SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

Traffic Flow theory

av

Xiaoyan Zhang

2012 - No 22

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

Traffic Flow theory

Xiaoyan Zhang

Självständigt arbete i matematik 30 högskolepoäng, avancerad nivå

Handledare: Sara Maad

2012

Abstract

The purpose of this paper is to introduce how to model traffic problems by using applied mathematics, We are not interested in individual cars, instead, we study the continuously distributed quantity ρ, which is the linear density of traffic on the road, measured in cars per kilometer. By the law of conservation of cars, and using several different speed-concentration models, we obtain the traffic flow model in PDE form. We solve these PDEs by the method of characteristics and numerically. We also introduce how to use this model to predict the timing of traffic lights.

Contents

1 Introduction **1**

2 Derivation of the model **2**

3 Speed-Concentration Models **3**
3.1 Normalized Models 5

4 The method of characteristics **6**

5 Characteristic curves for Greenshield's model **7**

6 Weak solutions **9**
6.1 Shock waves 13
6.2 Finding the shock wave by a numerical method 14

7 Entropy condition **19**

8 The Lax–Oleinik formula **22**
8.1 Entropy condition revisited 23
8.2 Uniqueness of entropy solutions 24

9 Riemann's problem **28**
9.1 Greenshields's model 30
9.2 Riemann problem for Edie's model 30

10 An application of the model: The timing of traffic lights **31**
10.1 Greenshields's model 32
10.2 Edie's model for traffic light 35

11 Acknowledgements **39**

1 Introduction

Traffic flow, in mathematics and civil engineering, is the study of interactions between vehicles, drivers, and infrastructure (including highways, signage, and traffic control devices), with the aim of understanding and developing an optimal road network with efficient movement of traffic and minimal traffic congestion problems. Traffic phenomena are complex and nonlinear, depending on the interactions of a large number of vehicles. Due to the individual reactions of human drivers, vehicles do not interact by simply following the laws of mechanics, but rather show phenomena of cluster formation and shock wave propagation, both forward and backward, depending on vehicle's density in a given area. Scientists approach the problem in three main ways, corresponding to the three main scales of observation in physics.

- Microscopic scale: At the most basic level, every vehicle is considered as an individual. An equation can be written for each, usually an ordinary differential equation (ODE). Cellular automation models can also be used where the road is discretised into cells which each contain a car moving with some speed, or are empty. The Nagel-Schreckenberg model is a simple example of a such a model. As the cars interact it can model collective phenomena such as traffic jams.
- Macroscopic scale: Similar to models of fluid dynamics, it is considered useful to employ a system of partial differential equations, which balance laws for some gross quantities of interest; e.g., the density of vehicles or their mean velocity.
- Mesoscopic (kinetic) scale: A third, intermediate possibility, is to define a function which expresses the probability of having a vehicle at a certain time in given position which runs with velocity. This function, following methods of statistical mechanics, can be computed using an integral-differential equation, such as the Boltzmann equation.

In this paper we will show you how to approach the problem by using the second method we mentioned above. By deriving a differential equation from physical principles and common sense, solving the equation, and then interpreting the answer as it refers to the phenomenon being modeled. The traffic flow model illustrates a widely-used approach to dynamic problems, an approach which often leads to system of partial differential equations in the form of conservation laws. In the case of my example here, the quantity is cars on a highway which is quite obviously discrete. However, suppose that we are not interested in the motion of individual cars, but only in some averaged quantities, for example, the carrying capacity of the road, which is the maximum number of cars per hour which the road can accommodate, or the number of times a car keeping up with the traffic will be stopped by a red light on a given stretch of road. In cases like this, one gets a good approximation, which appears to agree with the data by considering, instead of individual cars, a continuously distributed quantity ρ, the linear density of traffic on the road, measured in cars per kilometer. We suppose there is only one road that it is straight and uniform, and that traffic moves along it in one direction (left to right): a straight one-way highway with no intersections. Other embellishments can be added.

2 Derivation of the model

By the conservation law, we know that when physical quantities remain the same during some process, these quantities are said to be conserved. In our case, the number of cars in a segment of a highway are our physical quantities, and the process is to keep them fixed (i.e., the number of cars coming in equals the number of cars going out of the segment). If we assume that we have a stationary, one dimensional, infinite road, and we measure position along the road x denote $\rho(x,t)$ to be the number of cars at a point x and time t, and then examine a controlled section between x and $x+h$, since the number of cars in the controlled section is the integral of the density, the fundamental equation is

$$\frac{d}{dt}\int_x^{x+h} \rho(y,t)\mathrm{d}y = q(x,t) - q(x+h,t). \tag{1}$$

The quantity q is the linear flux of traffic. Flux is the number of cars passing the point x at time t per unit of time. So one can think of the units of q as the amount of traffic per hour, measured in cars per hour. We have a simple way of relating the flux to more familiar quantities in traffic: if the speed of the traffic is υ, in units of length per hour, then the amount of traffic passing a given x is just $\rho\upsilon$ cars per hour. Of course, the actual traffic, is composed of a lot of vehicles of different sizes, all traveling at different rates of speed, and so υ is a composite quantity which might be difficult to calculate. But in this simple model, just as there is a single quantity which represents density, there is also a single velocity, $\upsilon(x,t)$, at each point x and time t. Applying the mean value theorem for integrals for the left side of (1), we obtain

$$\frac{\partial}{\partial t}(h\rho(x^*,t)) = q(x+h,t) - q(x,t)$$

The value x^* is a point in the interval $(x, x+h)$. Divide by h to get

$$\frac{\partial}{\partial t}((\rho(x^*,t)) = \frac{q(x+h,t) - q(x,t)}{h}$$

and finally take the limit $h \to 0$, to obtain the fundamental conservation principle in the form of the partial differential equation:

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = \frac{\partial \rho}{\partial t} + \frac{\partial(\rho\upsilon)}{\partial x} = 0 \tag{2}$$

where we have substituted $q = \rho\upsilon$ to get the second form. Now we can see that the total amount of the traffic is exactly conserved by equation (2). However, we have obtained only an incomplete model, since equation (2) is a single equation with two unknowns, ρ and υ. In order to obtain a solvable model from (2), we need some other assumption. To be specific, in this case we assume that the velocity is a function of the density, that is $\upsilon = \upsilon(\rho)$, which means that on this particular road, the speed at which traffic moves is completely determined by its density. Such a model is an oversimplification of actual traffic, as it does not take into account the variations of individual drivers or different kinds of vehicles. However, in some respect, we find it

is quite realistic .

A number of such models have been proposed, but there is as yet no general agreement on which is "right" one. Roughly speaking, all of the models agree for certain types of traffic situation, but the following postulate is intuitively reasonable: v is a monotonically decreasing function of ρ , with a maximum at $\rho = 0$. At some value ρ_0, v become zero: ρ_0 is a critical density above which traffic cannot move.

3 Speed-Concentration Models

Now let us see several well known Speed-Concentration Models which we will use in this paper:

- Greenshield's model: the simplest and also perhaps the most obvious of such relationships is the linear relationship as proposed by Greenshield

  $$v = v_f(1 - \rho/\rho_j)$$

  where v_f is the free-flow speed and ρ_j is the jam density,

  $$q = \rho v = \rho v_f(1 - \rho/\rho_j).$$

  By differentiating this equation, setting $q' = 0$ we obtain conditions for maximum flow, and defining $q_m =$ maximum, $\rho_m =$ concentration at maximum flow, $v_m =$ speed at maximum flow, we can get

  $$\rho_m = \rho_j/2, \qquad v_m = v_f/2, \qquad q_m = v_f\rho_j/4 = v_m\rho_j/2.$$

  This model is simple to use and several investigators have found good correlation between the model and the field data.

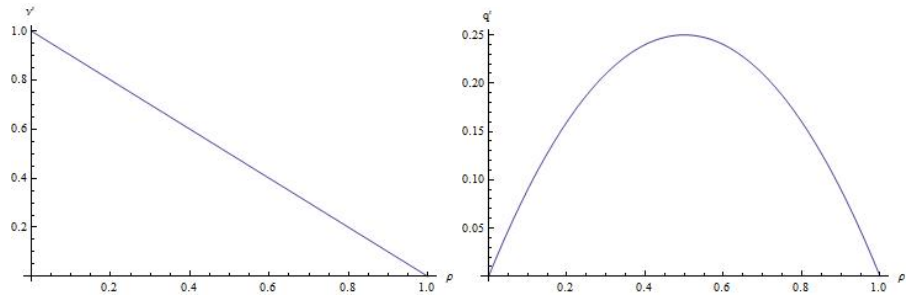

Figure 1: Greenshield's model

- Underwood's model: Underwood has demonstrated a model of the form

  $$v = v_f e^{-\rho/\rho_m}$$

where ρ_m is the concentration at maximum flow, and

$$q = \rho v = \rho v_f e^{-\rho/\rho_m}.$$

By using the same method we can get the conditions for maximum flow

$$\rho_m = \rho_m, \qquad v_m = v_f/e, \qquad q_m = v_f \rho_m/e$$

this model also has shortcomings in that it does not represent zero speed at high concentration. Here ρ_m is a parameter.

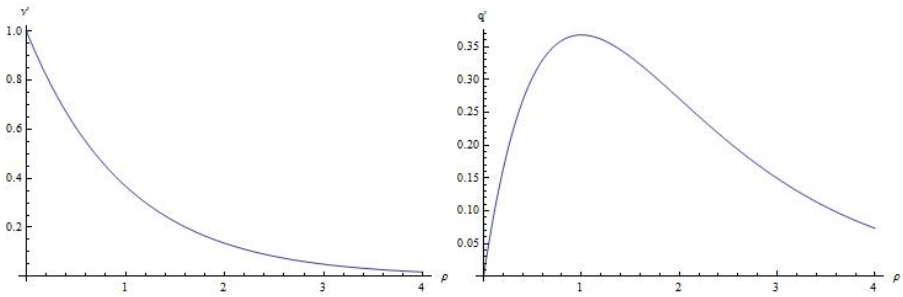

Figure 2: Underwood's model

- Greenberg's model: Greenberg, using a theoretical background, has postulated a speed-concentration model of the form

  $$v = v_m \log(\rho_j/\rho),$$

  where v_m is the speed at maximum flow,

  $$q = \rho v = \rho v_m \log(\rho_j/\rho).$$

  Again using differentiation to obtain conditions for maximum flow,

  $$\rho_m = \rho_j/e, \qquad q_m = v_m \rho_j/e.$$

  Greenberg found good agreement between this model and field data for congested flow. This model, however, breaks down at low concentrations, as may be seen by letting $\rho = 0$ in the model equation.

- Edie's model: as noted, Greenberg's model is useful for high concentrations and not for low concentrations. Conversely Underwood's model is useful for low concentrations and not for high concentrations, so Edie described a model that is a combination of Greenberg's model where useful for high concentrations and Underwood's model where useful for low concentrations. Note from Underwood and Greenberg's model we have $\rho_m = \rho_j/e$ and $v_m = v_f/e$, if we plug these two equations into Underwood and Greenberg's model respectively , we obtain $v = v_m(1 - \log(\rho/\rho_m))$ and $v = v_m e^{1-\rho/\rho_m}$. Notice that when $\rho = \rho_m$, $v = v_m(1 - \log(\rho/\rho_m)) = v_m e^{1-\rho/\rho_m} = v_m$, the two models become tangent.

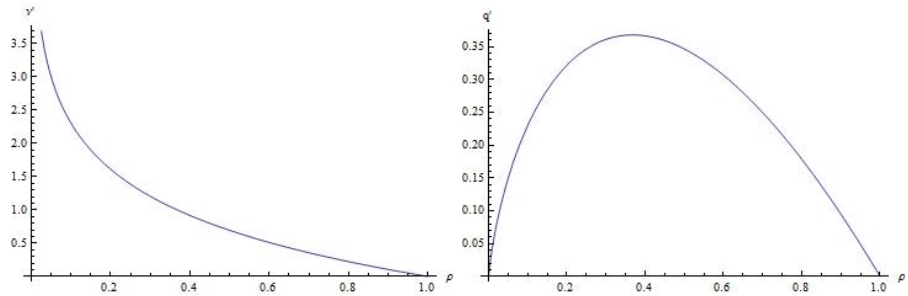

Figure 3: Greenberg's model

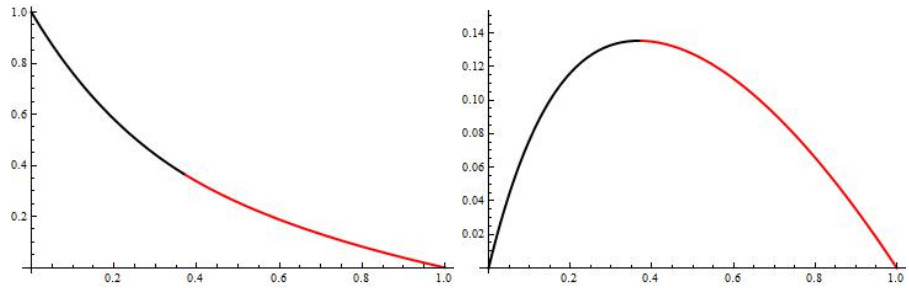

Figure 4: Edie's model

3.1 Normalized Models

Now I would like to show how to normalize the concentration and speed, and show the conservation law in dimensionless form i.e normalized form. Let us start by defining the dimensionless density

$$\tilde{\rho} = \frac{\rho}{\rho_j}$$

By substituting this into the Greenshield's model conservation law we obtain

$$\partial_t(\rho_j\tilde{\rho}) + \partial_x(\upsilon_m\rho_j\tilde{\rho}(1-\tilde{\rho})) = 0.$$

Remembering that ρ_j and υ_m are constant, we have

$$\partial_t\tilde{\rho} + \upsilon_m\partial_x(\tilde{\rho} - \tilde{\rho}^2) = 0.$$

Introducing now a reference length L and a reference time T defined as $T = L/\upsilon_m$, we define the dimensionless length $\tilde{x}$ and time $\tilde{t}$

$$\tilde{x} = x/L$$
$$\tilde{t} = t/T$$

With this change of variables, the partial derivatives transform as follows

$$\partial_t = \partial_{\tilde{t}}\frac{d\tilde{t}}{dt} = \frac{\upsilon_m}{L}\partial_{\tilde{t}}$$

and

$$\partial_x = \partial_{\tilde{x}} \frac{d\tilde{x}}{dx} = \frac{1}{L} \partial_{\tilde{x}}$$

The dimensionless equation of traffic conservation in advection form, after multiplying with $\frac{L}{v_m}$ becomes

$$\partial_{\tilde{t}} \tilde{\rho} + \partial_{\tilde{x}} (\tilde{\rho} - \tilde{\rho}^2) = 0. \tag{3}$$

We apply the same method to Underwood's and Greenberg's model but use a different reference length eL, we can obtain the dimensionless traffic equation for Underwood's model

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial (\tilde{\rho} e^{(-e\tilde{\rho})})}{\partial \tilde{x}} = 0 \tag{4}$$

And dimensionless traffic equation for Greenberg's model Greenberg's model

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \frac{\partial (\tilde{\rho}(1/e - \log(\tilde{\rho})))}{\partial \tilde{x}} = 0 \tag{5}$$

For the origin Greenberg's and Underwood's models, they are tangent at $\rho = \rho_m$, if we divide both side of this equation by ρ_j,i.e., normalizing ρ_m, then we can get $\tilde{\rho} = \rho_m/\rho_j = 1/e$, which implies that the normalized model tangent at $\tilde{\rho} = 1/e$, then we obtain Edie's model in the following form: v $= \begin{cases} e^{(-e\rho)} & \text{if } 0 \leq \rho \leq 1/e, \\ -\frac{1}{e} \log \rho & \text{if } 1/e < \rho \leq 1 \end{cases}$

Notice that in the rest of paper we only use the normalized model which request that $\rho \leq 1$ and for simplicity of notation we still denote ρ as normalized density.

4 The method of characteristics

Now we consider equation (2) together with a bounded initial condition $\rho_0 = \rho_0(x_0, 0)$. Since $q(\rho)$ is a known, differentiable function of ρ, equation (2) becomes an example of a first-order quasilinear partial equation in conservation form, also known as a scalar conservation law.

As the first step we rewrite (2), and carry out the differentiation.

$$\frac{\partial \rho}{\partial t} + \frac{\partial q(\rho)}{\partial x} = \frac{\partial \rho}{\partial t} + q'(\rho) \frac{\partial \rho}{\partial x} = 0 \tag{6}$$

Now $q'(\rho)$ is another known function of the density, called the wave speed. We will get into more details about this wave speed later. First we need to solve equation (6). We can see that it is a quasilinear partial differential equation: First we can reduce this problem to an ODE along curves $x(t)$ along which

$$\frac{d}{dt} \rho(x(t), t) = \frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} \frac{dx}{dt} = 0 \tag{7}$$

By comparing with equation (7), we see that (8) holds if $x'(t) = q'(\rho)$, where ρ is a solution of (7). Since ρ is constant along such curves, also $q'(\rho)$ is constant and equal to $q'(\rho_0)$, where $\rho_0 = \rho_0(x_0)$, so we obtain

$$x = q'(\rho)t + x_0$$

Since ρ has the value $\rho_0(x_0)$ at the x-intercept, we must have the solution

$$\rho(x,t) = \rho_0(x - q'(\rho)t) \tag{8}$$

which gives $\rho(x,t)$ implicitly as long as we can solve the equation $x = q'(\rho)t + x_0$ for x_0 as a function of x (for fixed t). This we can do if x is a monotonically increasing function of x_0 (again for a fixed value of t). Now,

$$\frac{dx}{dx_0} = 1 + q''\rho_0' t \tag{9}$$

by the chain rule, where $q'' = q''(\rho_0(x_0))$ and $\rho_0' = \rho_0'(x_0)$. Note that at $t = 0$ the right side of the equation equals unity, and so the expression remains positive, at least for a small t. Notice that the solution correspond to "transporting" (without change) the initial data $h(x)$ along the x-axis at a speed $x'(t) = q'(\rho)$, The lines

$$x = q'(\rho(x_0))t + x_0 \tag{10}$$

are called the characteristic curves for

$$\frac{d\rho}{dt} + q'(\rho)\frac{d\rho}{dx} = 0.$$

Notice that the method of characteristic that introduced in this section is just a special case of it, For the general case see [1] Chapter 3.

5 Characteristic curves for Greenshield's model

If we apply the normalized Greenshield's model to our equation, then equation (6) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho - \rho^2)}{\partial x} = \frac{\partial \rho}{\partial t} + (1 - 2\rho)\frac{\partial(\rho)}{\partial x} = 0 \tag{11}$$

Here $q'(\rho) = 1 - 2\rho$ and our characteristic equation is

$$x = (1 - 2\rho_0(x_0))t + x_0 \tag{12}$$

Now we consider a smooth initial condition $\rho_0(x_0) = e^{-x_0^2}$. By equation (13) we can get the characteristic equation for Greenshield's model

$$x = (1 - 2e^{-x_0^2})t + x_0$$

According to the characteristic method and by equation (9) we can get the solution as long as we can solve the characteristic equation to find x_0 as a function of x (for fixed t).

By equation (10) we have

$$\frac{dx}{dx_0} = -4tx_0e^{-x_0^2} + 1$$

Note that $x'(x_0)$ is never 0 for $x_0 \geq 0$, for $x_0 < 0$ it becomes 0 at x_0 when $t = -e^{-x_0^2}/4x_0$. We want to find the first time for which $x'(x_0) = 0$ for some x_0, so we need to find the when derivative of the RHS of $t = -e^{x_0^2}/4x_0$ is equal to 0, so we get $x_0 = \pm 1/\sqrt{2}$, we are only interested in $x_0 < 0$, so we achieve $x_0 = -1/\sqrt{2}$ and $t = \sqrt{2}e^{1/2}/4$. In Figure 5, see that the critical point of $x_0 \mapsto -e^{-x_0^2}/4x_0$ is a global minimum for $x_0 \leq 0$.

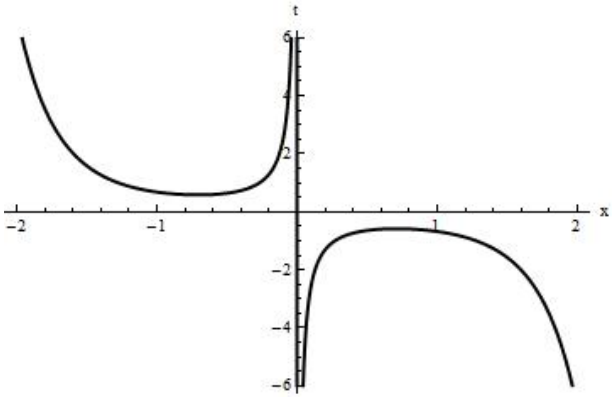

Figure 5: $t = -e^{-x_0^2}/4x_0$ (minimum t)

we have proved that $x_0 \mapsto (1 - 2\rho_0(x_0))t + x_0$ is monotonically increasing when $t \leq t_0 = \sqrt{2}e^{1/2}/4$ we have the solution

$$\rho(x,t) = \rho_0(x - (1 - 2e^{-x_0^2})t)$$

By the Figure 6, we can see that after $\frac{\sqrt{2}e^{1/2}}{4}$, the characteristics start crossing each other, which means, after a certain time, even for a smooth initial data, the solution $\rho(x,t) = \rho_0(x - (1 - 2e^{-x_0^2})t)$ is not valid .

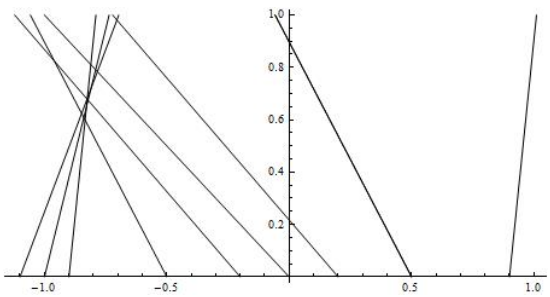

Figure 6: Characteristics

Why do the characteristic curves start crossing each other? What caused this problem?

Actually, for a quasilinear PDE, no matter the initial condition, smooth or not, the PDE itself can produce discontinuities to the solution. This discontinuity, called a shock, in the traffic problem, it is a demarcation line between regions of light traffic and of heavy traffic. In the next section we will introduce how can we get the solution for this kind of quasilinear PDE and how to deal with this shock phenomenon.

Remark 5.1. *For the initial condition is not C^1, the quasilinear PDE does not admit a smooth solution. However, even when ρ_0 is not continuous, the formula (9) does provide a reasonable candidate for a solution, and we call this kind of solution a* ***weak solution****. This makes sense even if ρ_0 is discontinuous in which case ρ will also be discontinuous. This notion is very useful in nonlinear PDEs, we will get into more details in next section.*

6 Weak solutions

Now we have several questions in mind. What does the intersections of the characteristics mean? And what will happen after the intersection? How can we get a reasonable solution for the quasilinear PDE?

In order to understand and compute these discontinuous solutions, we need to extend the notion of solution itself. Now we will introduce the notion of a solution ρ to a partial differential equation where ρ may not even be continuous. This type of solution will be known as weak solution of PDE. First, however, we define the notion of classical solution of the PDE: If the function ρ is C^1 and satisfies (13), we say that ρ is a classical solution. However, as described in the example above, sometimes a classical solution does not always exist, or you may want to allow for "solutions" which are not differentiable or even continuous. What do we mean by this?

In this section we will define the notion of a weak solution of a first-order, quasilinear initial-value problem of the form

$$\begin{cases} \rho_t + G(\rho)_x = 0 & x \in \mathbb{R}, t > 0 \\ \rho(x,0) = h(x). \end{cases} \tag{13}$$

here in order to be consistent with other material we write G as q . Before doing so, however, we give some preliminary definitions. We say that a subset of $\mathbb{R}$ is compact if it is closed and bounded. We say that a function $v : \mathbb{R}^2 \to \mathbb{R}$ has compact support if $v \equiv 0$ outside some compact set. We say that all smooth functions $v \in C^\infty(\mathbb{R} \times [0, \infty))$ with compact support are called test function. Now, we say that ρ is a weak solution of (13) if

$$-\int_0^\infty \int_{-\infty}^\infty [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t = \int_{-\infty}^\infty h(x) v(x,0) \, \mathrm{d}x \tag{14}$$

for all test functions $v \in C^\infty(\mathbb{R}\times[0,\infty))$. Notice that a function ρ need not be differentiable or even continuous to satisfy (15). Functions ρ which satisfy (15) may not be classical solutions of (13). However, they should satisfy (13) in some sense. Where did (15) come from? So far, we have just made a definition. We will now prove that if ρ is a classical solution of (13), then ρ is a weak solution of (13); that is, ρ will satisfy (15). In this sense, condition (15) is a natural extension of the notion of a "solution" to (13).

Theorem 6.1. *If ρ is a classical solution of* (13)*, then ρ is a weak solution of* (13)*, moreover if ρ is a weak solution which is C^1, then ρ is a classical solution of (14).*

Proof. If ρ is a classical solution of (13), then ρ is continuously differentiable and

$$\begin{cases} \rho_t + G(\rho)_x = 0 & t > 0 \\ \rho(x,0) = h(x). \end{cases}$$

Hence, for any smooth function $v : \mathbb{R} \times [0,\infty] \to \mathbb{R}$ with compact support,

$$\int_0^\infty \int_{-\infty}^\infty [\rho_t + G(\rho)_x] v \, \mathrm{d}x\mathrm{d}t = 0 \tag{15}$$

Integrating (16) by parts and using the fact that v vanishes at infinity, we see that

$$-\int_0^\infty \int_{-\infty}^\infty [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t = \int_{-\infty}^\infty h(x) v(x,0) \, \mathrm{d}x$$

But this is true for all functions $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support. Therefore, ρ is a weak solution of (13). The second part of the theorem we can prove it by simply rewrite the above proof the other way around. □

As mentioned above, the notion of weak solution allows for solutions ρ which need not even be continuous. However, weak solutions ρ have some restrictions on types of discontinuities, etc. For example, suppose that ρ is a weak solution of (13) such that ρ is discontinuous across some curve $x = \xi(t)$, but ρ is smooth on either side of the curve. Let ρ_l be the limit of ρ approaching $(\xi(t), t)$ from the left and let ρ_r be the limit of ρ approaching $(\xi(t), t)$ from the right. We claim that the curve $x = \xi(t)$ cannot be arbitrary, but rather there is a relation between $x = \xi(t)$, ρ_l and ρ_r.

Theorem 6.2. *If ρ is a weak solution of* (13) *such that ρ is discontinuous across the curve $x = \xi(t)$ but ρ is smooth on either side of $x = \xi(t)$, then ρ must satisfy the condition*

$$\xi'(t) = \frac{G(\rho_r) - G(\rho_l)}{\rho_r - \rho_l} \tag{16}$$

across the curve of discontinuity, where ρ_l is the limit of ρ approaching (x,t) from the left and ρ_r is the limit of ρ approaching (x,t) from the right.

Proof. If ρ is a weak solution of (13), then

$$\int_0^\infty \int_{-\infty}^\infty [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t = \int_{-\infty}^\infty h(x) v(x,0) \, \mathrm{d}x,$$

for all smooth function $v : \mathbb{R} \times [0,\infty] \to \mathbb{R}$ with compact support. Let v be a smooth function such that $v(x,0) = 0$, and break up the first integral into the regions Ω^-, Ω^+ where

$$\Omega^- \equiv \{(x,t) : 0 < t < \infty, -\infty < x < \xi(t)\}$$
$$\Omega^+ \equiv \{(x,t) : 0 < t < \infty, \xi(t) < x < +\infty\}$$

Therefore,

$$\int_0^\infty \int_{-\infty}^\infty [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t + \int_{-\infty}^\infty h(x) v(x,0) \, \mathrm{d}x$$
$$= \iint_{\Omega_-} [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t + \iint_{\Omega_+} [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t. \quad (17)$$

Combining the Divergence Theorem with the fact that v has compact support and $v(x,0) = 0$, we have

$$\iint_{\Omega_-} [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t = -\iint_{\Omega_-} [\rho_t + G(\rho)_x] v \, \mathrm{d}x\mathrm{d}t + \int_{x=\xi(t)} [\rho^- v \nu_2 + G(\rho^-) v \nu_1] \, \mathrm{d}S, \quad (18)$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to Ω_-. Similarly, we see that

$$\iint_{\Omega_+} [\rho v_t + G(\rho) v_x] \, \mathrm{d}x\mathrm{d}t = -\iint_{\Omega_+} [\rho_t + G(\rho)_x] v \, \mathrm{d}x\mathrm{d}t - \int_{x=\xi(t)} [\rho^+ v \nu_2 + G(\rho^+) v \nu_1] \, \mathrm{d}S \quad (19)$$

By assumption, ρ is a weak solution of

$$\rho_t + G(\rho)_x = 0$$

and ρ is smooth on either side of $x = \xi(t)$. Therefore, ρ is a classical solution on either side of the curve of discontinuity. Consequently, we see that

$$\iint_{\Omega_-} [\rho_t + G(\rho)_x] v \, \mathrm{d}x\mathrm{d}t = 0 = -\iint_{\Omega_+} [\rho_t + G(\rho)_x] v \, \mathrm{d}x\mathrm{d}t.$$

Combining this fact with (18), (19) and (20), we see that

$$\int_{x=\xi(t)} [\rho^- v \nu_2 + G(\rho^-) v \nu_1] \, \mathrm{d}S - \int_{x=\xi(t)} [\rho^+ v \nu_2 + G(\rho^+) v \nu_1] \, \mathrm{d}s = 0.$$

Since this is true for all smooth functions v, we have

$$\rho^- \nu_2 + G(\rho^-) \nu_1 = \rho^+ \nu_2 + G(\rho^+) \nu_1,$$

which implies

$$\frac{G(\rho^-) - G(\rho^+)}{\rho^- - \rho^+} = -\frac{\nu_2}{\nu_1}.$$

Now the curve $x = \xi(t)$ has slope given by the negative reciprocal of the normal to the curve; that is

$$-\frac{\nu_2}{\nu_1} = \frac{dt}{dx} = \frac{1}{\xi'(t)}$$

Therefore

$$\xi'(t) = -\frac{\nu_1}{\nu_2} = \frac{G(\rho^-) - G(\rho^+)}{\rho^- - \rho^+}.$$

Therefore, by theorem 6.1 we know that if the solution ρ has a discontinuity along a curve $x = \xi(t)$, then the solution ρ and the curve $x = \xi(t)$ must satisfy the condition

$$\xi'(t) = \frac{G(\rho^-) - G(\rho^+)}{\rho^- - \rho^+}.$$

And if we use direction to denote ρ^+ and ρ^- we can write the condition as

$$\xi'(t) = \frac{G(\rho_r) - G(\rho_l)}{\rho_r - \rho_l}. \tag{20}$$

□

For shorthand notation, we define

$$[\rho] = \rho_r - \rho_l$$

$$G[(\rho)] = G(\rho_r) - G(\rho_l)$$

$$\sigma = \xi(t)$$

We call $[\rho]$ and $[G(\rho)]$ the jumps of ρ and $G(\rho)$ across the discontinuity curve $x = \xi(t)$ and σ the speed of the curve of discontinuity. Therefore, if ρ is a weak solution with discontinuity along a curve $x = \xi(t)$, the solution must satisfy

$$[G(\rho)] = \sigma[\rho].$$

This is called the Rankine-Hugoniot jump condition.

6.1 Shock waves

Now since we have the weak solution of the quasilinear PDE, we can now consider a piecewise linear function as initial data for Greenshield's model:

$$\rho_0(x_0) = \begin{cases} 1/2 & \text{if } |x_0| \geq 1, \\ \frac{2-|x_0|}{2} & \text{if } |x_0| \leq 1. \end{cases} \tag{21}$$

and

$$\rho_0'(x_0) = \begin{cases} 0 & \text{if } |x_0| > 1, \\ -1/2 & \text{if } 0 < x_0 < 1, \\ 1/2 & \text{if } -1 < x_0 < 0. \end{cases}$$

The first shock occurs at the smallest positive t, for which there is an x_0 such that

$$\frac{dx}{dx_0} = 1 - q''\rho_0' t = 0.$$

We can find this t by using equation (9), which in this case is

$$\frac{dx}{dx_0} = \begin{cases} 1 & \text{if } |x_0| < 0, \\ 1+t & \text{if } 0 < x_0 < 1, \\ 1-t & \text{if } -1 < x_0 < 0. \end{cases}$$

We observe that the only possible shock occurs at $1 - t = 0$ when $t = 1$ in $x_0 \in (-1, 0)$, so a shock occurs at $t = 1$ which can also observe in Figure 7.

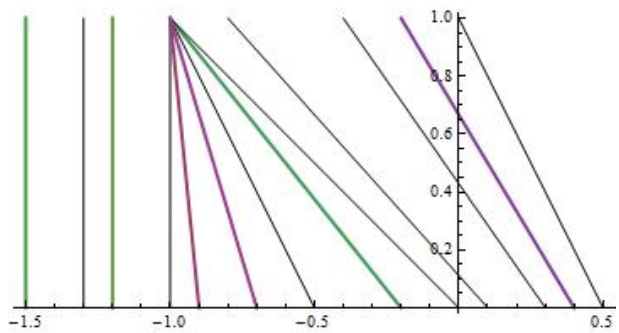

Figure 7: Time when shock occurs

We can see from Figure 7 that for times after 1, the lines start crossing each other, which means, in particular, that the implicitly given solution given by (9) will no longer valid. However, by Theorem 6.2 we know that if the solution satisfies the Rankine-Hugoniot jump condition we can still get a weak solution.

Notice for Greenshield's model that we have that $G(\rho) = \rho - \rho^2$, and since the shock occurs in the interval (-1,0) we can get $\rho_l = 1/2$ and $\rho_r = 1$ from the initial data, so we have

$$\xi'(t) = \frac{G(\rho_r) - G(\rho_l)}{\rho_r - \rho_l} = 1 - \rho_r - \rho_l = 1 - 1/2 - 1 = -1/2$$

and if we plug $t = 1$ into the characteristic equation we can get the point where the shock wave starts $(x, t) = (-1, 1)$, i.e., the initial condition of ODE $\xi'(t)$, so finally we achieve the shock wave

$$x = -1/2t - 1/2$$

We can observe this in Figure 8, where the red line is the shock wave.

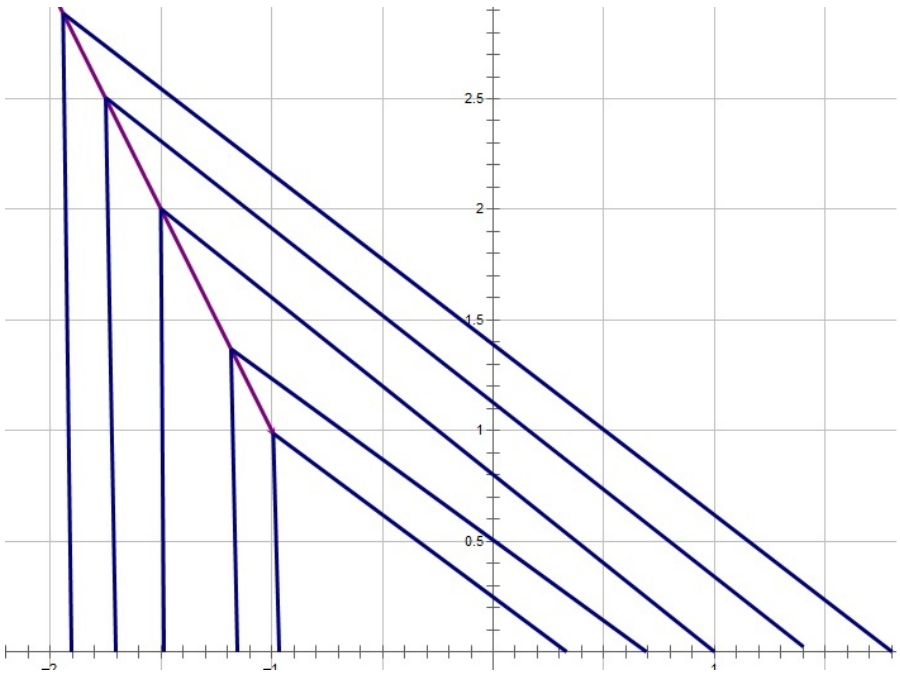

Figure 8: Shock wave for piecewise linear initial data

6.2 Finding the shock wave by a numerical method

Now we go back to our first initial data $\rho_0 = e^{-x_0^2}$, and look at $t > t_0 = \frac{\sqrt{2e}}{4}$,

We know that for Greenshield's model

$$\xi'(t) = \frac{G(\rho_r) - G(\rho_l)}{\rho_r - \rho_l} = 1 - \rho_l - \rho_r$$

where $\rho_l = e^{-x_0^{-2}}$ and $\rho_r = e^{-x_0^{+2}}$, but how can we decide x_0^- and x_0^+?

For fixed $t > t_0 = \frac{\sqrt{2e}}{4}$, the graph of the characteristic equation looks like in Figure 9(left figure), where we show x_0 on the horizontal axis, and x on the vertical axis. Notice that for a small or large x outside the $[x_{min}(t),\ x_{max}(t)]$, the characteristic equation only has one solution, but for x in the interval $(x_{min}(t), x_{max}(t))$, the equation has three solutions, for $x = x_{min}(t)$ or $x = x_{max}(t)$, the equation has two solutions. When $x_{min}(t)$ and $x_{max}(t)$ are plotted in (x, t)-plane, we obtain a cusp domain between $(x_{min}(t), x_{max}(t))$. Inside of the domain the the characteristic equation has three solutions and outside of the domain it has one solution.

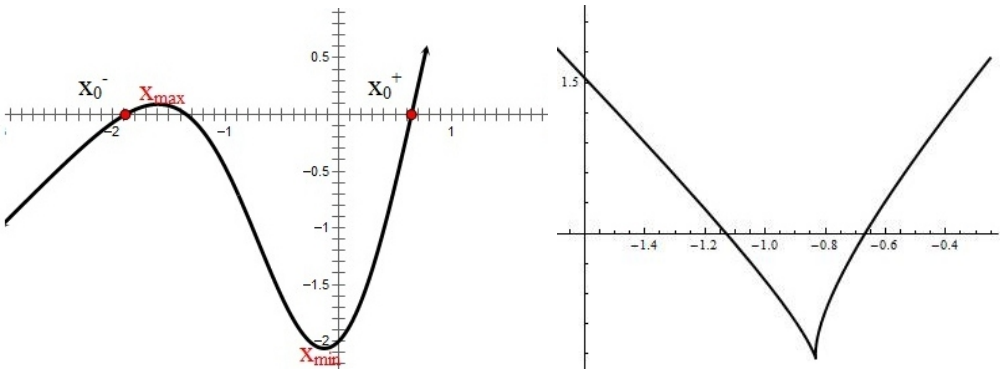

Figure 9: Characteristic equation for a fixed t and the cusp domain

The two points x_0^- and x_0^+ are the smallest and largest solutions respectively of the characteristic equation $x = (1 - 2e^{-x_0^2})t + x_0$. Since we have x_0^- and x_0^+ in the ODE so we can not solve it analytically, but by using the starting point of the shock curve we can solve it numerically by using Mathematica. We use Euler's method to solve the ODE, and apply the Newton-Raphson method in each step to find the largest and smallest solutions of the characteristic equation $x = (1 - 2e^{-x_0^2})t + x_0$.

Before we start programming, however, how can we know that a solution to the initial value problem exists and is unique? For a fixed x, we let $F(x_0, t) = (1 - 2e^{-x_0^2})t + x_0 - x$ and then $F_{x_0}(x_0, t) = 4tx_0 \cdot e^{-x_0^2} + 1$. By the **Implicit function theorem**, see [5], we know that except on the points when $F(x_0, t) = F_{x_0}(x_0, t) = 0$, we can always get a unique continuously differentiable function for x_0^- and x_0^+ respectively.

Notice by parameterizing x and t with x_0 as a parameter using $F(x_0, t) = F_{x_0}(x_0, t) = 0$, we can get the following two equations:

$$x = x_0 - \frac{e^{x_0}}{4x_0} + \frac{1}{2x_0}$$
$$t = -\frac{e^{x_0}}{4x_0}$$

which decides the critical curve for the characteristic equation which is the same as the cusp domain in Figure 9. By the implicit function theorem, x_0^- and x_0^+ are smooth functions of

x and t except on the cusp curve, we need to prove that it is also continuous on the closure of the cusp domain, for this we need to introduce the following theorem.

Theorem 6.3. *Let $\bar{\Omega} \subset \mathbb{R}^2$ be a closed set, let $f : \bar{\Omega} \mapsto f(\bar{\Omega}) \subset \mathbb{R}$ be continuous, and suppose that $x_0 < \zeta_0$, $(x_0, t) \in \bar{\Omega}$, $(\zeta_0, t) \in \bar{\Omega}$, f is monotonically increasing or decreasing in the x_0 variable. Let $f^{-1}(\cdot, t)$ be the inverse function of $f(\cdot, t)$, where t is considered as a parameter, Then $f^{-1} : f(\bar{\Omega}) \mapsto \bar{\Omega}$ is continuous.*

Proof. By theorem 3.5.2 in [6], we know that $f^{-1}(\cdot, t)$ is continuous for fixed t. Let $(x_0, t) \in \bar{\Omega}$. We need to show that for every $\epsilon > 0$, $|x_0 - \zeta_0| < \epsilon$ provided that , there exists a $\delta > 0$ such that

$$\begin{aligned} |f(x_0, t) - f(\zeta_0, t)| &< \delta, \\ |t - \tau| &< \delta, \end{aligned}$$

where $(x_0, t) \in \bar{\Omega}$, $(\zeta_0, t) \in \bar{\Omega}$.
We know from theorem 3.5.2 of [6] that for all $\epsilon_1 > 0$, there exists exists $\delta_1 > 0$ such that $|f(x_0, t) - f(\zeta_0, t)| < \delta_1$ which implies that $|x_0 - \zeta_0| < \epsilon_1$
Since f is continuous in t variable, we must have that for all $\epsilon_2 > 0$, there exists a $\delta_2 > 0$ when $|t - \tau| < \delta_2$ such that $|f(x_0, t) - f(x_0, \tau)| < \epsilon_2$.
By the Triangle inequality we have

$$|f(x_0, t) - f(\zeta_0, t)| \leq |f(x_0, t) - f(\zeta_0, \tau)| + |f(\zeta_0, \tau) - f(\zeta_0, t)|$$

If we want to $|f(x_0, t) - f(\zeta_0, t)| < \delta$ we must have

$$|f(x_0, t) - f(\zeta_0, t)| \leq |f(x_0, t) - f(\zeta_0, \tau)| + |f(\zeta_0, \tau) - f(\zeta_0, t)| < \delta + \epsilon_2$$

for $|t - \tau| < \delta_2$. Now if we require that for $\epsilon_1 = \epsilon$: $\delta < \delta_1/2$ and $\epsilon_2 < \delta_1/2$, then we have

$$|f(x_0, t) - f(\zeta_0, t)| < \delta_1$$

and so $|x_0 - \zeta_0| < \epsilon_1 = \epsilon$, which give us a ϵ that provide δ.

□

By the above theorem we can obtain that $f(\xi(t), t)$ is continuous on the closed cusp domain for the following initial value problem

$$\begin{cases} \xi'(t) = f(\xi(t), t), \\ \xi(t_*) = x_*. \end{cases} \tag{22}$$

where (t_*, x_*) is the cusp.

Theorem 6.4. *(Peano's existence theorem) Suppose f is bounded and continuous in x, and t. Then, for some value $\epsilon > 0$, there exists a solution $x = x(t)$ to the initial value problem within the range $[t_0 - \epsilon, t_0 + \epsilon]$.*

proof and other details see [4].

So far, we can achieve the existence of (23) by Peano's Existence theorem, however, in oder to get the uniqueness of the solution we need to introduce the **Peano's Uniqueness Theorem**.

Note that Picard existence and uniqueness theorem is not applicable here, since f is not locally Lipschitz on a rectangle which include (x_*, t_*).

Theorem 6.5. *(Peano's Uniqueness Theorem see [4])*
Consider a initial value problem

$$x' = f(t, x), \qquad x(t_*) = x_*$$

Let $f(t, x)$ be continuous in $t_0 \leq t \leq t_0 + a$, $|x - x_0| \leq b$ and and non increasing in x for each fixed t in $t_0 \leq t \leq t_0 + a$. Then the initial value problem has at most one solution in $t_0 \leq t \leq t_0 + a$, where a and b are positive integers.

Proof. Suppose $x_1(t)$ and $x_2(t)$ are two solutions of the initial value problem in $t_0 \leq t \leq t_0 + a$ which differ somewhere in $t_0 \leq t \leq t_0 + a$. We assume that $x_2(t) > x_1(t)$ in $t_1 < t < t_1 + \epsilon < t_0 + a$, while $x_1(t) = x_2(t)$ in $t_0 \leq t \leq t_1$, i.e., t_1 is the greatest lower bound of the set A consisting of those t for which $x_1(t) > x_2(t)$. This greatest lower bound exists because the set A is bounded below by x_0 at least. Thus, for all $t \in (t_1, t_1 + \epsilon)$ we have $f(t, x_1(x)) \geq f(t, x_2(t))$; i.e., $x_1'(t) \geq x_2'(t)$. Hence, the function $z(t) = x_2(t) - x_1(x)$ is non-increasing, since if $z(t_1) = 0$ we should have $z(t) \leq 0$ in $t \in (t_1, t_1 + \epsilon)$. This contradiction proves that $x_1(t) = x_2(t)$ in $t_0 \leq t \leq t_0 + a$. □

In order to apply Peano's Uniqueness Theorem, for getting uniqueness of the solution, we only need to prove that for a fixed t, f is decreasing in the x the x variable, i.e. if we have two solutions for (23) and $x_2 > x_1$, then $f(x_2(t)) - f(x_1(t)) \leq 0$

Proof. Recall that

$$f(\xi(t), t) = 1 - \rho_l - \rho_r = 1 - e^{-x^2{}_0^-(x,t)} - e^{-x^2{}_0^+(x,t)}$$

By Theorem 6.3 we know that $f(x, t)$ is continuous on the closed cusp domain. In order to let $f(x, t)$ continuous on the whole $\mathbb{R}^2$ we need to extend it to the whole plane. So for $t < t_0$, let $f(x, t) = f(x_0, t_0)$; For $t \geq t_0$: When $x > x_{max}$ let $f(\xi(t), t) = f(x_{max}(t), t)$; When $x < x_{min}$, let $f(x, t) = f(x_{min}(t), t)$. Now suppose we have two solutions x_2 and x_1 for (23), where $x_1 > x_2$, and we know that x_2 and x_1 also satisfy

$$\begin{aligned} x_1 &= (1 - 2e^{-x_0^2})t + x_0 \\ x_2 &= (1 - 2e^{-x_0^2})t + x_0 \end{aligned} \tag{23}$$

Let $x_1 0^-$ and $x_1 0^+$ be the smallest and largest solutions with respect to x_1 and $x_2 0^-$ and $x_2 0^+$ be the smallest and largest solution with respect to x_2. We know that the cusp curve is the critical curve of the characteristic equation $x = (1 - 2e^{-x_0^2})t + x_0$, so the smallest

and largest solution of characteristic equation must be outside of the cusp domain, notice that out side of cusp domain we have

$$x_0'(x) = \frac{1}{4x_0 t e^{-x_0^2} + 1}$$

which is greater then 0. So if $x_1 > x_2$ we have $x_1 0^- > x_2 0^-$ and $x1_0^+ > x2_0^+$, see the following Figure.

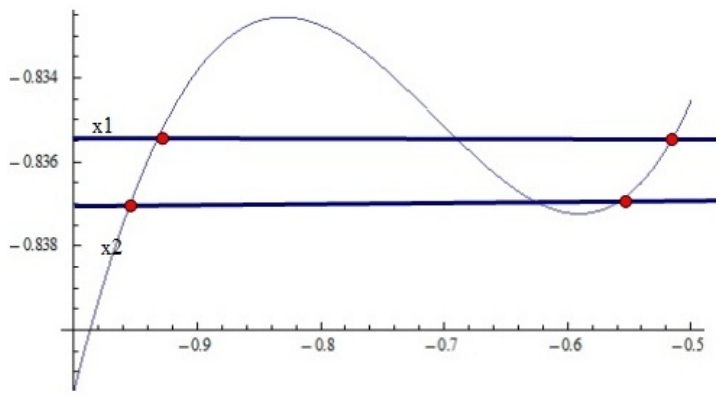

Figure 10: the smallest and largest solutions for $x_1 > x_2$

$$f(x_1(t)) - f(x_2(t)) = (1 - e^{-x_1 0^-} - e^{-x_1 0^+}) - (1 - e^{-x_2 0^-} - e^{-x_2 0^+}) \leq 0$$

so together with the **Peano's Uniqueness Theorem** we can achieve the uniqueness of solution . □

The following is a Mathematica code for solving the ODE for Rankine-Hugoniot condition with initial condition $x_* = -1/\sqrt{2}$ and $t_* = \sqrt{2}e^{1/2}/4$:

```
maxpunkt=20;
maxerr=0.0000000001;
h=0.01;
t[1]=0.25*Sqrt[2]*(Exp[0.5]);
Do[t[j]=t[1]+(j-1)*h,{j,2,maxpunkt}];
x[1]=Sqrt[2]*(0.25*(Exp[0.5])-1);
x[2]=x[1]+h*(1-2Exp[-0.5]);
n=2;
Do[{a=0,
b=-2.5,
eps=-((1-2Exp[-a^2])t[n]+a-x[n])/(4  t[n]*a*Exp[-a^2]+1),
While[Abs[eps]>maxerr,{a=a+eps,
```

```
eps=-((1-2Exp[-a^2])t[n]+a-x[n])/(4  t[n]*a*Exp[-a^2]+1)}],
eps=-((1-2Exp[-b^2])t[n]+b-x[n])/(4  t[n]*b*Exp[-b^2]+1),
While[Abs[eps]>maxerr,{b=b+eps,
eps=-((1-2Exp[-b^2])t[n]+b-x[n])/(4  t[n]*b*Exp[-b^2]+1)}],
n++,
x[n]=x[n-1]+h*(1-Exp[-a^2]-Exp[-b^2])},{n,2,maxpunkt}];
T=Array[t,maxpunkt];
X=Array[x,maxpunkt];
S=Thread[{X,T}];
ListPlot[S, Joined -> True, Mesh -> All, PlotStyle -> {Thick, Red}];
```

We can finally get the shock curve for initial data $\rho_0 = e^{-x^2}$. In Figure 11 we have plotted the shock curve together with some characteristics.

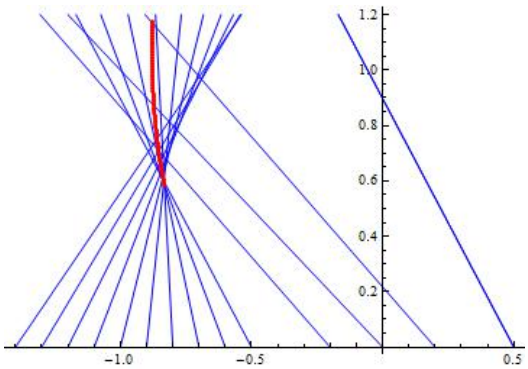

Figure 11: shock curve for e^{-x^2}

7 Entropy condition

We now try to solve a similar problem by the same techniques. We consider Greenshield's model but take different initial data,

$$\rho_0(x_0) = \begin{cases} 1 & \text{if } x_0 < 0, \\ 0 & \text{if } x_0 > 0. \end{cases} \tag{24}$$

If we apply method of characteristics we can obtain a solution as follows:

$$\rho_1(x,t) = \begin{cases} 1 & \text{if } x < 0, \\ 0 & \text{if } x > 0. \end{cases} \tag{25}$$

and the characteristics equation is

$$x = \begin{cases} x_0 - t & \text{if } x_0 < 0, \\ x_0 + t & \text{if } x_0 > 0. \end{cases}$$

We can see that this time the method of characteristic does not produce any intersections on the characteristic curves, however, as seen in the Figure 12, it does not provide information within a wedge.

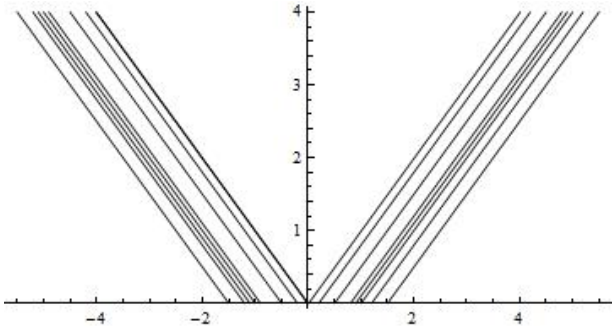

Figure 12: Solution with wedge

But it is easy to check that solution ρ_1 is a weak solution of (14) and (25) which still satisfy the Rankine-Hugoniot jump condition. However, we can create another such solution by writing

$$\rho_2(x,t) = \begin{cases} 1 & \text{if } x < -t, \\ 1/2(1 - x/t) & \text{if } -t < x < t \\ 0 & \text{if } x > t. \end{cases} \tag{26}$$

The function ρ_2, called a rarefaction wave, is also a continuous weak solution of (14) and (25).

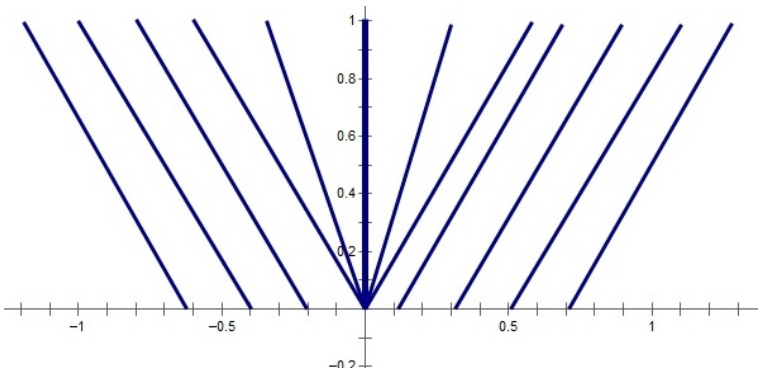

Figure 13: Rarefaction wave

In fact, we can create a whole continuum of solutions by combining shocks and rarefaction waves. Thus the weak solutions are in general not unique. So we need a "selection

criterion" that picks out the physically reasonable solution from the many possible weak solutions. However, can we find some criterion which ensures uniqueness?

A number of such criteria have been proposed, but there is as yet no general agreement on which is the "right" one.

Peter Lax has given a condition which tells which discontinuities are admissible for our traffic model. Called the geometric entropy condition, Lax's condition states that shocks are admissible if he characteristics on either side run into the shock in the direction of forward time. So the shock in Figure 14 is not admissible, since the characteristics on either side flow away from each other as t increasing.

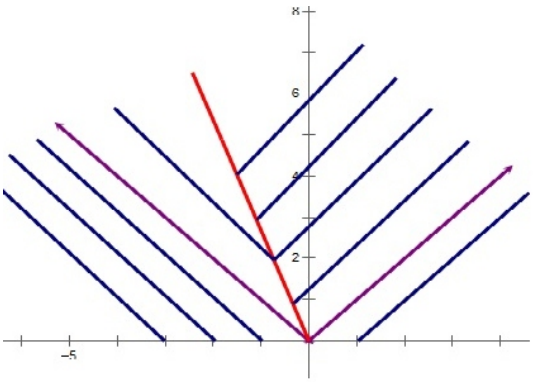

Figure 14: Shock curve filling in the wedge

Let's make these ideas more precise. In particular, for an equation of the form

$$\rho_t + G'(\rho)\rho_x = 0$$

we only allow for a curve of discontinuity in our solution $\rho_2(x,t)$ if the wave to the left is moving faster than the wave to the right. That is, we only allow for a curve of discontinuity between ρ_l and ρ_r if

$$G(\rho_l) < \sigma < G(\rho_r) \tag{27}$$

This is known as the entropy condition. We say that a curve of discontinuity is a shock curve for a solution ρ if the curve satisfies the Rankine-Hugoniot jump condition and the entropy condition for that solution ρ. Therefore, to eliminate the physically less realistic solutions, we only "accept" solutions ρ for which curves of discontinuity in the solution are shock curves. Now another question comes up: how can we easily just find the acceptable solution? Is there any solution formula for the general PDE which satisfies both of the conditions? The answer is yes.

8 The Lax–Oleinik formula

Consider the initial value problem (14), if ρ is a solution , then the planar vector field $(\rho, -G(\rho))$ is curl-free and therefore, if ρ is smooth, there exists a potential $\omega = \omega(x,t)$ such that

$$\omega_x = \rho \quad \omega_t = -G(\rho)$$

Thus, ω is a solution of the Hamilton-Jacobi equation

$$\omega_t + G(\omega_x) = 0. \tag{28}$$

The initial data $\rho(x,0) = h(x)$ transforms into the initial data

$$\omega(x,0) =: g(x) = \int_0^x h(y)\,\mathrm{d}y \qquad \forall x \in \mathbb{R}. \tag{29}$$

If ω is a solution of a initial-value problem of the Hamilton-Jacobi equation, then $\rho = \omega_x$ solves (14).

All the items illustrated below and the theory of Hamilton-Jacobi equation , excellently explained in [1].

- Assume that the flux function G is **uniformly convex**. With no loss of generality we may as well take
$$G(0) = 0$$
- The Legendre transform of G is
$$G^*(p) = \sup_{q\in\mathbb{R}^n}\{p.q - G(q)\} \qquad (p \in \mathbb{R})$$
- The Hopf–Lax formula for the solution of (28)
$$\omega(x,t) = \min_{y\in\mathbb{R}}\left(tG^*(\frac{x-y}{t}) + g(y)\right), \forall x \in \mathbb{R}, t > 0$$
We know that ω is not smooth in general . But we know that ω is differentiable a.e.(see [1]) Consequently,
$$\rho(x,t) = \frac{\partial}{\partial x}[\min_{y\in\mathbb{R}} tG^*(\frac{x-y}{t}) + g(y)] \tag{30}$$
is defined for a.e.(x,t) and is presumably a leading candidate for some sort of weak solution of the initial-value problem (14).
- **Notation**. For all the traffic flow models we covered in this paper $G(\rho)$ is uniformly concave, but for the simplicity of illustrating the proof and formula, we will take $G(\rho)$ as uniformly convex for now. Now G is uniformly convex, G' is strictly increasing and onto. Write
$$F = (G')^{-1} \tag{31}$$
for the inverse of G'.

Theorem 8.1. *Assume $G : \mathbb{R} \to \mathbb{R}$ is smooth, uniformly convex, and $h \in L^\infty(\mathbb{R})$.*

1. *For each time $t > 0$, there exists for all but at most countably many values of $x \in \mathbb{R}$ a unique point $y(x,t)$ such that*

$$\min_{y\in\mathbb{R}} \left(tG^*(\frac{x-y}{t}) + g(y) \right) = tG^*(\frac{x-y(x,t)}{t}) + g(y(x,t))$$

2. *The mapping $x \mapsto y(x,t)$ is nondecreasing.*
3. *For each time $t > 0$, the function ρ defined by (31) is*

$$\rho(x,t) = F(\frac{x-y(x,t)}{t}) \tag{32}$$

In particular, formula (33) holds for a.e.$(x,t) \in \mathbb{R} \times (0,\infty)$

Proof See [1] Chapter 3 .

Definition. Equation (33) is called the **Lax-Oleinik formula** for the solution of (14), where g is defined by (30).

Theorem 8.2. *Under the assumption of Theorem 8.1, the function ρ defined by **Lax-Oleinik formula** is a weak solution of (14).*

Proof See [1]

8.1 Entropy condition revisited

We know that the weak solutions are not generally unique. Since we believe the Lax-Oleinik formula does in fact provide the correct solution of this initial-value problem, we must see if it satisfies the Entropy condition discussed earlier.

Lemma 8.3. *(Another version of entropy condition) . Under the assumptions of Theorem 8.1, there exists a constant C such that the function ρ defined by the Lax-Oleinik formula satisfies the inequality*

$$\rho(x+z,t) - \rho(x,t) \leq \frac{C}{t} z \tag{33}$$

for all $t > 0$ and $x, z \in \mathbb{R}, z > 0$.

Remark 8.4. *The above inequality is another version of the entropy condition with respect to the Lax-Oleinik formula.It follows from (34) that for $t > 0$ the function $x \mapsto \rho(x,t) - \frac{Cx}{t}$ is nondecreasing, and consequently has left and right limits at each point. Thus also $x \mapsto \rho(x,t)$ has left and right hand limits at each point, with $\rho_l(x,t) \geq \rho_r(x,t)$. In particular, in the original form of the entropy condition (28) holds at any point of discontinuity.*

Proof. Since $F = (G')^{-1}$, and F is nondecreasing, thus if $z > 0$,

$$\begin{aligned}\rho(x,t) - \rho(x+z,t) &= F\left(\frac{x - y(x,t)}{t}\right) - F\left(\frac{x - z - y(x+z,t)}{t}\right) \\ &\geq F\left(\frac{x - y(x+z,t)}{t}\right) - F\left(\frac{x - z - y(x+z,t)}{t}\right)\end{aligned}$$

since $z > 0$ and $y(x,t)$ are nondecreasing. Recall that in computing the minimum in (31) (see [1]) we only need to consider those y such that $|\frac{x-y}{t}| \leq C$, for some constant C. Let $\tilde{G} = G\,|_B c(0)$, where $G\,|_B c(0)$ denote the boundary of $G(c(0))$ Then $\tilde{G}$ is Lipschitz, and by the above inequality,

$$\rho(x,t) - \rho(x+z,t) \geq -\frac{\texttt{Lip}(\tilde{G})z}{t}$$

□

8.2 Uniqueness of entropy solutions

We now establish the important assertion that a weak solution which satisfies the entropy condition is unique.

Definition 8.5. *We say that a function $\rho \in L^\infty(\mathbb{R} \times (0,\infty))$ is an entropy solution of the initial-value problem*

$$\begin{cases} \rho_t + G(\rho)_x = 0 & \text{in} \quad \mathbb{R} \times (0,\infty), \\ \rho = h & \text{on} \quad \mathbb{R}^n \times t = 0. \end{cases} \tag{34}$$

provided

$$\int_0^\infty \int_{-\infty}^\infty \rho v_t + G(\rho) v_x \, dxdt + \int_{-\infty}^\infty hv \, dx|_{t=0} = 0 \tag{35}$$

for all test functions $v : \mathbb{R} \times [0,\infty) \mapsto \mathbb{R}$, and

$$\rho(x+z,t) - \rho(x,t) \leq C(1 + \frac{1}{t})z \tag{36}$$

for some constant $C \geq 0$ and a.e. $x, z \in \mathbb{R}, t > 0$, with $z > 0$.

Theorem 8.6. *(Uniqueness of entropy solutions Proof rewrite from Evans [1]). Assume that G is convex and smooth. Then there exists–up to a set of measure zero-at most one entropy solution of (35).*

Proof. **1**. Assume that ρ and $\tilde{\rho}$ are two entropy solutions of (35), and write $\omega = \rho - \tilde{\rho}$. Observe for any point (x,t) that

$$
\begin{aligned}
G(\rho(x,t)) - G(\tilde{\rho}(x,t)) &= \int_0^1 \frac{d}{dr} G(r\rho(x,t) + (1-r)\tilde{\rho}(x,t))\, dr \\
&= \int_0^1 G'(r\rho(x,t) + (1-r)\tilde{\rho}(x,t))\, dr(\rho(x,t) - \tilde{\rho}(x,t)) \\
&= : b(x,t)\omega(x,t).
\end{aligned}
\tag{38}
$$

Consequently if v is a test function as above,

$$
0 = \int_0^\infty \int_{-\infty}^\infty (\rho - \tilde{\rho})v_t + [G(\rho) - G(\tilde{\rho})]v_x \, dxdt = \int_0^\infty \int_{-\infty}^\infty \omega[v_t + bv_x]\, dxdt \tag{39}
$$

2. Now take $\epsilon > 0$ and define $\rho^\epsilon = \eta_\epsilon * \rho$, $\tilde{\rho}^\epsilon = \eta_\epsilon * \tilde{\rho}$, where η_ϵ is the standard mollifier in the x and t variables. Here a standard mollifier is a family of functions η_ϵ based on η given by$\eta_\epsilon(x,t) = \epsilon^{-k}\eta(x/\epsilon, t/\epsilon)$,for all $\epsilon > 0$. Then according to $C4$ in [1]

$$
||\rho^\epsilon||_{L^\infty} \leq ||\rho||_{L^\infty}, \quad ||\tilde{\rho}^\epsilon||_{L^\infty} \leq ||\tilde{\rho}||_{L^\infty} \tag{40}
$$

$$
\rho^\epsilon \to \rho, \quad \tilde{\rho}^\epsilon \to \tilde{\rho} \;\; a.e., \;\; as \;\; \epsilon \to 0 \tag{41}
$$

Furthermore the entropy inequality (37) implies

$$
\rho_x^\epsilon(x,t), \tilde{\rho}_x^\epsilon(x,t) \leq C(1 + \frac{1}{t}) \tag{42}
$$

for an appropriate constant C and all $\epsilon > 0$, $x \in \mathbb{R}, t > 0$.
3. Write

$$
b_\epsilon(x,t) := \int_1^0 G'(r\rho^\epsilon(x,t) + (1-r)\tilde{\rho}^\epsilon(x,t)\, dr
$$

Then (40) becomes

$$
0 = \int_0^\infty \int_{-\infty}^\infty \omega(v_t + b_\epsilon v_x)\, dxdt + \int_0^\infty \int_{-\infty}^\infty \omega(b - b_\epsilon)v_x \, dxdt \tag{43}
$$

4. Now select $T > 0$ and any smooth function $\varphi : \mathbb{R} \times (0,T) \mapsto \mathbb{R}$ with compact support. We choose v to be the solution of the following terminal-value problem for a linear transport equation :

$$\begin{cases} v_t^\epsilon + b_\epsilon v_x^\epsilon = \varphi \quad in \ \mathbb{R} \times (0,T) \\ v = 0 \quad on \ \mathbb{R} \times t = T. \end{cases} \tag{44}$$

Let us solve (45) by the method of characteristics. For this , fix $x \in \mathbb{R}$, $0 \le t \le T$, denote by $x_\epsilon(\cdot)$ the solution of the ODE

$$\begin{cases} \dot{x}_\epsilon(s) = b_\epsilon(x_\epsilon(s), s)(s \ge t) \\ x_\epsilon(t) = x \end{cases} \tag{45}$$

and set

$$v^\epsilon(x,t) = -\int_t^T \varphi(x_\epsilon(s), s)\, ds \quad (x \in \mathbb{R}, 0 \le t \le T). \tag{46}$$

Then v^ϵ is smooth and is the unique solution of (45). Since $|b_\epsilon|$ is bounded and φ has compact support, v^ϵ has compact support in $\mathbb{R} \times [0,T)$.

5. We now claim that for each $s > 0$, there exists a constant C_s such that

$$|v_x^\epsilon| \le C_s \quad on \ \mathbb{R} \times (s,T). \tag{47}$$

To prove this, first note that if $0 < s \le t \le T$, then

$$b_{\epsilon,t}(x,t) = \int_0^1 G''(r\rho^\epsilon + (1-r)\tilde{\rho}^\epsilon)(r\rho_x^\epsilon + (1-r)\tilde{\rho}_x^\epsilon)\, dr \le \frac{C}{t} \le \frac{C}{s} \tag{48}$$

by (33), since G is convex .
Next, differentiate the PDE in (45) with respect to x:

$$v_{xt}^\epsilon + b_\epsilon v_{xx}^\epsilon + b_{\epsilon,x} v_x^\epsilon = \varphi_x. \tag{49}$$

Now set $a(x,t) := e^{\lambda t} v_x^\epsilon(x,t)$, for

$$\lambda = \frac{C}{s} + 1 \tag{50}$$

Then by(50)

$$a_t + b_\epsilon a_x = \lambda a + e^{\lambda t}[v_{xt}^\epsilon + b_\epsilon v_{xx}^\epsilon] = \lambda a + e^{\lambda t}[-b_{\epsilon,x} v_x^\epsilon + \varphi_x] = [\lambda - b_{\epsilon,x}]a + e^{\lambda t}\varphi_x. \tag{51}$$

Since v^ϵ has compact support, a attains a nonnegative maximum over $\mathbb{R} \times [s,T]$ at some finite point (x_0, t_0). If $t_0 = T$, then $v_x = 0$. If $0 \le t_0 < T$, then

$$a_t(x_0,t_0) \le 0, a_x(x_0,t_0) = 0.$$

Consequently equation (52) gives

$$[\lambda - b_{\epsilon,x}]a + e^{\lambda t}\varphi_x \le 0 \quad at(x_0,t_0). \tag{52}$$

But since $b_{\epsilon,x} \leq C/s$ and λ is given by (51), inequality (53) implies

$$a(x_0, t_0) \leq -e^{\lambda t_0}\varphi_x \leq e^{\lambda T}||\varphi_x||_{L\infty}$$

A similar argument shows that

$$a(x_1, t_1) \geq -e^{\lambda T}||\varphi_x||_{L\infty}$$

at any point (x_1, t_1) where a attains a non positive minimum. These two estimates and the definition of a imply (48).

6. We will need one more inequality, namely

$$\int_{-\infty}^{\infty} |v_x^\epsilon(x,t)|\, dx \leq D \tag{53}$$

for all $0 \leq t \leq \tau$ and some constant D, provided τ is small enough.
To prove this, choose $\tau > 0$ so small that $\varphi = 0$ on $\mathbb{R} \times (0, \tau)$. Then if $0 \leq t \leq \tau$, we see from (47) that v is constant along the characteristic curve $x_\epsilon(\cdot)$ (solving (46))for $t \leq \tau$. Select any partition $x_0 < x_1 < \cdots < x_N$. Then $y_0 < y_1 < \cdots < y_N$, where $y_i := x_i(s)(i = 1, \cdots, N)$ for

$$\begin{cases} \dot{x}_\epsilon(s) = b_\epsilon(x_\epsilon(s), s) & (t \leq s \leq \tau) \\ & (x_\epsilon(t) = x_i. \end{cases}$$

As v^ϵ is a constant along the characteristic curve $x_i(\cdot)$, we have

$$\sum_{i=1}^{N} |v^\epsilon(x_i, t) - v^\epsilon(x_{i-1}, t)| = \sum_{i=1}^{N} |v^\epsilon(y_i, \tau) - v^\epsilon(y_{i-1}, \tau)| \leq \texttt{var}\ \ v^\epsilon(\cdot, \tau),$$

"var" denoting variation with respect to x. Taking the supremum over all such partitions, we find

$$\int_{\infty}^{-\infty} |v_x^\epsilon(x,t)|\, dx = \texttt{var} v^\epsilon(\cdot, t) \leq \texttt{var} v^\epsilon(\cdot, \tau) = \int_{-\infty}^{\infty} |v_x^\epsilon(x,\tau)|\, dx \leq C$$

Since v^ϵ has constant support and estimate (43) is valid for $s = \tau$.

7. Now at last we complete the proof by setting $v = v^\epsilon$ in (44) and substituting, using (45)

$$\begin{aligned} \int_0^\infty \int_{-\infty}^{\infty} \omega\varphi\, dxdt &= \int_0^\infty \int_{-\infty}^{\infty} \omega[b_\epsilon - b]v_x^\epsilon\, dxdt \\ &= \int_T^\tau \int_{-\infty}^{\infty} \omega[b_\epsilon - b]v_x^\epsilon\, dxdt \\ &+ \int_\tau^0 \int_{-\infty}^{\infty} \omega[b_\epsilon - b]v_x^\epsilon\, dxdt \\ &= : I_\tau^\epsilon + J_\tau^\epsilon \end{aligned}$$

Then in view of (42),(48), and the Dominated Convergence Theorem,

$$I_\tau^\epsilon \longrightarrow 0$$

as $\epsilon \longrightarrow 0$ for each $\tau > 0$. On the other hand, if $0 < \tau < T$, we see by (54)

$$|J_\tau^\epsilon| \leq \tau C \max_{0<t<\tau} \int_{-\infty}^{\infty} |v_x^\epsilon| \, dx \leq \tau C$$

Thus

$$\int_{\infty}^{0} \int_{-\infty}^{\infty} \omega\varphi \, dxdt = 0$$

for all smooth function φ as above, and so $\omega = \rho - \tilde{\rho} = 0$ a.e. □

9 Riemann's problem

The initial value problem (35) with the piecewise-constant initial data for the case $G(\rho)$ is uniformly convex is

$$h(x_0) = \begin{cases} \rho_l & \text{if } x_0 < 0, \\ \rho_r & \text{if } x_0 > 0. \end{cases} \tag{54}$$

This initial-value problem (35), (55) is known as **Riemann's problem**.

Theorem 9.1. *(See Evans, Chap3) For G uniformly convex, there exists a unique weak, admissible solution to Riemann's problem (14), (55).*

1. *If $\rho_l > \rho_r$, then the admissible solution has a shock curve of the speed σ and the solution is given by*

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } \frac{x}{t} < \sigma, \\ \rho_r & \text{if } \frac{x}{t} > \sigma. \end{cases} \tag{55}$$

   where $\sigma = \frac{G(\rho_l) - G(\rho_r)}{\rho_l - \rho_r}$.

2. *if $\rho_l < \rho_r$, then the solution has a rarefaction wave and the solution is given by*

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } \frac{x}{t} < G'(\rho_l), \\ F(x/t) & \text{if } G'(\rho_l < \frac{x}{t} < G'(\rho_r), \\ \rho_r & \text{if } \frac{x}{t} < G'(\rho_r) \end{cases} \tag{56}$$

Here F is defined in (32)

Remarks.

1. In the first case the states ρ_l and ρ_r are separated by a shock wave with constant σ. In the second case the states ρ_l and ρ_r are separated by the rarefaction wave.

2. Riemann problem is a nice example of Lax-Oleinik formula, and also a nice illustration of the uniqueness assertion.

Proof. 1. Assume $\rho_l > \rho_r$. Clearly ρ defined by (56) is a weak solution of our PDE. In particular since $\sigma = [G(\rho)]/[\rho]$, the Rankine-Hugoniot condition holds . Furthermore note that

$$G'(\rho_r) < \sigma = \frac{G(\rho_l) - G(\rho_r)}{\rho_l - \rho_l} = \int_{\rho_r}^{\rho_l} G'(r)\, dr < G'(\rho_l)$$

in accordance with (28). Since $\rho_l > \rho_r$, the entropy condition holds as well. Uniqueness follows from Theorem 8.4.
2. Assume $\rho_l < \rho_r$. We must first check that ρ defined by (57) solves the conservation law in the region $\{G'(\rho_l) < x/t < G'(\rho_r)\}$. To verify this, let us ask the general question as to when a function ρ of the form

$$\rho(x,t) = v(\frac{x}{t})$$

solves (35). We compute

$$\rho_t + G(\rho)_x = \rho_t + G'(\rho)\rho_x = -v'(\frac{x}{t})\frac{x}{t^2} + G'(v)v'(\frac{x}{t})\frac{1}{t} = v'(\frac{x}{t})\frac{1}{t}[G'(v) - \frac{x}{t}] \tag{57}$$

Thus, assuming that v' never vanishes, we find $G'(v(\frac{x}{t})) = \frac{x}{t}$. Hence

$$\rho(x,t) = v(\frac{x}{t}) = F(\frac{x}{t})$$

solves the conservation law. Now $v(x/t) = \rho_l$ provided $x/t = G'(\rho_l)$; and similarly $v(x/t) = \rho_r$ provided $x/t = G'(\rho_r)$.
As a consequence we see that the rarefaction wave ρ defined by (57) is continuous in $\mathbb{R} \times (0, \infty)$, and is a solution of our PDE in each of its regions of definition. It is easy to check that ρ is thus an weak solution of (35),(55). Furthermore, since as noted in section 8.1 we may as well assume F is Lipschitz continuous, we have

$$\rho(x+z,t) - \rho(x) = F(\frac{x+z}{t}) - F(\frac{x}{t}) \leq \frac{Lip(F)z}{t}$$

if $G'(\rho_l)t < x < x + z < F'(\rho_r)t$. This inequality implies that ρ satisfies the entropy condition. Uniqueness is once more a consequence of Theorem 8.5. □

Remark 9.2. *Notice that for the simplicity of proving all of the proof above are based on that G is uniformly convex. However, the G related to all traffic flow models in this paper are all uniformly concave, so if we let*

$$\rho = -\hat{\rho}$$

So for Greenshields model we have

$$\bar{G}(\hat{\rho}) = \hat{\rho} + \hat{\rho}^2$$

$$\hat{\rho}_t + (\hat{\rho} + \hat{\rho}^2)_x = 0$$

where $\hat{\rho} + \hat{\rho}^2$ is uniformly convex.

9.1 Greenshields's model

Now we can compute **Riemann's problem** for the Greenshield's traffic flow model.

1. If $\rho_l < \rho_r$, then the admissible solution has a shock curve of the speed $1 - \rho_l - \rho_r$ and the solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } x/t < 1 - \rho_l - \rho_r, \\ \rho_r & \text{if } x/t > 1 - \rho_l - \rho_r. \end{cases} \tag{58}$$

2. $\rho_l > \rho_r$, then the solution has a rarefaction wave and the solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } \frac{x}{t} < 1 - 2\rho_l, \\ 1/2(1 - x/t) & \text{if } 1 - 2\rho_l < \frac{x}{t} < 1 - 2\rho_r, \\ \rho_r & \text{if } \frac{x}{t} > 1 - 2\rho_r \end{cases} \tag{59}$$

Now if we go back and look *Section* 7, we can explain how can we get the solution (26) and (27) with the initial data (25).

9.2 Riemann problem for Edie's model

From Section 3 we know that in Edie's model we have

$$v = \begin{cases} e^{(-e\rho)} & \text{if } 0 \le \rho \le 1/e, \\ -\log \rho/e & \text{if } 1/e \le \rho \le 1. \end{cases}, \tag{60}$$

and together with the initial Riemann's initial data (55) we can get the Edie-Riemann shock wave solution

1. If $0 \le \rho_l \le \rho_r \le 1/e$, then the admissible solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } \frac{x}{t} < \frac{\rho_l e^{-e\rho_l} - \rho_l e^{-e\rho_r}}{\rho_l - \rho_r}, \\ \rho_r & \text{if } \frac{x}{t} > \frac{\rho_l e^{-e\rho_l} - \rho_l e^{-e\rho_r}}{\rho_l - \rho_r}. \end{cases} \tag{61}$$

2. If $0 \le \rho_l \le 1/e < \rho_r \le 1$, then the admissible solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } \frac{x}{t} < \frac{\rho_l e^{-e\rho_l} + \rho_r \log \rho_r/e}{\rho_l - \rho_r}, \\ \rho_r & \text{if } \frac{x}{t} > \frac{\rho_l e^{-e\rho_l} + \rho_r \log \rho_r/e}{\rho_l - \rho_r}. \end{cases} \tag{62}$$

3. If $1/e \le \rho_l \le \rho_r \le 1$, then the admissible solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } \frac{x}{t} < \frac{-\rho_l \log \rho_l/e + \rho_r \log \rho_r/e}{\rho_l - \rho_r}, \\ \rho_r & \text{if } \frac{x}{t} > \frac{-\rho_l \log \rho_l/e + \rho_r \log \rho_r/e}{\rho_l - \rho_r}. \end{cases} \tag{63}$$

The rarefaction wave solutions look as follows:

1. $1/e \geq \rho_l \geq \rho_r \geq 0$, then the solution has a rarefaction wave and the solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } x < t(e^{-e\rho_l} - \rho_l e^{1-e\rho_l}), \\ a^{-1}(x/t) & \text{if } t(e^{-e\rho_l} - \rho_l e^{1-e\rho_l}) < x < t(e^{-e\rho_r} - \rho_r e^{1-e\rho_r}), \\ \rho_r & \text{if } x > t(e^{-e\rho_r} - \rho_r e^{1-e\rho_r}). \end{cases} \tag{64}$$

   where a^{-1} is the inverse function of $a(\rho) = e^{-e\rho} - \rho e^{1-e\rho}$.

2. $1 \geq \rho_l \geq 1/e \geq \rho_r \geq 0$, then the solution has a rarefaction wave and the solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } x < t\frac{-1-\log\rho_l}{e}, \\ e^{(-1-ex/t)} & \text{if } t\frac{-1-\log\rho_l}{e} < x < 0, \\ a^{-1}(x/t) & \text{if } 0 < x < t(e^{-e\rho_r} - \rho_r e^{1-e\rho_r}), \\ \rho_r & \text{if } x > t(e^{-e\rho_r} - \rho_r e^{1-e\rho_r}). \end{cases} \tag{65}$$

   where $a^{-1}(x/t)$ is the inverse function of $a(\rho) = e^{-e\rho} - \rho e^{1-e\rho}$ with respect to x/t.

3. $1 \geq \rho_l \geq \rho_r \geq 1/e$, then the solution has a rarefaction wave and the solution is given by

$$\rho(x,t) = \begin{cases} \rho_l & \text{if } x < t\frac{-1-\log\rho_l}{e}, \\ e^{(-1-ex/t)} & \text{if } t\frac{-1-\log\rho_r}{e} > x > t\frac{-1-\log\rho_l}{e}, \\ \rho_r & \text{if } x > t\frac{-1-\log\rho_l}{e}. \end{cases} \tag{66}$$

A special case which is interesting for us is that applying the initial condition (25), in which case the rarefaction wave is given by (66) as below

$$\rho(x,t) = \begin{cases} 1 & \text{if } x < \frac{-t}{e}, \\ e^{(-1-ex/t)} & \text{if } \frac{-t}{e} < x \leq 0, \\ a^{-1}(x/t) & \text{if } 0 < x < t, \\ 0 & \text{if } x > t \end{cases} \tag{67}$$

10 An application of the model: The timing of traffic lights

One of the most important things this model can do is predict, what will happen at a traffic light, it can also tell us how lights should be timed, depending on how heavy the traffic is. It also explains why, for an individual driver, there is no perfect staging of lights.

Let us consider, first, a single traffic light which is red for a time interval T_R and then green for an interval of duration T_G. The density of traffic everywhere on the road can be calculated explicitly by using traffic flow models. First assume that the incoming density has the constant value $\rho_0 < 1$ and traffic is running free when the light changes.

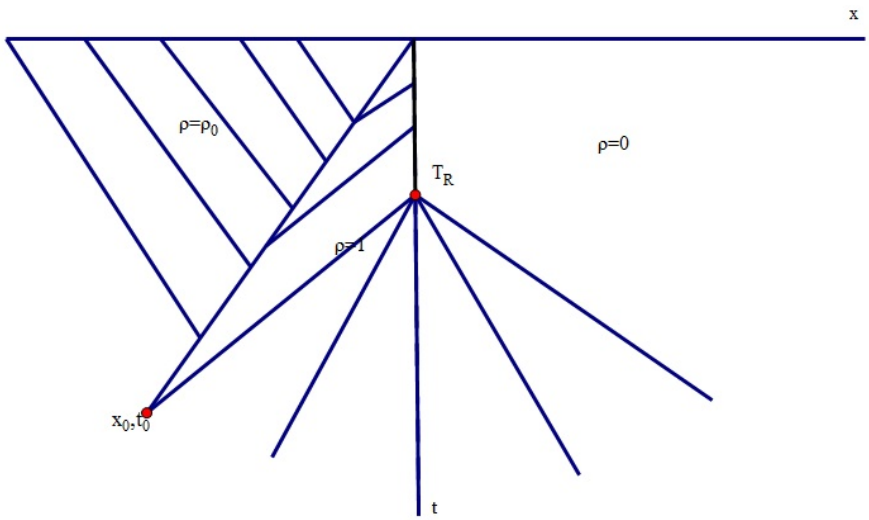

Figure 15: The traffic light cycle(red)

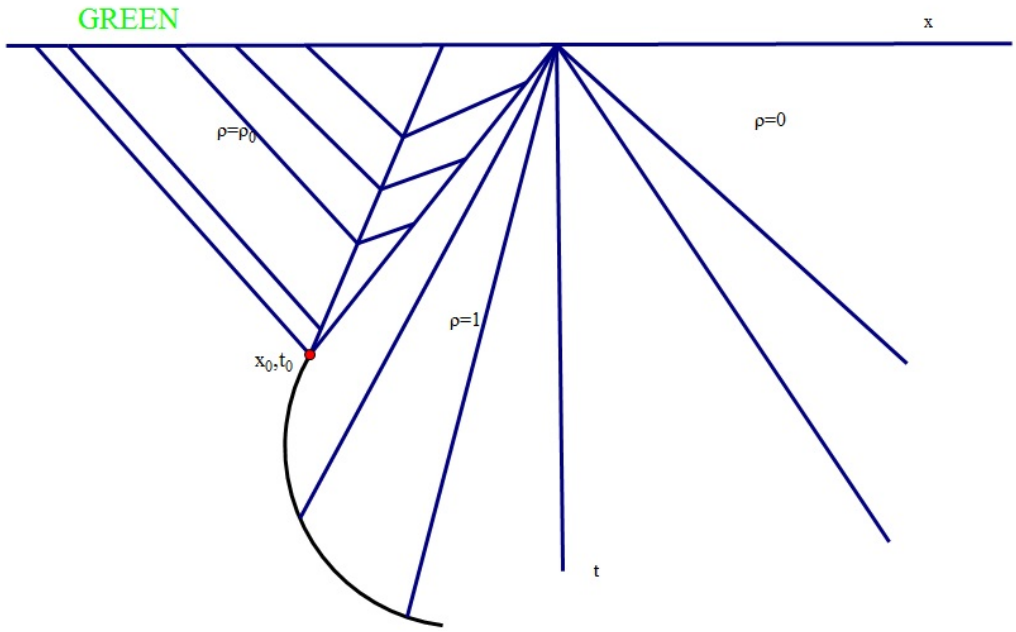

Figure 16: The traffic light cycle(Green)

10.1 Greenshields's model

1. First step (Red phase) $0 < t < T_R$:

Since the incoming density $\rho_0 < \rho_1 = 1$, and cars back up behind the light ($x < 0$), a shock forms and moves up the street with speed

$$\sigma = \frac{G(\rho_1) - G(\rho_0)}{\rho_1 - \rho_0} = 1 - (\rho_0 + \rho_1) = -\rho_0$$

The solution now followes from (59)

$$\rho(x,t) = \begin{cases} \rho_0 & \text{if } \frac{x}{t} < -\rho_0, \\ 1 & \text{if } \frac{x}{t} > -\rho_0 \end{cases}$$

2. Second step (Green Phase) $t > T_R$: When the light turns green, traffic begins to move at the light. Since behind the red traffic light $\rho = 0$, we have that the density on the left side is bigger than on the right side, so the rarefaction forms; if we take this instant to be the origin of time, then the rarefaction is centered at the origin, as seen in Figure 16, and has head and tail speeds $1 = 1 - 2 \times 0$ and $-1 = 1 - 2 \times 1$ respectively. The tail of the rarefaction runs into the shock at

$$t_0 = \frac{\rho_0}{1-\rho_0} T_R \qquad\qquad x_0 = -\frac{\rho_0}{1-\rho_0} T_R$$

which is the intersection point of the shock wave and the left tail of the rarefaction wave, and the solution follows from (60):

$$\rho(x,t) = \begin{cases} \rho_0 & \text{if } x < -t\rho_0, \\ 1 & \text{if } -t\rho_0 \le x < T_R - t, \\ 1/2(1 - x/t) & \text{if } T_R - t \le x \le t - T_R, \\ -1 & \text{if } x > t - T_R, \end{cases}$$

This solution valid as long as $-t\rho_0 < T_R - t$.

3. Third step $t > t_r$ ($t_r > T_R$):

Now the speed of the shock changes. If we let $\phi(t)$ denote the position of the shock, then the shock speed is $\sigma = d\phi/dt$ and ϕ satisfies the differential equation

$$\frac{d\phi}{dt} = 1 - \left(\rho_0 + \frac{1}{2}(1 - \frac{\phi}{t})\right)$$

since ρ_1 is the density to the right of the shock, is given by equation (60). Rearranging terms gives a linear, first-order ordinary differential equation

$$\frac{d\phi}{dt} - \frac{\phi}{2t} = \frac{1}{2} - \rho_0$$

which we can solve. Applying the initial condition $\phi(t_0) = x_0 = -t_0$ we get

$$\phi(t) = (1 - 2\rho_0)t - 2\sqrt{\rho_0(1-\rho_0)tT_R}$$

as the equation of the arc the shock describes in Figure 16.

Now there are two cases:

Case 1: If $\rho_0 \geq 1/2$, then$\frac{d\phi}{dt} \leq 0$, for all $t > t_0$, and $\phi(t) \to -\infty$ as $t \to \infty$, that is, if the road is at greater than half its carrying capacity, then the shock continues to move to the left for all time. The velocity of the shock line is thus reduced from $-\rho_0$ to $1/2-\rho_0$. Clearly the discontinuity step at $\phi(t)$ goes to zero as $t \to \infty$. However, the drivers observe the shock even a long time after the traffic disturbance. This agrees with practical experience.
Case 2: If $\rho_0 < 1/2$, $\phi(t) \to \infty$ as $t \to \infty$, which means the shock speed eventually becomes positive, and the shock crosses the point $x = 0$ at a time

$$t_r = \frac{\rho_0(1-\rho_0)T_R}{(\frac{1}{2}-\rho_0)^2}$$

provided the light is still green. For example if we assume that after the time $t = 2T_R(T_R = T_G)$ the traffic light changes from green to red. How long should the green phase be to eliminate the shock, i.e., we need $t_r \leq 2T_R$ where $\phi(t_r) = 0$. It turns out to make a big difference whether the shock gets through the intersection or not before the light changes again.

First suppose that the timing of red and green lights are such that the shock crosses the intersection before the light turns red again; that is, $T_G > t_r$. If we assume that the cycle of lights is fixed, this means that

$$\rho_0 \leq \frac{1}{2}\left(1 - \sqrt{\frac{T_G}{T_R + T_G}}\right) \tag{68}$$

Since the density behind the shock is again ρ_0 when the light turns red again, then in this situation after the red-green cycle we have the same initial condition as before, and to the left of the light the flow is periodic in time. It is also periodic beyond the light, and the average density a short distance downstream must also be ρ_0: the effect of the light is to impose on the traffic a temporary slowdown, from which it recovers.

However, let us now suppose that the shock does not get through the intersection. We can make the following calculation of the flux at the light itself: during the green cycle, the density is precisely 1/2 (since it is the center of the rarefaction) and so is the velocity, while during the red cycle the flux is zero, the average flux over a period is

$$\bar{q} = \frac{1}{4}\frac{T_G}{T_R + T_G}$$

Now, this must be the average flux anywhere along the road (since there are no sources or sinks), and so the average density is found by $\rho_\rho^2 = \bar{q}$. Solving this equation has two roots, but the smaller is precluded by the fact that inequality (64) is violated. Hence we have

$$\bar{\rho} = \frac{1}{2}\left(1 + \sqrt{\frac{T_G}{T_R + T_G}}\right)$$

as the value of the average density. The effect of a small change in incoming density from a value satisfying (64) to one just violating it has been to move the average density from the lower to the upper half of the density-flux diagram, greatly increasing the travel time for drivers.

The actual numbers are dramatic: Richards [7] points out that if $T_R = T_G$, then the greatest incoming density that permits free flow at the light is

$$\rho_0 \leq \frac{1}{2}(1 - \sqrt{\frac{T_G}{T_R + T_G}}) = (1 - 1/\sqrt{2})/2 \approx 0.15.$$

which is followed by (69); While the average density when the shock does not get through the light is

$$\bar{\rho} = \frac{1}{2}(1 + \sqrt{\frac{T_G}{T_R + T_G}}) = (1 + /1\sqrt{2})/2 \approx 0.85$$

.

There is a second feature of roads with traffic lights, also familiar to drivers, can be derived from this model. This is also illustrated in [2]. This has to do with the timing of successive lights along a road. If we look at the flow behind a well-timed light, then it is periodic in time, with period $T_R + T_G$. However, beyond the light the mass that has crossed the light in a single green cycle (which corresponds to a set of drivers, as well) spreads out, since the velocities vary between 1 and $1 - 2\rho_0$. If there a second light on the road, and its cycle has the same period, then the only way to arrange its cycle so that the entire mass gets through the second light on a single cycle is to make the ratio T_G/T_R greater for the second light. Even then, this works only if the second light is sufficiently close to the first. Realistically, a series of lights will be timed so that only a part of this packet will get through later lights. Which part? Reasoning on the basis of the fluxes, it is optimal to plan so that the later part of the packet, where the density is greater, will cross the second light without waiting. Thus it is to be expected that the first drivers through a green light will get caught at the next light. A light that is timed for the benefit of the first phalanx of drivers will have the effect of cutting off the last drivers through the previous light.

10.2 Edie's model for traffic light

What will we get if we apply the Edie's model in traffic light problems?
From section 3 we know that Edie' model is as below

$$v = \begin{cases} e^{(-e\rho)} & \text{if } 0 \leq \rho \leq 1/e, \\ -\log \rho/e & \text{if } 1/e \leq \rho \leq 1. \end{cases} \tag{69}$$

So we divide this model into two different cases by the difference of the incoming density:
Case 1: Incoming density $\rho_0 < 1/e < 1$
1. First step (Red phase) $0 < t < T_R$: Now cars back up behind the light, and a shock forms and moves up the street with speed

$$\sigma = \frac{G(\rho_0) - G(\rho_1)}{\rho_0 - \rho_1} = \frac{\rho_0 e^{-e\rho_0} + \rho_1 \log \rho_1 / e}{\rho_0 - \rho_1} = \frac{\rho_0 e^{-e\rho_0}}{\rho_0 - 1},$$

so the solution is

$$\rho(x,t) = \begin{cases} \rho_0 & \text{if } \frac{x}{t} < \frac{\rho_0 e^{-e\rho_0}}{\rho_0 - 1}, \\ 1 & \text{if } \frac{x}{t} > \frac{\rho_0 e^{-e\rho_0}}{\rho_0 - 1} \end{cases}$$

by (62).

2. Second step (Green Phase) $t > T_R$:

When the light turns green, traffic begins to move at the light, and a rarefaction forms; we do the same thing as in Greenshield's model, we take this instant to be the origin of time, then the rarefaction is centered at the origin, and has head and trail speed 1 and $-1/e$. The tail of the rarefaction runs into the shock at

$$t_0 = T_R \frac{-\rho_0 e^{-e\rho_0}}{\rho_0 e^{-e\rho_0} + \rho_0 - 1} \qquad x_0 = -T_R \frac{-\rho_0 e^{-e\rho_0}}{\rho_0 e^{-e\rho_0} + \rho_0 - 1}.$$

and the solution follows from (68).

3. Third step $t > t_r (t_r > T_R)$:

Now the speed of the shock changes. we also let $\phi(t)$ denote the position of the shock, then the shock speed is $\sigma = d\phi/dt$ and ϕ satisfies the differential equation

$$\frac{d\phi}{dt} = \frac{\rho_0 e^{-e\rho_0} + \rho_r \log(\rho_r)/e}{\rho_0 - \rho_r}$$

where ρ_r is given by equation (68), note for $x < 0$, the ODE is given by

$$\rho_0 e^{-e\rho_0} + e^{(-1-e\phi/t)}(-1 - e\phi/t)/e \frac{}{\rho_0 - e^{(-1-e\phi/t)} \text{if } \frac{-t}{e} < x < 0}$$

This equation is not linear as for Greenshield's model, but by introducing the new dependent variables $y = \phi/t$, we can solve the equation as follows: Let $\phi/t = y$, and denote the RHS of ODE as $H(y)$. Then we can get a new ODE as

$$y' = \frac{H(y) - y}{t}$$

This equation is separable and by integrating both side of above equation we get

$$t = t_0 e^{\int_{-1}^{0} \frac{1}{H(y) - y} dy} \tag{70}$$

we know that when $x = 0$ we have $y = 0$, and so we can get that the shock cross the point $x = 0$ at time

$$t_r = -T_R \frac{\rho_0(-e^{e+e\rho_0} + \rho_0 e^2(1 + e^{e\rho_0}))}{(\rho_0 e^2 - e^{e\rho_0})(\rho_0 - e^{e\rho_0} + \rho_0 e^{e\rho_0})} \tag{71}$$

Observe that as $\rho_0 \to \frac{1}{e}$, $\lim_{\rho_0 \to \frac{1}{e}} t_r = \infty$ which implies that for $\rho_0 \geq 1/e$ the shock will never cross $x = 0$. But when $\rho_0 < 1/e$, we can always get a t at which the shock cross $x = 0$. If we take $\rho_0 = 1/5$ and $T_R = 1/2$, we can observe the difference of the solution of $\phi(t)$ by using Greenshield's and Edie's model by the following figure:

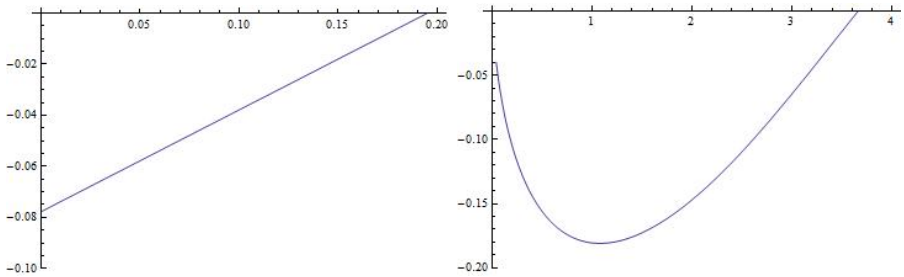

Figure 17: Shock curve for Greenshield's(left) and Edie's(right) model

Case 2: Incoming density $\rho_0 > 1/e$
Now cars back up behind the light, and a shock forms and moves up the street with speed

$$\sigma = \frac{G(\rho_0) - G(\rho_1)}{\rho_0 - \rho_1} = \frac{-\rho_0 \log(\rho_0)/e + \rho_1 \log(\rho_1)/e}{\rho_0 - \rho_1} = \frac{-\rho_0 \log(\rho_0)/e}{\rho_0 - 1}$$

when the light turns green, traffic begins to move at the light, and a rarefaction forms; we do the same thing as in case 1, we take this instant to be the origin of time, then the rarefaction is centered at the origin, and has head and trail speed 1 and $-1/e$. The tail of the rarefaction runs into the shock at

$$t_0 = T_R \frac{\rho_0 \log(\rho_0)/e}{-\rho_0 \log(\rho_0)/e + \rho_0 - 1} \qquad x_0 = -T_R \frac{\rho_0 \log(\rho_0)/e}{-\rho_0 \log(\rho_0)/e + \rho_0 - 1}$$

and now the speed of the shock changes. we also let let $\phi(t)$ denote the position of the shock, then the shock speed is $\sigma = d\phi/dt$ and ϕ satisfies the differential equation

$$\frac{d\phi}{dt} = \frac{-\rho_0 \log(\rho_0)/e + \rho_r \log(\rho_r)/e}{\rho_0 - \rho_r}$$

where ρ_r is given by equation (68), and with the initial condition $\phi(t_0) = -t_0 = x_0$, we use the same method as in **Case 1**, we can get same equation as (72) but different $H(y(t))$, however, for $\rho_0 > \frac{1}{e}$, the integral of (72) does not converge on $[-1, 0]$ which means for $\rho_0 > \frac{1}{e}$ the shock will never cross $x = 0$

By these two cases we can achieve that for Edie's model the greatest incoming density that permits free flow at the light ρ_0 is not equal to $1/2$ but $1/e$.

Now suppose that the incoming density is less than $1/e$, which means the shock can eventually cross the $x = 0$. It is also periodic beyond the light, and the average density a short distance downstream must also be ρ_0.

And the average density when the shock does not get through the light can be computed as below:

$$\bar{q} = \frac{1}{e^2} \frac{T_G}{T_G + T_R}$$

So the average density is given by

$$\bar{\rho} = \frac{1}{2} + \frac{\sqrt{1 - 4T_G/e^2(T_G + T_R)}}{2}$$

Now we try again let $T_R = T_G$,

$$\bar{\rho} = \frac{1}{2} + \frac{\sqrt{1 - 4T_G/e^2(T_G + T_R)}}{2} = 1/2 + \sqrt{1 - 2/e^2}/2 \approx 0.93$$

which also has dramatic difference if we compare to $1/e \approx 0.38$.

By the above calculation we can see that the result we get by applying different models are different, even though we apply the same incoming density, the shock curve appears huge different, however as we said earlier, there is no "right" solution, all of the models agree for certain types of traffic situations.

11 Acknowledgements

Firstly, I would like to thank my parents for sending me all the way to Sweden studying mathematics. Secondly, I would like to thank my advisor Sara Maad Sasane for introducing me a wonderful project to work on and for carefully explaining all of my questions. Thirdly, I would like to thank Professor Annemarie Luger for reading the manuscript carefully and for her valuable comments. Fourthly , I would like to thank my friend Isabelle and Anu for their great help of Latex and Mathematica.

List of Figures

1 Greenshield's model 3
2 Underwood's model 4
3 Greenberg's model 5
4 Edie's model 5
5 $t = -e^{-x_0^2}/4x_0$ (minimum t) 8
6 Characteristics 8
7 Time when shock occurs 13
8 Shock wave for piecewise linear initial data 14
9 Characteristic equation for a fixed t and the cusp domain 15
10 the smallest and largest solutions for $x_1 > x_2$ 18
11 shock curve for e^{-x^2} 19
12 Solution with wedge 20
13 Rarefaction wave 20
14 Shock curve filling in the wedge 21
15 The traffic light cycle(red) 32
16 The traffic light cycle(Green) 32
17 Shock curve for Greenshield's(left) and Edie's(right) model 37

References

[1] Lawrence C.Evans, Partial Differential Equations, American Mathematical Society, 1998.

[2] Barbara Lee Keyfitz, Hold That Light, Six themes on variation, American Mathematical Society, 2000.

[3] Daniel L.Gerlough and Matthew J.Huber, Traffic Flow theory A Monograph, Transportation Research Board, 1975.

[4] Ravi P.Agarwal, and Donal O'sRegan, An Introduction to Ordinary Differential Equation, Springer.

[5] Steven G.krantz and Harold R.parks, The Implicit function Theorem, Birkhauser 2003.

[6] Andre I. Khuri Advanced Calculus With Applications in Statistics, Second edition, WILEY, INTERSCIENCE, 2003

[7] P.I Richards , Shock waves on the high way , Operation research, 4:42-51, 1956.

[8] Michael Renardy Robert C. Rogers, An introduction to Partial Differential Equations, Springer, 1992.