'}$ since we have a restriction on this graph.

3.1. **Wheels.** A graph consisting of an outer cycle and an inner vertex such that all vertices on the cycle is adjacent to the inner vertex will be called a **wheel**. The vertices on the cycle will be called outer vertices. We will first prove the bunkbed conjecture E_3^T to be true for wheels.

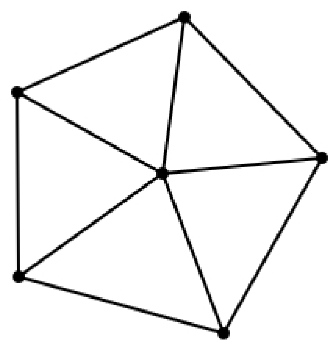

FIGURE 3.4. An example of a wheel.

Lemma 6. *If G is a minimal counterexample to conjecture 3 (BBCE_3^T) then G does not have a non-transcendental vertex $x \in G$ such that x has degree three and such that $x \neq u, v$ belongs to two triangles.*

Proof. Assume that G is a minimal counterexample, and that there exists a vertex x in G such that $x \notin T$, $x \neq u, v$ and $\deg(x) = 3$. Applying the Δ-operation on one of the triangles containing x in G we have

$$P_G(u_0 \leftrightarrow v_0) = \frac{1}{4} P_{G_1^\Delta}(u_0 \leftrightarrow v_0) + \ldots + \frac{1}{4} P_{G_4^\Delta}(u_0 \leftrightarrow v_0).$$

Since $G_i^\Delta, i = 1, 2, 3$ are minors of G we already know that

$$P_{G_i^\Delta}(u_0 \leftrightarrow v_0) \geq P_{G_i^\Delta}(u_0 \leftrightarrow v_1)$$

for $i = 1, 2, 3$ by assumption, and hence we may assume that all the edges in one of the triangles containing x have the same color, λ_1. Further by using the restricted Δ-operation we may by the same reasoning assume the two non colored edges of the other triangle containing x have the same color, λ_2. We can now contract the edge between x and the other vertex contained in both triangles without changing any probability. To realize that the probabilities are contained the importance of the degree of x comes in. If $deg(x) > 3$ contracting the edge as above could arise new paths. We end up with a minor of the graph for which the conjecture is known to be true. Note that if $x = u, v$ the contraction might change the probability of the events. $\square$

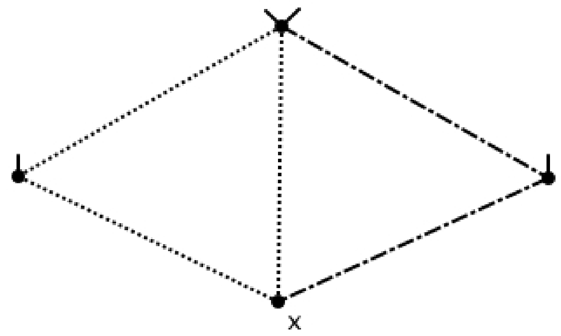

FIGURE 3.5. An illustration of the case in lemma 6.

Theorem 3. *Let G be a finite wheel such that each edge is coloured either red or blue wiht equal probability. Then*

$$P_{E_3^T}(u_0 \leftrightarrow v_0) \geq P_{E_3^T}(u_0 \leftrightarrow v_1),$$

where 0 and 1 stands for starting/ending on a red or blue edge respectively, or u/v transcendental. In other words, BBCE_3^T is true for wheels.

Proof. Assume that G is a minimal counterexample to BBCE_3^T. By Lemma 6 we know that G can't contain any outer, non-transcendental vertex $x \neq u, v$. Also, two transcendental vertices can not be neighbours in a minimal counterexample by Lemma 3. So whenever two transcentental outer vertices are adjacent we use the T-operation to contract the edge between them. Also if u or v (or both of them) are transcendental the theorem holds by Lemma 2. If there are five (or more) outer vertices at least three of them must be non-transcendental. If the center vertex is transcendental no outer vertex can be transcendental. Hence by Lemma 6 only u and v can be outer vertices. This gives an outerplanar graph for which BBCE_3^T is already known to be true. These things gives us that if G is a minimal counterexample it must be one of the following two graphs:

(1) G has four outer vertices, u, v and two trancendental nodes.
(2) G has three outer vertices, u, v and one trancendental node.

The second case can not be a minimal counterexample by Lemma 6 applied around the center vertex. Computing the probability to get from u to v ending up on a red edge and comparing it with the probability that the last edge is blue in the first case completes the proof. □

To prove that also BBC holds for wheels one must prove BBCE_3^T also for all minors of wheels. By using Lemma 6, the T-operation and the following lemmas on all the minors of wheels we can find that it is only finitely many cases to check to prove the bunkbed conjecture for wheels.

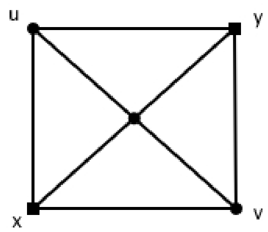

FIGURE 3.6. An illustration of (1) where $x, y \in T$. We have $P(u_0 \leftrightarrow v_0) = \frac{95}{128}$ and $P(u_0 \leftrightarrow v_1) = \frac{89}{128}$.

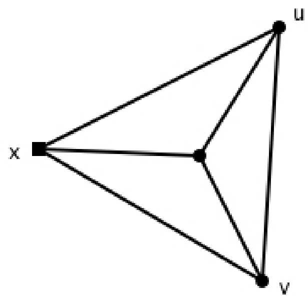

FIGURE 3.7. An illustration of (2) where $x \in T$. We have $P(u_0 \leftrightarrow v_0) = \frac{3}{4}$ and $P(u_0 \leftrightarrow v_1) = \frac{1}{2}$.

Lemma 7. *If G is a minimal counterexample to $BBCE_3^T$ then there can be no non-transcendental vertex of degree three, different from u and v, with two transcendental neighbours.*

Proof. Let G be a finite graph with a non-transcendental vertex x of degree three, different from u and v, and such that x has at least two transcendental neighbours (see Figure 3.8).

By using the Y-operation around x we find that if G is a minimal counterexample all the edges from x must have the same color (G_1^Y, G_2^Y and G_3^Y gives smaller graphs without changing any probabilities). If the three edges from x have the same color we can always walk freely between y and z, going in and out with any color. By contracting xy and xz we do not change any probability, and the smaller graph has no restrictions. Hence G can not be a minimal counterexample. □

For the next lemma we will use the following operation:

$Y\Delta-$operation: Let G be a graph with a non-transcendental vertex, x, of degree three, different from u and v, with neighbours y, z, w such that the

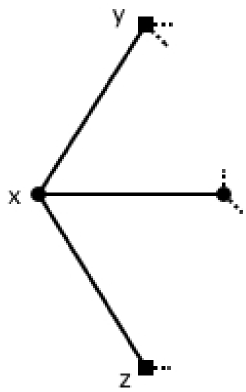

FIGURE 3.8. An illustration of the case in Lemma 7, where y and z are in T.

edges xy, xz and xw all have the same color. Then we can remove x, and put out new edges yz, zw and wy all with the same color, without changing the probability to get from u to v. The operation is illustrated in Figure 3.9.

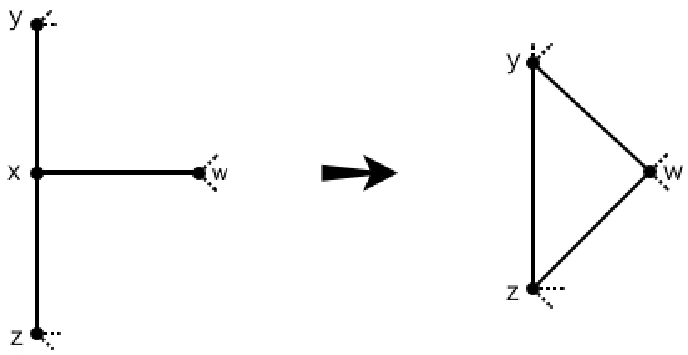

FIGURE 3.9. With all edges in the figure having the same color this illustrates how the $Y\Delta-$operation works.

Lemma 8. *Let G be a minimal counterexample to $BBCE_3^T$. Then we can not have a triangle xyz with x non-transcendental and both x and y having degree three, different from u and v. And such that the vertices, x' and y', adjacent to x and y respectively are transcendental.*

Proof. Let G be a finite graph with a triangle xyz with x non-transcendental, x and y having degree three, different from u and v. And such that the vertices, x' and y', adjacent to x and y respectively are transcendental. Assume G is a minimal counterexample.

By Lemma 7 y must be non-transcendental. And from Lemma 6 the edges $x'z$ and $y'z$ can not exist (so if G is a wheel x' and y' must have degree two). By using the $Y-$operation around x we find that for G to be a minimal counterexample all the edges from x must have the same color. Now by using the $Y\Delta-$operation around x we end up with two edges between y and z. If they have the same color we may remove one of them without changing any probability and then G can not be a minimal counterexample by lemma 7. If they on the other hand have different colors we may contract the edges yz without changing the probability to get from u to v. The lemma follows. $\square$

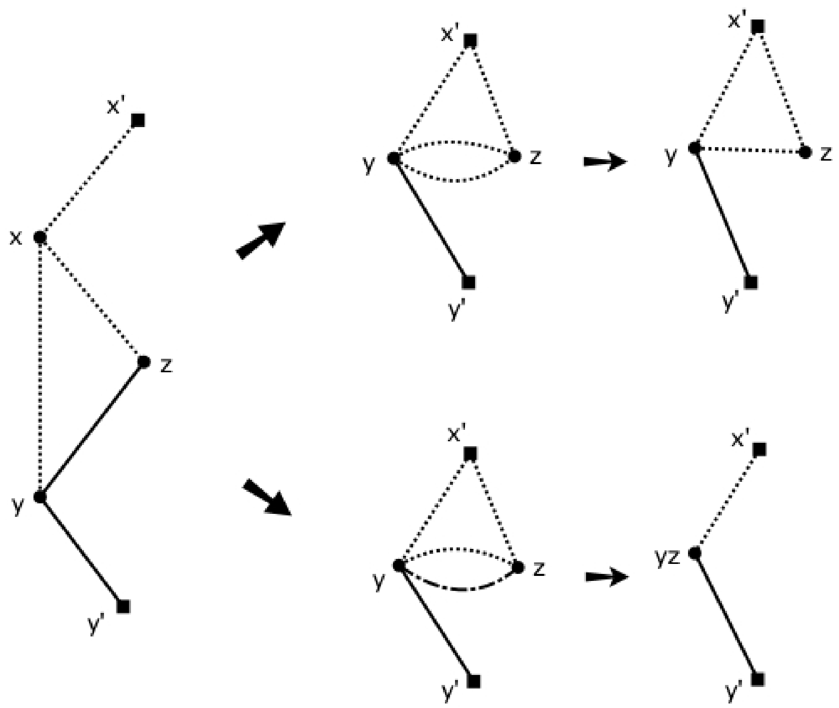

FIGURE 3.10. An illustration of Lemma 8.

Lemma 9. *$BBCE_3^T$ is true when $|T| = 0, 1$.*

The proof is analogues to the one Linusson gave in [10] for model $E_2^{p,T}$.

Proof. If we don't have any transcendental node we are done since it is impossible to use a blue edge, and hence we can not reach v with a blue edge. Now assume that we have exactly one transcendental node, x. We will condition on wether or not there is a path from u_0 (a path starting from u in a red edge) to x_0 (and ending in x with a red edge). If such a path does not exist we can never find a path ending in v with a blue edge, and we are done. And if such a path exists we have that

$$P_{E_3^T}(x_0 \leftrightarrow v_0) \geq P_{E_3^T}(x_0 \leftrightarrow v_1),$$

since we can only effect the probability ($P_{E_3^T}(x_0 \leftrightarrow v_0)$) in a positive way by conditioning on a red path to exist. $\square$

3.2. **BBC for wheels.** Now to one of the main result of this thesis, which is a new result.

Theorem 4. *$BBCE_3^T$ holds for all minors of wheels and hence BBC holds for wheels.*

To prove the theorem we will look at a construction of minors of wheels. To understand the construction, and verify that all minors are included we need the following lemma.

Lemma 10. *For a minor of a wheel to be a minimal counterexample to $BBCE_3^T$ the center vertex must be non-transcendental.*

Proof. Assume G is a minor of a wheel, with the center vertex transcendental. Then by Theorem 1 G can not be outerplanar. By Lemma 3 there can be no transcendental outer vertex of degree three since an outer vertex of degree three is adjadent to the center vertex and we can have no transcendental neighbors. Now assume we have a transcedental outer vertex, x, of degree two. Then x can not be adjacent to a vertex different from u and v of degree two by Lemma 5. We also have that x can not be adjacent to a vertex of degree three, different from u and v by Lemma 7. Hence the only possible case is the graph in Figure 3.11.

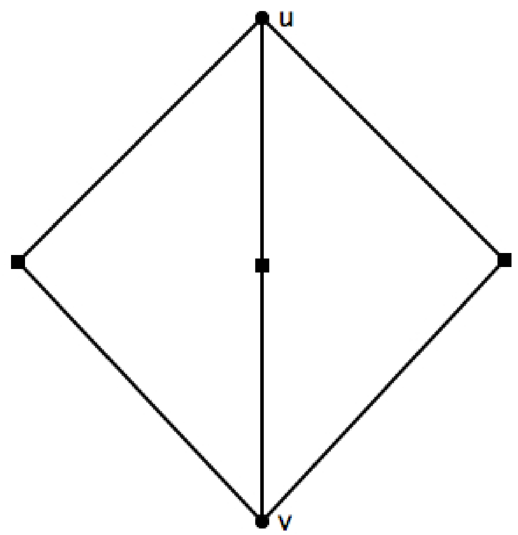

FIGURE 3.11. The squares denote the transcendental nodes. In this figure all nodes but u and v are transcendental.

By Lemma 2 there can be no cutset separating u from v in a minimal counterexample. Hence the graph in Figure 3.11 can not be a minimal counterexample. □

Now to the construction. Every wheel can be constructed from L-formed pieces, see Figure 3.12, we call the L-formed pieces the first construction pieces.

We want to find the construction pieces for minors of wheels, which are possible minimal counterexamples. The minors of wheels can be constructed

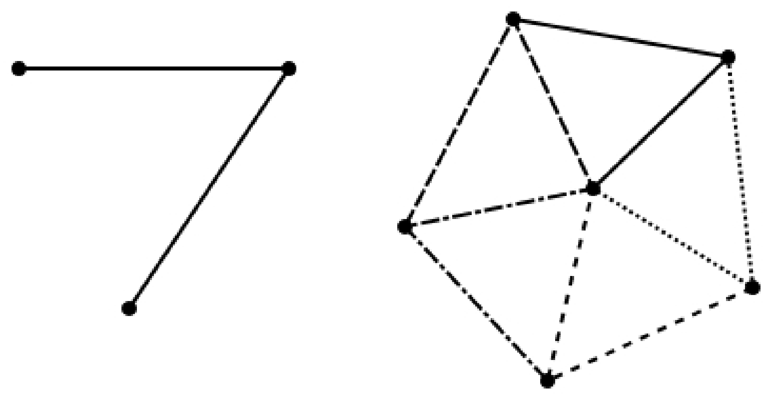

FIGURE 3.12. The wheel to the right is constructed from five L-formed pieces such as the one to the left.

by minors of the first construction pieces. If we from one L-formed piece, in a minor of a wheel G, remove the outer edge the graph, G, will become outerplanar. Thus for a graph (a minor of a wheel) to be a possible minimal counterexample the outer edge of each L-formed piece must exist. It is possible to remove the other edge from some of the pieces. We must also consider all combinations of transcendental nodes, but we know that the center vertex can not be transcendental by Lemma 10. Also by Lemma 3 we can not have both outer vertices of one L-formed piece transcendental. We get the new construction pieces shown in Figure 3.13, we call them the second construction pieces.

When we construct a minor of a wheel we will always start with the vertex u, then some construction pieces, v, some more pieces and then end in u. For example $uacbvdu$ is a minor of a wheel. First we will only consider the cases when the center vertex is different from u and v, later we will find this enough. We will use combinations of the second construction pieces to find new parts, which we will refer to as the construction pieces.

We can not have b first in the string of construction pieces since in a minimal counterexample u must be non-transcendental. The only parts that fits with b are c and e. We can thus remove b and instead put fb (number 1 in Figure 3.14) and cb (number 2 in Figure 3.14). Further we can not have c at the end of the string of construction pieces (between u and v) since for the graph to be a minimal counterexample v must be non-transcendental. c fits with e only (since we removed b), thus we can remove c and put ce instead (number 6 in Figure 3.14). Again we can never use either of e and f alone but we can combine them to fe (number 5 in Figure 3.14) and remove both of them. Now we have found the construction pieces (see Figure 3.14) we will use to find all the minors of wheels that are possible minimal counterexamples.

We have the following theorem.

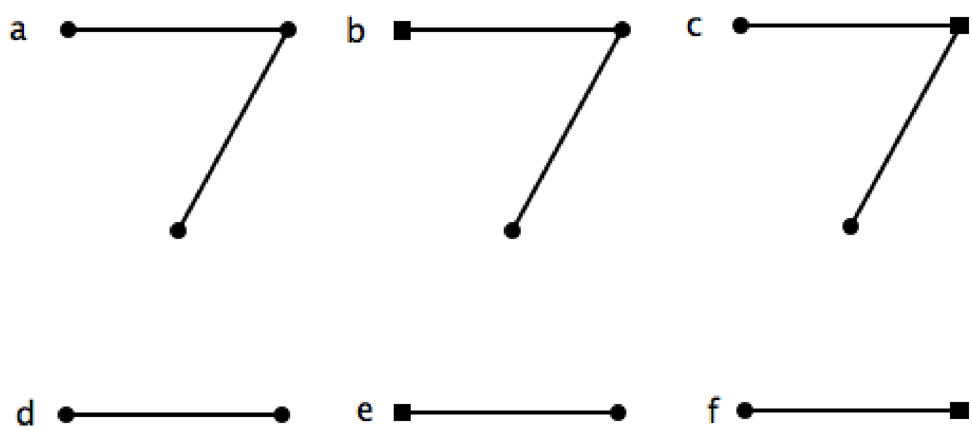

FIGURE 3.13. These are the second construction pieces. The squares stands for transcendental nodes and the circles for non-transcendental nodes.

Theorem 5. *All minors of wheels which are possible minimal counterexamples can be constructed by combinations of the pieces in Figure 3.14. A combination is of the form* $ux_1, \ldots x_k v y_1 \ldots y_l$ *where* x_i *and* y_i *are construction pieces.*

We can glue the pieces together in different ways but by using some of the properties we have found for minimal counterexamples we find only a few to be valid. When we glue the pieces together we will start with the vertex u then some construction pieces, v, some more pieces and end in u. For example we have the possible minimal counterexample $u3314v5u$ in Figure 3.15.

Now we want to know how can we combine these construction pieces to build possible minimal counterexamples to BBCE_3^T. We will use the following lemmas to exclude some combinations.

Lemma 11. *4, 5 and 6 can not be followed by any piece in a minimal counterexample to* $BBCE_3^T$ *for minors of wheels.*

Proof. If we put any piece after 4, 5 or 6 we end up with a non-transcendental vertex of degree 2 (different from u and v), which can not exist in a minimal counterexample by Lemma 4. □

Lemma 12. *Neither of the combinations 12, 15, 16, 21, 25, 26, 11, 22 can exist in a minimal counterexample to* $BBCE_3^T$ *for minors of wheels.*

Proof. For the case 12 we have that the part in Figure 3.16 exist in the graph. There is a vertex ($x \neq u, v$) of degree three with two transcendental neighbours which can not exist in a minimal counterexample by Lemma 7. The proofs of the other cases are analogoues to the proof of the case 12. □

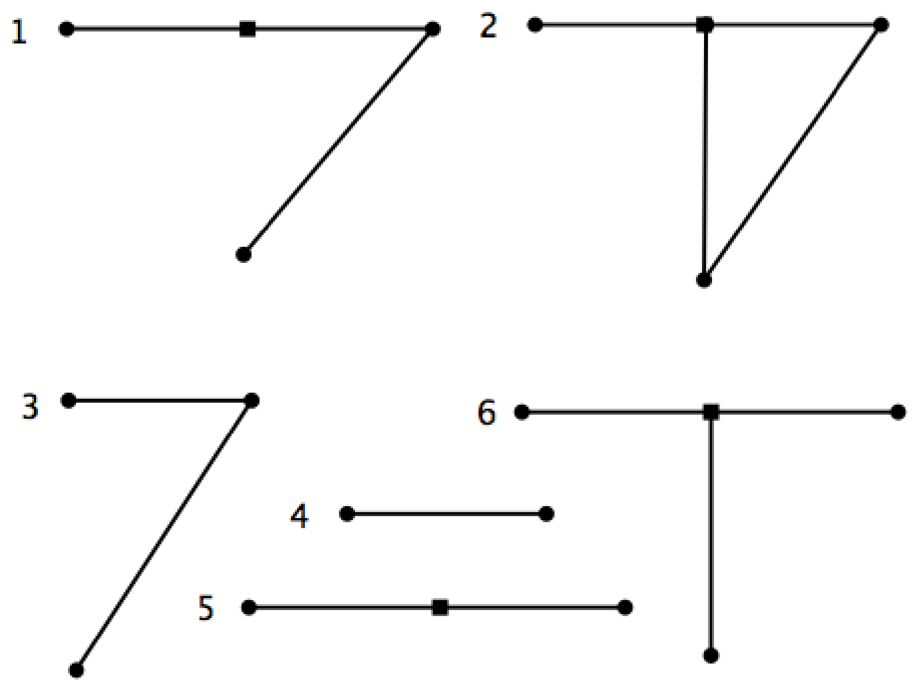

FIGURE 3.14. The six construction pieces for minors of wheels. The squares stands for transcendental nodes and the circles for non-transcendental nodes.

Lemma 13. *The combinations 133, 23, 332, 336 and 333 can not exist in a minimal counterexample to* $BBCE_3^T$ *for minors of wheels.*

Proof. Again we will only prove the lemma rigorously for one case, the other follows in the same way. Take 133, the graph in Figure 3.17. We have a nontranscendental vertex of degree three, different from u end v, contained in two triangles. Such a vertex can not exist in a minimal counterexample due to Lemma 6.

□

Lemma 14. *The combinations 131, 132, 231, 135, 136 and 232 can not exist in a minimal counterexample to* $BBCE_3^T$ *for minors of wheels.*

Proof. All of the combinations (131, 132, 231, 135, 136 and 232) gives a triangle xyz with x non-transcendental and both x and y having degree three, different from u and v. And such that the vertices, x' and y', adjacent to x and y respectively are transcendental. This can not exist in a minimal counterexample to $\mathrm{BBC}E_3^T$ by Lemma 8. □

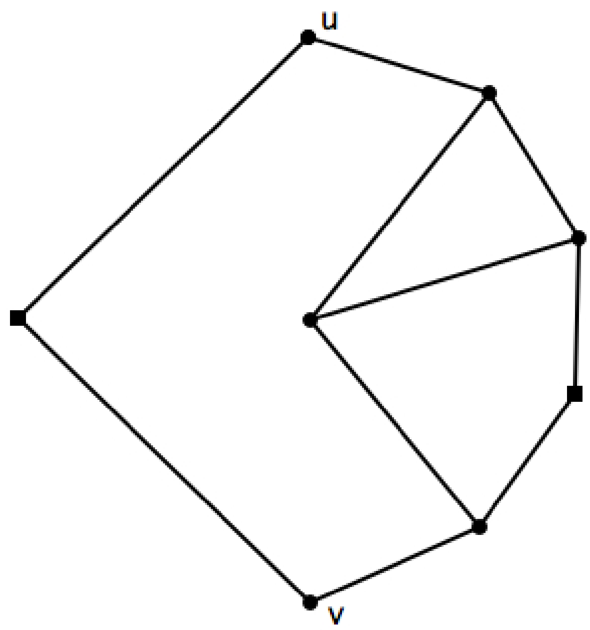

FIGURE 3.15. The construction $u3314v5u$, where the squares are considered transcendental and the circles non-transcendental.

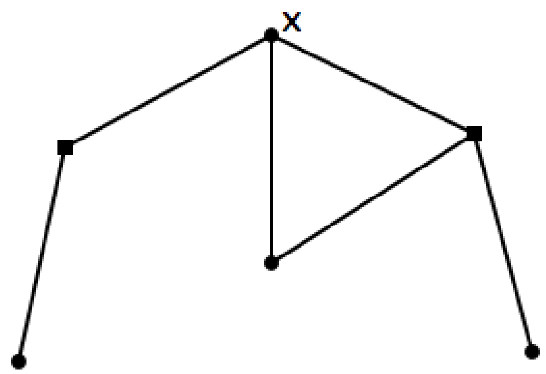

FIGURE 3.16. The combination 12, as usual the squares markes the transcendental nodes.

We can now state the theorem below.

Theorem 6. *All possible minimal counterexamples to $BBCE_3^T$ that are minors of wheels (not wheels) is of the form $uXvYu$ where X and Y are some of the combinations listed below, and u and v are different from the center.*

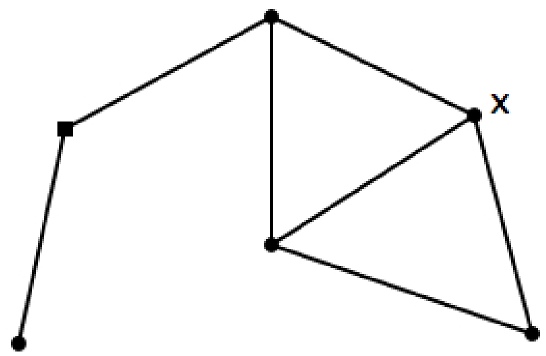

FIGURE 3.17. The combination 133, as usual the squares markes the transcendental nodes.

1	*5*	*33*	*34*	*334*	*134*	*3313*
2	*6*	*35*	*13*	*335*	*314*	*33134*
3	*31*	*14*	*36*	*313*	*3314*	
4	*32*	*24*	*324*	*331*	*3134*	

Proof. We will go through all combinations of pieces by start looking at all start pieces.

- **1:** For the first case, let us start by 1. We can of course have 1 alone. By Lemma 12 1 can not be followed by 1, 2, 5 or 6. Thus we can only continue by 3 or 4.
  - **13:** If we start by 13 we have of course the combination 13. By Lemma 14 we can not continue with 1, 2, 5 or 6 and by Lemma 13 the combination 133 is not allowed. The only way to continue is with 4.
    - **134:** We have the combination 134 and can not continue due to Lemma 11.
  - **14:** 14 is a valid combination and we can not continue due to Lemma 11.
- **2:** We have 2 alone as a valid piece. We can not continue with 1, 2, 5 or 6 by Lemma 12. Also 23 is a non-valid combination by Lemma 13. We can only continue with 4.
  - **24:** 24 is a valid combination and we can not continue due to Lemma 11.
- **3:** 3 alone is valid. We can continue with any of the six pieces.
  - **31:** From above (when we list the combinations starting with 1) we can continue with 3 and 4. 31 alone is of course also valid.
    - **313:** 313 is a valid combination. By Lemma 14 we can not add 1, 2, 5 or 6. By Lemma 13 we can not continue by 3 either. 4 is the only piece possible to continue with.

3134: We have 3134 but can not continue by Lemma 11.

314: 314 is a valid combination but by Lemma 11 we can not continue.

32: From above we find that 2 can only be followed by 4.

324: We have the combination 324 but we can not add any piece by Lemma 11.

33: We have 33 alone. By Lemma 13 we can not continue with 2, 3 or 6. 1, 4 and 5 are valid to continue with.

331: 331 alone is valid. According to above we can continue with 3 and 4.

3313: 3313 is a valid combination. We can not add 1, 2, 5 or 6 by Lemma 14. By Lemma 13 we can not continue with 3. The only possibility is to add 4.

33134: 33134 is valid and by Lemma 11 we can not continue.

3314: 3314 is a valid combination. By Lemma 11 we can not continue.

334: We have the combination 334 but can not add another piece by Lemma 11.

335: 335 is a valid combination but we can not continue by Lemma 11.

34: 34 is valid. By Lemma 11 we can not continue with any piece.

35: 35 is a valid combination but we can not continue according to Lemma 11.

36: We have 36 but can not continue by Lemma 11.

4: 4 alone is valid but according to Lemma 11 we can not continue with any piece.

5: 5 is valid and by Lemma 11 we can not continue.

6: 6 alone is valid but we can not continue by Lemma 11. □

We find that in all 26 combinations above there are at most one transcendental nodes. In 3 and 4 there is no transcendental nodes and hence if X or Y is equal to 3 or 4 the graph $uXvYu$ can not be a minimal counterexample to BBCE_3^T by Lemma 9 since the graph, $uXvYu$, has at most one transcendental node. Now we have at most 24^2 graphs that can be minimal counterexamples to BBCE_3^T for wheels (we choose two, not necessarily different, combinations X and Y).

Proof. **BBC for wheels.** Let G be a minor of a wheel, and a minimal counterexample to BBCE_3^T. First, a wheel can not be a minimal counterexample to BBCE_3^T by Lemma 3. By Lemma 10 we have that the center vertex in G can not be transcendental. A possible graph with u (or v) in the center can be written uXu (vXv). By Lemma 9 neither u nor v can be in the center since every combination X contains at most one transcendental node. We have 24^2 cases left according to Theorem 6, some of them are outerplanar or can be omitted by some other lemma. To make it easy all these 24^2 cases was checked by computer simulations. □

As a direct consequence we also have:

Corollary 1. ***(To Theorem 4)*** *BBC holds for all minors of wheels.*

To prove the bunkbed conjecture also for graphs with only one inner vertex it is probably possible to use the lemmas of this section to get a finite number of possible minimal counterexamples to check. This does not seem like a very good method to prove BBC for larger graphs, it is mostly lots of computations.

3.3. **Series-parallel graphs.** While trying to prove the bunkbed conjecture for series-parallel graphs we found lots of interesting results. The results concerning the bunkbed conjecture will be presented in this section and some other results will be presented in Section 7. The definition of series-parallel graph that will be used here, and is the most common, is the following.

Definition 1. A multigraph G with no loops is a series-parallel (SP for short) multigraph if it can be generated from an edge by the operations of subdividing an edge, i.e. replacing it by two edges (series) and doubling an edge, i.e. replacing it by two edges (parallel).

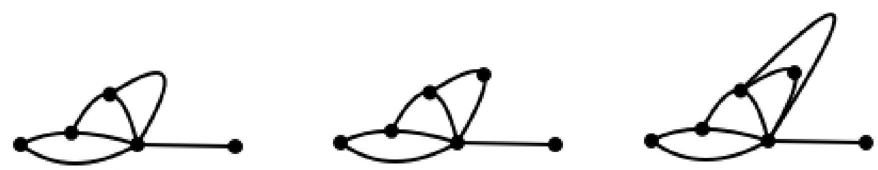

FIGURE 3.18. This is an example of a construction of a series-parallel graph using series and parallel operations.

Definition 2. A simple series-parallel graph is a series-parallel graph without multiple edges.

For the non-simple graphs we have the following lemma.

Lemma 15. *A non-simple graph can not be a minimal counterexample to $BBCE_3^T$.*

Proof. Let G be a minimal counterexample to $\mathrm{BBC}E_3^T$ and assume G is non-simple. Then there must be at least one double edge, $\{e_1, e_2\}$. We have two cases. If e_1 and e_2 have the same color we may just remove one of them to obtain a smaller graph for which the statement is true by assumption. If e_1 and e_2 have different color we may contract the edges and again obtain a smaller graph for which the statement is true by assumption. □

The alternative to this definition is that a series-parallel graph is constructed from a double edge using series and parallel operations. The alternative construction require the graph to be 2-connected. We have the following well-known theorem proved by Duffin in [4]. Recall that a subdivision of a graph G is a graph obtained by repeatedly subdividing edges of G.

Theorem 7. *A 2-connected graph is series-parallel if and only if it does not contain a subdivision of K_4.*

We will now introduce another useful way to look at series-parallel graphs, to be able to prove conjecture 3 for a subclass of SP graphs.

Lemma 16. *A 2-connected graph consisting of at least 3 vertices is series-parallel iff it can be constructed from a cycle with non-crossing cords (a 2-connected outerplanar graph), replacing some (or all) of the cords with connected series-parallel graphs.*

Note that a cord, that is just an edge between two vertices, is a series-parallel graph itself so we can say that we change all the cords to connected series-parallel graphs.

Proof. For the first direction, let G be a 2-connected series-parallel graph. Then G can be constructed using the operations of series and parallel. By taking the biggest outer cycle from this construction G can be constructed by putting in 2-connected series-parallel cords in the cycle.

For the other direction, we know that the cycle with some cords (only edges here) is series-parallel, since it is outerplanar. And since any 2-connected series-parallel graph can be constructed from one (double) edge using series and parallel operations the graph must be series-parallel, se Figure 3.19. □

We will use this characterization of series-parallel graphs to prove that $\mathrm{BBC}E_3^T$ is true for a subset of the series parallel-graphs.

Definition 3. The series-parallel graphs constructed from a cycle with cords, where connected series-parallel graphs are placed parallel to some cords will be called SP_0 graphs.

The difference from before is that we require the cord to stay, this will be necessary for the proof method we use.

We are now ready to state and prove one of the main results of this thesis, a generalization of the bunkbed conjecture for SP_0 graphs.

Theorem 8. *Let G be a SP_0 graph with every edge colored either red or blue with equal probability. Then G can not be a minimal counterexample to $BBCE_3^T$.*

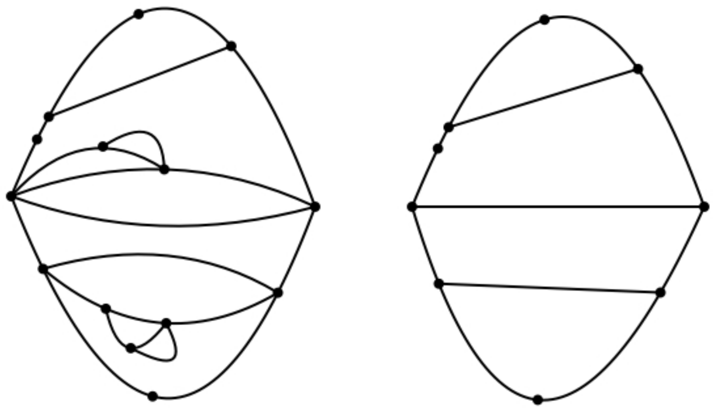

FIGURE 3.19. If the graph to the right is series-parallel then so is also the one to the left.

Proof. To get a contradiction, assume that G, together with some set of transcendental nodes, is a minimal counterexample. Call the two paths from u to v on the outer cycle outer paths. We start by proving that u must have degree two. Assume $\deg(u) > 2$, then there must be a cord (since we kept all the cords) that does not separate u from v, and one such x is as close to u as possible along the outer path. Then by using Lemma 4 we have that all vertices different from u and v of degree two are transcendental. And also by using the T-operation we find that there must be a triangle with two vertices on the outer path from u to x, with one vertex of degree two. By using the $\Delta-$operation we find that since G is a minimal counterexample all the edges in the triangle can be assumed to have the same color. We may then remove the vertex of degree two without altering any probability. Hence u must have degree two.

Let x and y be the two neighbours of u. Note that $u \notin T$. If any of the edges ux, uy are blue we may remove it (since they can never be used), and contract the red edge (if one) to obtain a smaller graph without alterning the probability, and G can not be a minimal counterexample. Hence both ux and uy must be red.

Now if xy is not an edge then one of x and y have degree two, say y. If not there is a cord not separating u from v and a minimal such. Since G is a minimal counterexample this is not possible for the same reason as above. By contracting uy we obtain a minor of G with the restriction of ux being red. This will be handled later.

If on the other hand xy is an edge in G we have two cases. Either the edge is red or it is blue. If xy is blue we may contract xy since we can walk between x and y along both a red and a blue path. We may also contract uxy, since the edge is red, and we obtain a minor of G for which the theorem is true. If xy is red we may contract uy (we can assume that y had degree 2) without altering the probability. Again we end up with a minor of G with the condition that ux is red.

For G to be a minimal counterexample then some of the minors with the restriction that one edge from u is red must be. And hence there must be a minimal such graph. For such a graph to be a minimal counterexample, again if the other edge from u is blue we may remove it and contract the red edge. Hence we have the case where both the edges from u are red. This is handled above. □

The idea of the proof above can of course be used on all graphs constructed from a cycle with cords, were any graph can be put parallel to the cords. Linusson used this idea to prove the conjecture for outerplanar graphs (and all minors that are again outerplanar).

3.4. **Minimal counterexamples - summary.** To summarize we now have an idea of how a minimal counterexample must look. We can not have any of the following:

(1) Adjacent transcendental nodes.
(2) $u \in T$.
(3) $v \in T$.
(4) Non-transcendental vertices of degree two, different from u and v.
(5) $T \subset V(G)$ containing a cutset of G that separates u from v.
(6) A non-transcendental vertex $x \in G$ such that x has degree three and such that $x \neq u, v$ belongs to two triangles.
(7) A non-transcendental vertex of degree three, different from u and v, with two transcendental neighbours.
(8) A triangle xyz with x and y non-transcendental of degree three, different from u and v, and such that the other vertices, x' and y', adjacent to x and y respectively are transcendental.
(9) An outerplanar graph.
(10) A SP_0 graph.
(11) A wheel.
(12) A minor of a wheel.
(13) A non-2-connected graph.
(14) A triangle with a vertex of degree two, different from u and v.
(15) $|T| = 0, 1$.

We can use these facts to see that small graphs can not be minimal counterexamples. For graphs with only two nodes it is clear. For a graph with three nodes we have either a triangle, no transcendental nodes or a transcendental cutset. Also graphs on three vertices are outerplanar. Continuing to larger graphs we have the following by using the other facts and computing the probabilities in a few small cases.

Lemma 17. *No graph with at most four vertices can be a minimal counterexample to* $BBCE_3^T$*, and hence BBC holds for graphs with at most four vertices.*

Proof. Let G be a minimal counterexample to $\mathrm{BBC}E_3^T$. All graphs on at most four vertices are outerplanar but K_4. Thus, if G is a minimal counterexample it must be K_4. Since we can not have a non-transcendental vertex different from u and v contained in two neighbour triangles by Lemma 6, both the vertices different from u and v must be transcendental. But since these two verices are neighbours G can not be a minimal counterexample by Lemma 3. $\square$

By using some of the criteria for minimal counterexamples, and some computer simulations we also have the theorem below.

Theorem 9. *BBC holds for graphs with at most five vertices.*

4. A set of start points

One can also analyze whether or not the bunkbed conjecture is true when we have a set of startpoints or endpoints (or both). To me the following two conjectures seems likely to be true.

Conjecture 4. *Let G be a finite graph and let $H = G \times K_2$ be the corresponding bunkbed graph then*

$$P_S(S_0 \leftrightarrow T_0) \geq P_S(S_0 \leftrightarrow T_1).$$

where S_0 and T_0 is a set of vertices in the zero-layer and T_1 in the one-layer.

And also we have the following conjecture for the red-blue model.

Conjecture 5. *Let G be a finite graph with every edge colored red or blue with equal probability. Then*

$$P_S(S_0 \leftrightarrow T_0) \geq P_S(S_0 \leftrightarrow T_1).$$

Where the index 0 (1) stands for starting / ending in a red (blue) edge or the start / end points are transcendental, analogous with earlier.

Intuitively it seems like the proof method used in the proof of Theorem 8 can be used for outerplanar graphs for the two conjectures above. The problem is that we can not always be sure there is a $u \in S$ of degree two with a neighbor of degree two.

5. Directed acyclic graphs

Another way to prove that the conjecture holds for a graph would be to define a map from all the colorings containing only blue paths to a subset of the set of colorings containing only red paths, and finding the inverse. This tends to be easy for most concrete examples of graphs but it seems hard to do it otherwise. For example for some graphs you may define the map to change color on the blue edges, on the blue paths from v and to the first trancendental node. If this gives any extra red paths we change the color to blue on some edges in those paths. This idea will be useful when talking about directed graphs.

Model 3. ***(Model*** *D) Let G be a finite acyclic directed graph with vertices u and v such that no edges are directed from v and no edges are directed to u, and such that all edges are colored either red or blue with equal probability. A path may change color at a vertex in $T \subset V(G)$.*

Note that we can assume that u is the unique source and that v is the unique sink. If there were another source or sink we may just remove it since it can never be used.

Conjecture 6. ***(BBCD).*** *Let G be a finite graph in model D. Then for any $u, v \in V(G)$, with the properties of model D, we have*

$$P_D(u_0 \leftrightarrow v_0) \geq P_D(u_0 \leftrightarrow v_1),$$

where v_0 means ending up in a red edge and v_1 in a blue (or $v \in T$). We always start with a red edge (or $u \in T$), as usual.

This conjecture also seems likely to be true. It is also likely that the conjecture is true for all directed graphs (not only acyclic). We will prove the conjecture to be true for some special sets of transcendental nodes.

Theorem 10. *If $T \subset V(G)$ contains a cutset separating u from v or $u \in T$ or $v \in T$ then*

$$P_D(u_0 \leftrightarrow v_0) \geq P_D(u_0 \leftrightarrow v_1)$$

holds.

Also this theorem can be proven using the same mirror argument Linusson used to prove Lemma 1. And for the case when $T \subset V(G)$ contains a cutset separating u from v we can prove it in the following way.

Proof. Let $A \subset T \subset V(G)$ be a cutset separating u from v. Define a function f to fix the color of all the edges between u and the vertices in A and change the color on all the other edges. The inverse is trivial. And if we apply f on all the colorings containing only blue paths the image will be a subset of the colorings containing only red paths from u to v. With a blue (red) path we mean a path starting in u with a red edge an ending in v with a blue (red) edge. □

Se for example Figure 5.1 to understand the function f.

Theorem 11. *Conjecture 5.1 is true when subset $S \subset T$ can be expanded to a cutset A such that no paths from u to v contains a vertex in $A \backslash T$.*

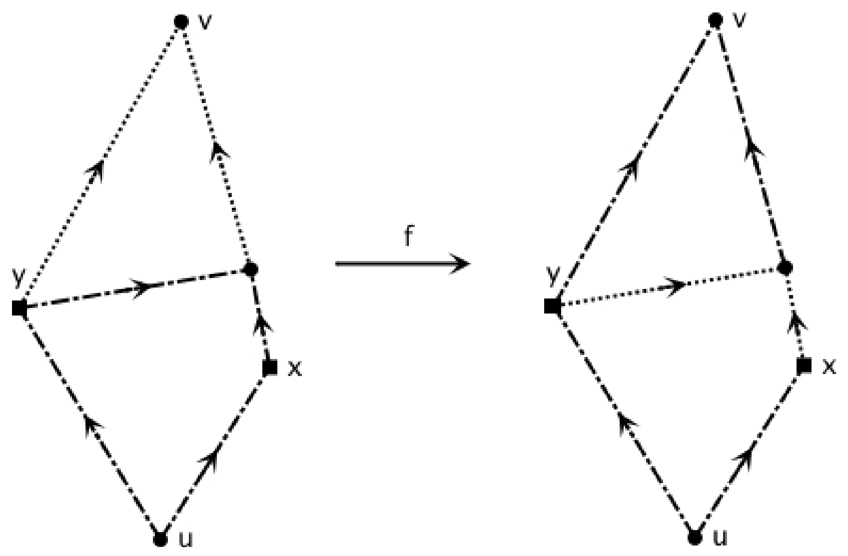

FIGURE 5.1. f is the function changing color on all the edges but them below the transcendental cutsen $\{x, y\}$.

Proof. We can use the same function as in the proof of Theorem 10 since the non-transcendental vertices in the cutset can never be used for a path from u to v. □

Theorem 12. *Conjecture 5.1 is true when T is an antichain.*

Proof. Let G be a finite graph in model D. Let T be the set of transendental nodes in G and assume that T forms an antichain of G. To prove the theorem for G we will define a function, f, from all the colorings containing only blue paths to (a subset of) the set of colorings containing only red paths.

Let f be the function changing the color on all edges included in paths (even though the path is not valid because of the coloring) from a transcendental node to v. The inverse is trivial. Since all blue paths must go through a transcendental node all blue paths will be changed into red, and no blue paths can arise. □

6. Another approach

To find a polynomial in p describing $P(u_0 \leftrightarrow v_0)$ and $P(u_0 \leftrightarrow v_1)$, it is easiest to count the number of ways to put out i edges in $\tilde{G}$ which gives paths $u \leftrightarrow v$, multiplying it by the probability of the graph having i edges and sum over all the possible number of edges in the graph. The probability that a bunkbed graph containing e_H edges, all of them existing with probability p, gets i edges is $p^i(1-p)^{e_H - i}$. But we are only interested in the coefficients in front of this number (for all i) to compare $P(u_0 \leftrightarrow v_0)$ and $P(u_0 \leftrightarrow v_1)$. Let us name the coefficients $\alpha_0, \dots, \alpha_{e_H}$ and $\beta_0, \dots, \beta_{e_H}$ respectively. We will call these coefficients probability constants.

Example 1. *Let, for giving an easy example, G be the graph consisting of two vertices and one edge connecting them. Let one of them be u and v the other.*

Figure 6.1. Here G consists of only two vertices with an edge between them, and the corresponding bunkbed graph, $\tilde{G}$ is in the figure.

The probability constants are listed below.

	$(1-p)^4$	$p(1-p)^3$	$p^2(1-p)^2$	$p^3(1-p)$	p^4
$P(u_0 \leftrightarrow v_0)$	*0*	*1*	*3*	*4*	*1*
$P(u_0 \leftrightarrow v_1)$	*0*	*0*	*2*	*4*	*1*

It seems likely to be true that all the probability constants for $u_0 \leftrightarrow v_0$ are bigger than the probability constants in front of $u_0 \leftrightarrow v_1$ respectively. This is computed to be true if the graph G contains no more than 4 vertices. Some random simulations has also been done, and they all gave the expected result. We formulate the following conjecture.

Conjecture 7. *Let G be a finite graph, and $\tilde{G}$ the corresponding bunkbed graph. Then for all the probability constants we have $\alpha_i \geq \beta_i$ for all choices of $u, v \in G$.*

Obviously BBC follows from this conjecture, and hence the bunkbed conjecture is true if the bunkbed graph contains no more than eight vertices.

Note also that the conjecture above is true for the first $n+1$ probability constants, where n is the least number of edges for a path from u_0 to v_0. We know that $\alpha_{n+1} \geq \alpha_n \cdot (|E(H)| - n)$ since we may put out the extra edge in all free positions without getting any repetitions. We also have that $\beta_{n+1} = \alpha_n \cdot (n+1)$, since we can choose any of the $n+1$ vertical edges to get to the 1-layer. We have at least $2n+n+1$ edges in the bunkbed graph and hence

$$\alpha_{n+1} \geq \alpha_n \cdot (|E(H)| - n) \geq \alpha_n \cdot (2n+1) > \alpha_n \cdot (n+1) = \beta_{n+1}.$$

It is not true in general that there are more ways to put out i edges in a graph and having a path from u to v than from u to x if the path with the least number of edges from u to v contains less edges than the one from u to x, or if the probability to get from u to v is greater than the probability to get from u to x. Take for example the graph in Figure 6.

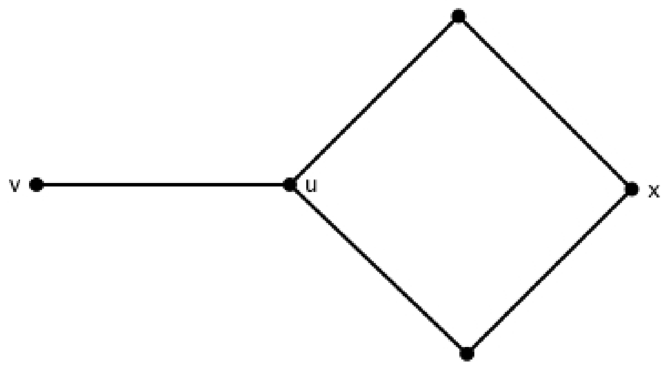

FIGURE 6.2. This graph is commonly used as a counterexample when talking about percolation and random walks.

In the case of figure 6 we have $P(u \leftrightarrow v) = p$ and $P(u \leftrightarrow x) = 2p^2 - p^4$ and hence for some p

$$P(u \leftrightarrow v) \geq P(u \leftrightarrow x).$$

Also we have $4 = \alpha_4 < \beta_4 = 5$ for the same example. Even though this is not true in general it seems likely to be true for bunkbed graphs.

7. Characterization of series-parallel graphs

While trying to prove the bunkbed conjecture for all series-parallel graphs we instead manage to prove the three theorems below. We also found a result (Theorem 16) that seemed useful. The idea was to use the definition of series-parallel graphs as a cycle with series-parallel cords, and use the theorem on the series-parallel cords but it failed for case (2).

The first result is most likely already known, but we have not found it anywhere. The same for Theorem 15.

Theorem 13. *A 3-connected graph with at least six edges can not be series-parallel.*

Proof. For the proof we use theorem 7 and find a K_4 minor. Let G be a 3-connected graph containing at least six edges and take any cycle C of G with minimal length. Since the graph G is 3-connected and has at least six edges, it must have a node, x, that does not lie on the cycle, C. By Menger's Theorem there are three node-disjoint paths from x to three distinct nodes of C, which we call a, b and c. By some deletions and contractions the cycle and the three paths can be reduced to a K_4 minor with x, a, b and c as vertices. $\square$

For the relations between series-parallel graphs and outerplanar graphs we found a nice way to prove the theorem below.

Theorem 14. *If G is a series-parallel graph without $K_{2,3}$ minors, then G is outerplanar.*

Proof. Let G be a connected SP graph without $K_{2,3}$ minors and let C be a cycle of G of maximum length. Suppose first that G has a node x that does not lie on C. Since G is 2-connected there exist two edge disjoint paths from x to distinct nodes y and z on C so that these paths have only the node x in common. If the edge yz exists and is an edge of C, then C can be extended to a longer cycle using the two paths which contradicts the maximality of C. If y and z are not joined by an edge of C then C and the two paths are easily reduced to a $K_{2,3}$ minor of G, which gives another contradiction. Hence all nodes of G must be nodes of C.

To prove that the graph is outerplanar we can, without loss of generality assume that G is a simple graph. Now draw C as a circle in the plane. Also draw the remaining edges as straight line segments. If any two such edges cross, then these edges plus C can be reduced to a K_4 minor of G which contradicts that G is series-parallel. Thus, no cords cross, and we have an outerplanar drawing of G in the plane. $\square$

Theorem 15. *Every series-parallel graph can be constructed from any two adjacent vertices of the graph, using operations of series and parallel.*

Proof. The proof is by induction over the number of operations done. Let G be any graph constructed by n operations, and assume that G can be constructed from any edge in the graph. Let G' be a graph constructed from G using one more operation. If it was the parallel operation it is clear that also G' can be constructed from any edge in the graph. If it was the series operation on an edge $ab \in G$ we now have edges ax and xb in G' where

x is the added vertex. We know that G' can be constructed from any edge but ax and xb so we only need to concentrate on these two edges.

G' can be constructed from ax by first double the edge ax and then place a new vertex, b, on that new edge. This gives us the edge ab and we already know that G' can be constructed from this edge. The construction steps are listed in figure 7.1. If we do the same construction as for G we will end up with the graph $G' \cup ab$. But it is easy to realize that when G was constructed from ab the first step was to double the edge. By skipping this step we can construct G' from ab. The same works for xb. □

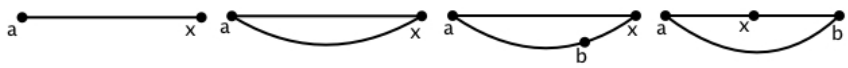

FIGURE 7.1. The construction stes used in theorem 15.

While trying to prove BBCE_3^T for series-parallel graphs we found the following theorem, in [9], that seemed useful.

Theorem 16. *Every (non-null) simple series-parallel graph G has one of the fol lowing:*

(1) a vertex of degree at most one,

(2) two distinct vertices of degree two with the same neighbors,

(3) two distinct vertices x, y and two not necessarily distinct vertices $w, z \in V(G)\backslash\{x, y\}$ such that the neighbors of y are x and w, and every neighbor of x is equal to y, w, or z, or

(4) ve distinct vertices y_1, y_2, x_1, x_2, w such that the neighbors of w are x_1, x_2, y_1, y_2, and for i = 1, 2 the neighbors of y_i are w and x_i.

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Department of Mathematics, Stockholm University, SE-106 91 Stockholm, Sweden

E-mail address: `madde@math.su.se`