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Pólya's inventory

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Abstract

This work aims at introducing Pólya's theory of enumeration. After an initial discussion regarding a general problem within combinatorial enumeration we devote some effort to group theory. Basic extracts from the theory of generating functions proves necessary to present, which serves to establish the concept of cycle index. Ultimately, we hope to reconcile the two main topics of this text: the Redfield-Pólya theorem as a continuation of Burnside's lemma.

Acknowledgements

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1

Introduction

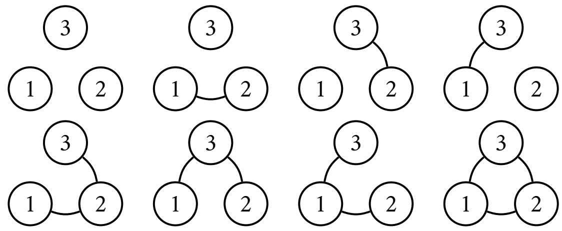
A tangible task such as a problem in enumeration — counting the number of things — can entail many inconveniences. Anyone that has ever opened a book on combinatorics can vouch for this. But the tricks of the trade are numerous, too, and here we shall provide at least one. In 1937 an article entitled *Kombinatorische Anzahlbestimmungen fur Gruppen, Graphen und chemische Verbindungen* came along. It was published in Acta Mathematica, Vol. 68, pp. 145 to 254. It's author was George Polya, and in it a theorem was to be found which gave method to solving a variety of problems related to enumeration. In short one can describe it as a way of counting — generate a sequence, even — of inequivalent mappings between finite sets: so-called patterns. This undertaking rests upon, and ties together, several areas within mathematics. Therefore, our discussion has to go in several directions throughout earlier parts of this text.

1.1 An Elucidating Example

A configuration is acquired by choosing elements of a (finite) set under certain conditions. In this text we deal with the problem of counting the number of configurations on a given set, not only under a prescribed combinatorial condition but also with respect to some imposed relation. An elucidating example is that of counting the number of undirected graphs with three vertices. In the usual state of affairs the condition that vertices are labelled is taken into account, providing us with the problem of finding all possible labelled graphs with three vertices, and counting them. The problem reduces firsty to that of specifying the number of edges in the graph while counting the number of possible graphs given this specific number of edges, and secondly to add up the results.

Consider a set $V = \{1, 2, 3\}$ of three labels, which shall serve as vertices: vertex 1, 2 and 3. The edge set E is a subset of $V \times V$, consisting of unordered pairs of elements in V, hence there are $\binom{3}{2} = 3$ possible edges we can use. Moreover, we specify how many edges must be in the graph we're considering. As shown in figure 1.1, there's only one possible graph with three labelled vertices and zero edges, three possible graphs with one and two edges respectively, and finally there is only one graph with three edges. Accounting for all

Figure 1.1: Every possible graph with the three labelled vertices 1, 2 and 3.



possibilities and summing them up,

$$\binom{\binom{3}{2}}{0} + \binom{\binom{3}{2}}{1} + \binom{\binom{3}{2}}{2} + \binom{\binom{3}{2}}{3} = 2^{\binom{3}{2}},$$

so provides us with 8 distinct graphs where $V = \{1, 2, 3\}$. Whilst this answer might be satisfactory there's a natural observation which can be made, namely that some of the graphs in figure 1.1 are up to isomorphism identical — permuting the labels of one graph yields another with the same number of edges. It is furthermore the case that starting with a specific graph, for example $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$, we obtain the two remaining graphs with two edges via a permutation. By use of $\sigma = (1, 2, 3)$, where $\sigma \in \mathfrak{S}_3$, the graph $G = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\})$ becomes $\sigma G = (\{1, 2, 3\}, \{\{1, 3\}, \{2, 3\}\})$. Yet another permutation using σ, this time on σG, yields $\sigma^2 G = (\{1, 2, 3\}, \{\{1, 2\}, \{1, 3\}\})$.

Figure 1.2: The graphs G, σG and $\sigma^2 G$ in the orbit of G.

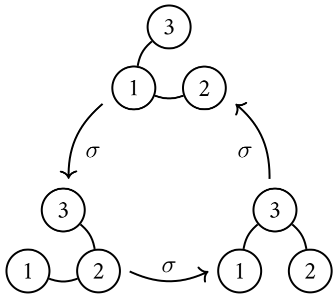
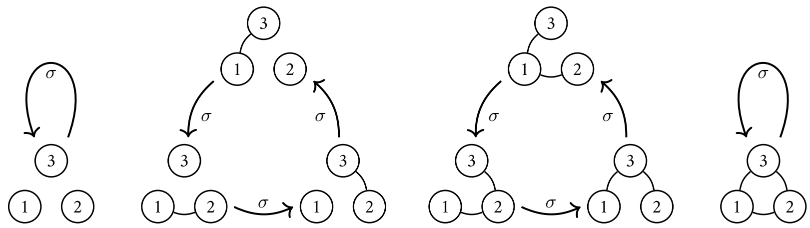


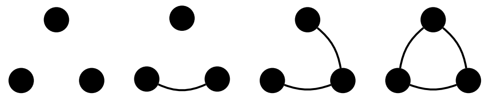
Figure 1.2 represents the orbit of G. Perhaps this situation is familiar. Under the group action of $\mathfrak{S}_3$ on the three letters 1, 2 and 3 (the vertex set), the set of all 8 distinct graphs reduces to that of four isomorphism classes, each representing an orbit corresponding to a graph of 0,1,2 or 3 edges, as shown in the figure (1.3) below.
Thus we have counted the number of configurations, namely the number of graphs with three vertices, under the imposed relation of isomorphism. This example illustrates our main concern throughout this text, namely that of counting equivalence classes of

Figure 1.3: The four orbits.



configurations. In Chapter 3 we shall elaborate on this concept in a general discussion on group actions and, in particular, in discussing a well known lemma attributed to William Burnside.

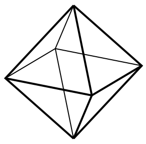
Figure 1.4: The four up to isomorphism distinct graphs with three vertices.



Before venturing any further there's something more to be mentioned. We will end this introduction with another, somewhat more involved, example which further elucidates the subject of this treatise. This example can be found in Pólya's original article [8].

1.2 Octahedron *We have at our disposal six balls with three different colours: three red balls, two blue balls and one yellow ball. Balls of the same color cannot be distinguished. The balls are to be assigned to the six vertices of an octahedron, which moves freely in space. In how many ways can this be done?*

Figure 1.5: Octahedron.

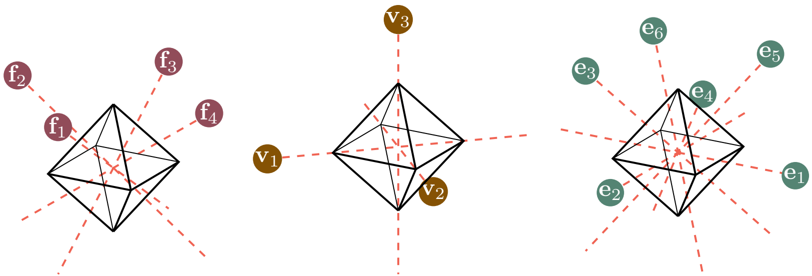


Under the ordinary combinatorial condition, such an arrangement corresponds to the multinomial

$$\binom{6}{3,2,1} = \frac{6!}{3!2!1!} = 60.$$

Here, however, we must also take into consideration arrangements which are equivalent under rotations of the octahedron. As in our previous example permutations are involved, this time we consider the permutation group of the octahedron, consisting of all possible transformations of the octahedron with respect to its symmetries. A transformation is a rotation about some axis of symmetry of the octahedron. In figure 1.6 the axes of symmetry are shown.

Figure 1.6: Symmetries of the octahedron.



There are 4 axes of symmetry, denoted $\mathbf{f}_n$*, going through the centers of two opposite faces. The axes of symmetry connecting opposite vertices are denoted* $\mathbf{v}_n$ *and those connecting the midpoints of two opposite edges are denoted* $\mathbf{e}_n$*.*

Rotating the octahedron about some axis, say $\mathbf{f}_4$, permutes the vertices on the opposite faces through which the axis runs. This corresponds to a permutation of cycle type $[3^2]$ — it consists of two cycles of order 3, which are disjoint. We assign the symbol x_k to a cycle of order k, hence the permutation about $\mathbf{f}_4$ that we're considering acquires the symbol x_3^2: *two cycles of order 3.* We note that every permutation about some axis $\mathbf{f}_n$ has the symbol x_3^2. In this manner we label every permutation with its appropriate symbol.

x_1^6: Doing nothing to the octrahedron is the same as rotating it 0° or 360° about some axis. This is the identity permutation, which consists of 6 cycles of order 1.

x_3^2: 120° rotation about an axis through two opposite faces. There are 4 axes of symmetry and rotation can be done clockwise or counterclockwise, hence in total there are 8 permutations of type f_3^2.

$x_1^2x_4$: 90° rotation about an axis through two opposite vertices. Yet again, rotation can be done clockwise or counterclockwise. There are 6 permutations of this type.

$x_1^2x_2^2$: 180° rotation about an axis through two opposite vertices. There are 3 permutations of this type.

x_2^3: 180° rotation about an axis through the midpoints of two opposite edges. There are 6 permutations of this type.

Thus, in the octahedral group there are 24 rotational symmetries accounting for the transformations we're interested in. By taking the arithmetic mean of the polynomial

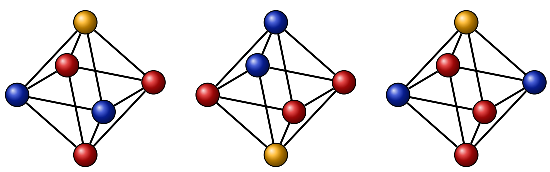
$$x_1^6 + 8x_3^2 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3 \tag{1.1}$$

we get what is called the *cycle index* of the octahedral group (the term was introduced by Pólya in [8]):

$$\frac{x_1^6 + 8x_3^2 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3}{24}. \tag{1.2}$$

The cycle index is crucial. Through substituting $x_1 = x + y + z$, $x_2 = x^2 + y^2 + z^2$, $x_3 = x^3 + y^3 + z^3$ and $x_4 = x^4 + y^4 + z^4$ into (1.2) and expanding in powers of x, y and z the solution to our problem is the coeffiecient before x^3y^2z, which turns out to be 3. Thus, when considering arrangements which are equivalent under rotational transformations there are 3 ways of assigning 3 red balls, 2 blue balls and one yellow ball to the vertices of the octahedron.

Figure 1.7: The three distinct assignments of colored balls to the vertices of the octahedron.



This example presents a remarkable concoction of different theories. As we've seen it utilizes concepts from group theory and extends on the lemma often attributed to Burnside (Burnside's lemma). A new idea is introduced, called the cycle index, which in an elegant way interacts with the theory of generating functions. In chapter 9, we begin in earnest our study of Pólya's Enumeration Theorem (PET), also called the Redfield-Pólya Theorem.

2

Permutations

> *AMONG the various notations used in the following pages, there is one of such frequent recurrence that a certain readiness in its use is very desirable in dealing with the subject of this treatise. We therefore propose to devote a preliminary chapter to explaining it in some detail.* (Burnside, [3])

2.1 On Permutations

In section 1.1 of chapter 1 we rearranged the three vertices of a graph. Specifically, we applied the operation of replacing each vertex by a different one, in such a way that no two vertices were replaced by one and the same vertex. In short, we applied an operation on the vertices called a permutation.

2.1.1 Definition. Let $a_1, a_2, a_3, \ldots, a_n$ be a set of n distinct letters. A permutation on the n letters is the operation of replacing each letter by another, which may be the same letter or a different one, under the condition that no two distinct letters be replaced by one and the same letter. A permutation will change any given arrangement $a_1, a_2, a_3, \ldots, a_n$, of the n letters, into a definite new arrangement $b_1, b_2, b_3, \ldots, b_n$ of the same letters. ◆

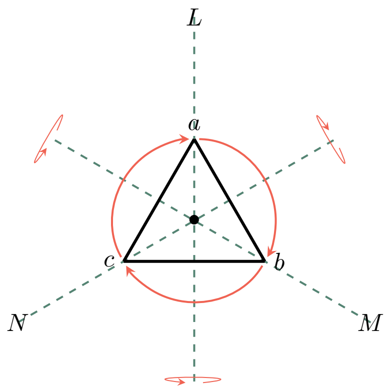
2.1.2 Definition. Let $S = \{a_1, a_2, a_3, \ldots, a_n\}$. A permutation on S can be defined, in an equivalent manner, as a mapping $\sigma : S \longrightarrow S$ which is 1-1 and onto. ◆

2.1.3 Definition. Let $S = \{a_1, a_2, a_3, \ldots, a_n\}$. A permutation σ of the set S can be written in Cauchy's two-line notation, where in a matrix one lists the letters of S in the first row, and the image of each letter in the second row:

$$\sigma = \begin{pmatrix} a_1 & a_2 & \ldots & a_n \\ \sigma(a_1) & \sigma(a_2) & \ldots & \sigma(a_n) \end{pmatrix}.$$ ◆

2.1.1 Example. The equilateral triangle $\triangle$ with vertices a, b, and c has rotational symmetry about its geometric centre •. The axes of symmetry are L, M and N. They are perpendicular to each edge, and passes through •. Picture rotating $\triangle$ about • by 120

Figure 2.1: An equilateral triangle and its symmetries.



degrees in the direction shown by the arrows. Label this transformation with the symbol σ. This transformation permutes the vertices: a is sent to b, and b to c. The resulting triangle coincides with the initial one, and the transformation sends $\triangle$ into itself. Denote σ by

$$\underbrace{\begin{pmatrix} a & b & c \\ b & c & a \end{pmatrix}}_{\sigma}.$$

Applying σ twice and three times to $\triangle$ yields

$$\underbrace{\begin{pmatrix} a & b & c \\ c & a & b \end{pmatrix}}_{\sigma^2} \text{ and } \underbrace{\begin{pmatrix} a & b & c \\ a & b & c \end{pmatrix}}_{\iota=\sigma^3}$$

where σ^3 is the transformation of rotating $\triangle$ by 360 or 0 degrees, doing nothing to $\triangle$. Label this transformation with the symbol ι. Reflection in some axis can be pictured as a rotation by 180 about the axis (flipping $\triangle$). Accounting for L, M, and N yields

$$\underbrace{\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}}_{\tau}, \underbrace{\begin{pmatrix} a & b & c \\ c & b & a \end{pmatrix}}_{\mu} \text{ and } \underbrace{\begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix}}_{\lambda}. \qquad \diamond$$

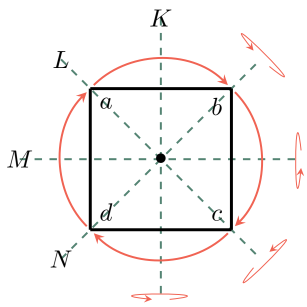
Remark. The transformations in example 2.1.1 can be done in composition. A clockwise rotation by 240 degrees of $\triangle$ followed by a flip with respect to the axis m would, in terms of the symbols σ^2 and μ, be the composition $\mu \circ \sigma^2$. The corresponding Cauchy two line notation

$$\underbrace{\begin{pmatrix} a & b & c \\ a & c & b \end{pmatrix}}_{\mu\circ\sigma^2},$$

is equivalent to performing the transformation τ on $\triangle$. Note that the composition $\mu \circ \sigma^2$ is read from right to left — the first transformation applied to $\triangle$ is σ^2, followed by the transformation μ applied to $\sigma^2(\triangle)$.

2.1.2 Example. Consider a square with vertices a, b, c, and d. It has rotational symmetry about its geometric centre •. The axes of symmetry are K, L, M, and N.

Figure 2.2: A square and its symmetries.



There are ways of transforming □ with respect to • or the axes K, L, M, or N. Picture rotating □ about • by 0, 90, 180, or 270 degrees in the direction shown by the arrows. Label these transformation by the symbols ι, σ, σ^2, or σ^3 respectively. Label a transformation by reflection in the axes K, L, M, and N by the symbols τ, μ, λ, and φ respectively. Listing all transformations in Cauchy's two line notation will suffice for this example. ◇

Table 2.1: Transformations of a square.

$$\underset{\iota}{\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}} \quad \underset{\sigma}{\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}} \quad \underset{\sigma^2}{\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}} \quad \underset{\sigma^3}{\begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}}$$

$$\underset{\tau}{\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}} \quad \underset{\mu}{\begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}} \quad \underset{\lambda}{\begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}} \quad \underset{\varphi}{\begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix}}$$

Remark. Here, too, transformations can be done in composition. A clockwise rotation by 90 degrees of □ followed by a flip with respect to the axis K would, in terms of the symbols σ and τ, be the composition $\tau \circ \sigma = \mu$.

2.1.4 Definition. Let $\sigma, \tau : S \longrightarrow S$ be permutations of a set S and let $x \in S$. Then $(\tau \circ \sigma)(x) = \tau(\sigma(x))$ and we define the product of permutations as $\tau\sigma(x) = \tau(\sigma(x))$. Hence permutations done in composition is the same as for composition of functions. ◆

2.1.3 Example. Let $\alpha, \beta : \{i\}_{i=1}^4 \longrightarrow \{i\}_{i=1}^4$ be permutations of $\{i\}_{i=1}^4 = \{1, 2, 3, 4\}$, where

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \text{ and } \beta = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Namely $\alpha(1) = 3,\ \alpha(2) = 4,\ \alpha(3) = 1,\ \alpha(4) = 2$, and $\beta(1) = 2,\ \beta(2) = 3,\ \beta(3) = 4,\ \beta(4) = 1$, so that the composite permutation $\alpha\beta$ is defined by $\alpha\beta(i) = \alpha(\beta(i))$:

$$\alpha\beta(1) = 4,\ \alpha\beta(2) = 1,\ \alpha\beta(3) = 2,\ \alpha\beta(4) = 3,\ \text{or}$$

$$\underset{\alpha\beta}{\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}}.$$ ◇

Remark. The permutations α and β in example 2.1.3 can be written in a different way, called *cycle notation*. In cycle notation α is written $(13)(24)$, and β is written (1234).

2.1.5 Definition. Let $\sigma : S \longrightarrow S$ be a permutation of a set S of n letters. The *cycle decomposition* of σ is obtained by choosing an letter $x \in S$, which begins the cycle, and thereafter applying σ repeatedly — first to x, then to $\sigma(x)$, and so on — so that for each successive time that σ is applied the image is entered as the next letter in the cycle. The cycle ends, and starts over, when an application of σ returns the original letter x. If the resulting cycle contains every letter of S it is exhaustive and we are done. Otherwise choose any letter $y \in S$ which does not belong to the resulting cycle, and repeat the process by constructing a cycle which begins with y. When all letters of S can be found in any of the cycles so created the set of cycles is exhaustive, and the cycles are disjoint. ◆

2.1.4 Example. Consider the permutation σ which in two line notation is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix},$$

viz. $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 4$, $\sigma(4) = 5$, and $\sigma(5) = 3$. Using definition 2.1.5 we obtain the cycle notation of σ.

i. Choose some letter, say 1, and apply σ repeatedly until 1 is returned: 1, $\sigma(1) = 2$, and $\sigma^2(1) = 1$, and so 1 is returned after two sucessive applications of σ, hence the process ends. We get the cycle (12).

ii. (12) does not contain the letter 3. So pick 3, and repeat the process: 3, $\sigma(3) = 4$, $\sigma^2(3) = 5$, and $\sigma^3(3) = 3$, and so 3 is returned after three sucessive applications of σ, hence the process ends. We get the cycle (345).

iii. Every letter 1, 2, 3, 4, and 5 is in some cycle, hence the set of cycles is exhaustive.

$$\sigma = (12)(345).$$ ◇

2.1.5 Example. Returning to the square □ in example 2.1.2, we once again consider the permutations ι, σ, σ^2, σ^3, τ, μ, λ, and φ — the transformations of □. Taking the product

Table 2.2: Transformations of a square.

$$\underset{\iota}{\begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}} \quad \underset{\sigma}{\begin{pmatrix} a & b & c & d \\ b & c & d & a \end{pmatrix}} \quad \underset{\sigma^2}{\begin{pmatrix} a & b & c & d \\ c & d & a & b \end{pmatrix}} \quad \underset{\sigma^3}{\begin{pmatrix} a & b & c & d \\ d & a & b & c \end{pmatrix}}$$

$$\underset{\tau}{\begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}} \quad \underset{\mu}{\begin{pmatrix} a & b & c & d \\ a & d & c & b \end{pmatrix}} \quad \underset{\lambda}{\begin{pmatrix} a & b & c & d \\ d & c & b & a \end{pmatrix}} \quad \underset{\varphi}{\begin{pmatrix} a & b & c & d \\ c & b & a & d \end{pmatrix}}$$

of any two permutations results in any of the above listed ones. This can be checked by means of a multiplication table. In it, the product is taken so that the rightmost factor is an permutation from the leftmost column while the leftmost factor is an permutation from the top row in the table, ie. $\varphi\,\mu = \sigma^2$.

Table 2.3: Product table for the transformations of a square.

□	ι	σ	σ^2	σ^3	τ	μ	λ	φ
ι	ι	σ	σ^2	σ^3	τ	μ	λ	φ
σ	σ	σ^2	σ^3	ι	μ	λ	φ	τ
σ^2	σ^2	σ^3	ι	σ	λ	φ	τ	μ
σ^3	σ^3	ι	σ	σ^2	φ	τ	μ	λ
τ	τ	φ	λ	μ	ι	σ^3	σ^2	σ
μ	μ	τ	φ	λ	σ	ι	σ^3	σ^2
λ	λ	μ	τ	φ	σ^2	σ	ι	σ^3
φ	φ	λ	μ	τ	σ^3	σ^2	σ	ι

◇

Remark. Table 2.3 is the muliplication table of D_4 — the dihedral group of order 4 — the group of rigid motions of a square.

2.1.6 Definition. Let S be a nonempty set. The set $\mathfrak{S}_S$ consists of all permutations of S.

2.1.7 Definition. Let $S = \{1, 2, 3, \dots, n\}$. The set $\mathfrak{S}_n$ is the set of all permutations of S. The cardinality of $\mathfrak{S}_n$ is $n!$, since there are $n!$ bijective mappings from S to S. ◆

2.1.6 Example. Let $S = \mathbb{N}_3 = \{1, 2, 3\}$, so that

$$\mathfrak{S}_3 = \{(1)(2)(3), (1)(23), (12)(3), (13)(2), (123), (132)\}\,.$$ ◇

2.1.1 Theorem. *$\mathfrak{S}_n$ has the following properties:*

I. *If τ and μ are permutations belonging to $\mathfrak{S}_n$, then $\tau\mu$ belongs to $\mathfrak{S}_n$ too;*

II. *For any permutations σ, τ, and μ belonging to $\mathfrak{S}_n$, their product is associative,*

$$(\sigma\tau)\mu = \sigma(\tau\mu);$$

III. *The identity permutation, denoted ι, belongs to $\mathfrak{S}_n$, so that for all $\sigma \in \mathfrak{S}_n$*

$$\iota\sigma = \sigma\iota = \sigma;$$

IV. *For every permutation $\sigma \in \mathfrak{S}_n$ there exists an inverse counterpart denoted σ^{-1}, in $\mathfrak{S}_n$, for which*

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = \iota.$$

Proof. I - IV follows immediately from the properties of bijective functions. ■

Remark. The properties which $\mathfrak{S}_n$ satisfies in theorem 2.1.1 are called *group axioms*, and $\mathfrak{S}_n$ is called *the symmetric group on n letters.*

2.1.7 Example. The product table for $\mathfrak{S}_3$ is the same as that for $\triangle$. ◇

Table 2.4: The product table of $\mathfrak{S}_3$.

$\mathfrak{S}_3$	(1)(2)(3)	(123)	(132)	(1)(23)	(13)(2)	(12)(3)
(1)(2)(3)	(1)(2)(3)	(123)	(132)	(1)(23)	(13)(2)	(12)(3)
(123)	(123)	(132)	(1)(2)(3)	(13)(2)	(12)(3)	(1)(23)
(132)	(132)	(1)(2)(3)	(123)	(12)(3)	(1)(23)	(13)(2)
(1)(23)	(1)(23)	(12)(3)	(13)(2)	(1)(2)(3)	(132)	(123)
(13)(2)	(13)(2)	(1)(23)	(12)(3)	(123)	(1)(2)(3)	(132)
(12)(3)	(12)(3)	(13)(2)	(1)(23)	(132)	(123)	(1)(2)(3)

Table 2.5: Replacing 1, 2, and 3 by a, b, and c table 2.4 becomes that of the rigid motions of $\triangle$.

$\triangle$	ι	σ	σ^2	τ	μ	λ
ι	ι	σ	σ^2	τ	μ	λ
σ	σ	σ^2	ι	μ	λ	τ
σ^2	σ^2	ι	σ	λ	τ	μ
τ	τ	λ	μ	ι	σ^2	σ
μ	μ	τ	λ	σ	ι	σ^2
λ	λ	μ	τ	σ^2	σ	ι

2.2 TYPE & CONJUGACY There are two basic but relevant topics on the theory of permutations which need to be adressed before we begin our section on group theory. Firstly we need to define the *cycle type* of a permutation, so that any permutation of a finite number of letters can be classified accordingly. Next comes a brief study of so called *conjugacy* which, in a nice way, relates to cycle types.

2.2.1 DEFINITION. Let $\sigma : S \longrightarrow S$ be a permutation of a finite set S. In cycle notation σ is written as a collection of cycles, where each cycle has a certain number of letters — the length of a cycle — and where there is a certain number of cycles of a specific length. The *type* of σ is a way of accounting for how many cycles of each length are present in the cycle decomposition of σ. We shall follow the notation used in [2]. ◆

2.2.1 EXAMPLE. Let σ be the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 7 & 2 & 5 & 4 & 8 & 1 & 6 & 9 \end{pmatrix},$$

which in cycle decomposition we write $\sigma = (1327)(45)(68)(9)$. The type of σ is expressed as an unordered list $[1,2,2,4]$: *one cycle of length 1, two cycles of length 2, and one cycle of length 4.* The list can be made more compact by introducing the notation $[1,2,2,4] := [1^1, 2^2, 4^1]$. ◇

2.2.2 DEFINITION. Two permutations $\sigma, \tau \in \mathfrak{S}_n$ are conjugate if there exists $\mu \in \mathfrak{S}_n$ such that $\mu\sigma\mu^{-1} = \tau$. ◆

2.2.2 EXAMPLE. In $\mathfrak{S}_6$, let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 4 & 1 \end{pmatrix} \quad \text{and} \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 6 & 1 & 4 \end{pmatrix},$$

viz. $\sigma = (1326)(45)$ and $\tau = (1235)(46)$. Then $\mu = (1)(4)(23)(56)$ is the permutation sought after for which $\mu\sigma\mu^{-1} = \tau$. Hence σ and τ are conjugate. ◇

2.2.1 THEOREM. *The permutations $\sigma, \tau \in \mathfrak{S}_n$ are conjugate if and only if they have the same cycle type.*

Proof. See [2]. ■

3

Groups

The ideas in the pages yet to be presented rest upon group theory. Of particular importance to us will be the study of group actions, orbits, and stabilizers which is presented in chapter 5. This undertaking necessitates a familiarity with the idea of a group. In the present chapter we shall review the most basic definitions and concepts and present them through examples.

3.1 Binary Operations

Foundational to the study of algebraic structures is the notion of binary operations. In example 2.1.5 in chapter 2 we found that the set of transformations of a square is closed under composition of transformations, which could be illustrated by use of table 2.3. Part of the reason for this is that composition of functions is a binary operation.

3.1.1 Definition. A *binary operation* $*$ on a nonempty set S is a mapping from the cartesian product $S \times S = \{(x, y) \mid x, y \in S\}$ into S:

$$\underset{x,y \in S}{(x, y)} \longmapsto *(x, y) \in S.$$

The element $*(x, y)$ in S is denoted $x * y$. ◆

3.1.1 Example. Ordinary addition $+$ and multiplication $\cdot$ are binary operations on the sets $\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$ (as is subtraction, except on the set $\mathbb{N}$). ◇

3.1.2 Example. Matrix addition in $M_{m \times n}(\mathbb{F})$ is a binary operation. Matrix multiplication in $M_{n \times n}(\mathbb{F})$ is a binary operation. ◇

3.1.3 Example. Taking the product of permutations in $\mathfrak{S}_n$ is a binary operation, as is composition of transformations of the polygons discussed in example 2.1.1 and 2.1.2. More generally, on the set of all functions from S to S, composition of functions is a binary operation. ◇

3.1.4 EXAMPLE. On $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ the functions $*(x, y) = \max\{x, y\}$, and $*(x, y) = \min\{x, y\}$ are binary operations. A somewhat exotic binary operation could be that of $*(A, B) = A \cap B$ where $A, B \in \wp(S)$, and $S = \{a, b, c\}$. ◇

3.1.5 EXAMPLE. An inner product $\langle , \rangle : \mathbf{V} \times \mathbf{V} \longrightarrow \mathbb{F}$ for some vector space $\mathbf{V}$ defined over $\mathbb{F}$ is not a binary operation. The distance function $d : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ is another example of a function which is not a binary operation. The specific reason being, in both cases, that the codomain is not a factor in the cartesian product which constitutes the domain. In the set $\mathbf{M}(\mathbb{F})$ of all matrices over some field $\mathbb{F}$, matrix addition is not a binary operation since matrix addition isn't even possible for matrices of different dimensions. ◇

3.1.2 DEFINITION. The binary operation $* : S \times S \longmapsto S$ is said to be *associative* if $x * (y * z) = (x * y) * z$, for all $x, y, z \in S$. If in S there exists an element e such that $e * x = x$ and $x * e = x$ then e is called an *identity element* for $*$. Given the existance of an identity element $e \in S$, if for $x \in S$ there exists a counterpart $y \in S$ such that $x * y = e$ and $y * x = e$ then y is said to be *an inverse* of x. ◆

3.1.6 EXAMPLE. In the set $\mathbf{GL}_n(\mathbb{F})$ of all invertible $n \times n$-matrices, both matrix addition and multipliciation is associative. Only for matrix multiplication an identity element exists, being I_n. Additive and multiplicative inverses exist for every $M \in \mathbf{GL}_n(\mathbb{F})$. ◇

3.1.7 EXAMPLE. In the set $\mathfrak{S}_n$, the product of permutations is an associative binary operation. This is merely a consequence of the fact that composition of functions is associative. The identity permutation belongs to $\mathfrak{S}_n$, and as we saw before there exists for every $\sigma \in \mathfrak{S}_n$ an inverse permutation σ^{-1}. ◇

3.2 GROUPS

A group is a set S equipped with a binary operation $*$ which satisfies the properties in definition 3.1.2, viz. $*$ is assocititive, has an identity element, and each element in S has an inverse. One often speaks of a set with a binary operation satisfying *the group axioms.*

3.2.1 DEFINITION (GROUP AXIOMS). A set G equipped with a binary operation $*$ is a *group* if the following properties hold.

I. If $x, y \in G$, then $x * y \in G$. (*Closure*);

II. For all $x, y, z \in G$, $x * (y * z) = (x * y) * z$. (*Associativity*);

III. There exists $e \in G$ so that $e * x = x$ and $x * e = x$ for all $x \in G$. (*Identity*);

IV. For each $x \in G$ there exists $y \in G$ so that $x * y = e$ and $y * x = e$. (*Inverse*). ◆

Remark. The correct way to denote a group is as an ordered pair $(G, *)$. Here the fancy symbol $\mathfrak{G}$ (black-letter G) will be used, admittedly for stylistic reasons but also as a kind of shorthand for $(G, *)$, and to distinguish a group from a graph. We allow for an abuse of notation by using $\mathfrak{G}$ when referring to the underlying set G.

3.2.1 EXAMPLE. In chapter 2 every example presented is a group. The set of all transformations of the equilateral triangle (example 2.1.1) is a group under composition of transformations, as is the set of all transformations of the square (example 2.1.2). In theorem 2.1.1 it is verified that $\mathfrak{S}_n$ is indeed a group. ◇

3.2.2 EXAMPLE. The set $\mathbf{GL}_n(\mathbb{F})$ of all invertible $n \times n$-matrices over the field $\mathbb{F}$ — *the general linear group* — is a group under matrix multiplication. So is $\mathbf{SL}_n(\mathbb{F})$ — *the special linear group* — consisting only of invertible $n \times n$-matrices with determinant equal to 1. ◇

3.2.3 EXAMPLE. The set $\mathbb{Q}^{\times} = \mathbb{Q} \setminus \{0\}$ is a group under ordinary multiplication. So are $\mathbb{R}^{\times}$, and $\mathbb{C}^{\times}$. ◇

3.2.4 EXAMPLE. For a non-empty set S the set of all permutations of S is Sym(S), which is a group under composition of functions. ◇

3.2.5 EXAMPLE. In example 2.1.7 in chapter 2, we saw that the product tables of $\mathfrak{S}_3$ and that of the rigid motions of △ coincided. This is because they are *isomorphic*, which means that $\mathfrak{S}_3$ and the group of rigid motions of △ represents the same group. From this perspective there is no reason to distinguish them other than for illustrative purposes. ◇

In dealing with groups, there are two basic properties which are central.

3.2.1 PROP. *For a group $\mathfrak{G}$, where $a, b, c \in \mathfrak{G}$, the following applies.*

I. *If $a * b = a * c$, then $b = c$.*

II. *If $a * c = b * c$, then $a = b$.*

Proof. See [1]. ■

3.2.2 PROP. *For a group $\mathfrak{G}$ where $a, b \in \mathfrak{G}$ the equations $a * x = b$ and $x * a = b$ has unique solutions.*

Proof. See [1]. ■

3.2.2 DEFINITION. A group $\mathfrak{G}$ is said to be *abelian* if $a * b = b * a$ for all $a, b \in \mathfrak{G}$. ◆

Remark. From now on the somewhat cumbersome notation of $*$ will be abandoned and replaced by the multiplicative notation, viz. $a * b$ will instead be written ab.

3.2.6 EXAMPLE. The set $\mathbf{M}_{m \times n}(\mathbb{R})$ is an abelian group under matrix addition. The set $\mathbb{Z}_5 = \{[0]_5, [1]_5, [1]_5, [3]_5, [4]_5\}$ of congruence classes *modulo 5* is an abelian group under addition of congruence classes, while $\mathbb{Z}_5^{\times} = \{[1]_5, [2]_5, [3]_5, [4]_5\}$ is an abelian group under multiplication of congruence classes. ◇

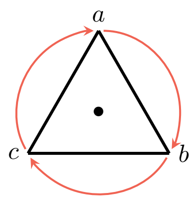
Remark. Oftentimes $\mathbb{Z}_n$ and $\mathbb{Z}_n^{\times}$ are written $\mathbb{Z}/n\mathbb{Z}$ and $\left(\mathbb{Z}/n\mathbb{Z}\right)^{\times}$.

3.2.3 DEFINITION. If in $\mathfrak{G} = (G, *)$ the set G is finite, $\mathfrak{G}$ is said to be a finite group and we denote the *order* of $\mathfrak{G}$ by $|\mathfrak{G}|$. ◆

3.2.7 EXAMPLE. The group of rigid motions of a regular n-gon is denoted $\mathfrak{D}_n$. As it has n rotational symmetries and n reflective symmetries $|\mathfrak{D}_n| = 2n$. It is therefore called *the dihedral group of order* $2n$. In examples 2.1.1, and 2.1.2 — the rigid motions of △ and □ — we are dealing with the dihedral groups $\mathfrak{D}_3$, and $\mathfrak{D}_4$ where $|\mathfrak{D}_3| = 6$, and $|\mathfrak{D}_4| = 8$. ◇

3.2.8 EXAMPLE. Returning to the equilateral triangle in example 2.1.1, we consider only the rotations with respect to its geometric centre. We denote these transformations by

Figure 3.1: Rotations of △ about •.



$\iota = (a)(b)(c)$, $\sigma = (abc)$, $\sigma^2 = (acb)$, and obtain $\mathfrak{C}_3 = \left(\left\{\iota, \sigma, \sigma^2\right\}, \circ\right)$, which is a group under composition of transformations. $\mathfrak{C}_3$ is short for *the cyclic group of order 3* and is commonly expressed in terms of some generator a as $\mathfrak{C}_3 = \left\langle a \mid a^3 = e\right\rangle$. This is an example of an abelian group. Moreover, it is an example of a *subgroup* of the rigid motions of △, and — as per example 3.2.5 — $\mathfrak{C}_3$ is a subgroup of $\mathfrak{S}_3$. ◇

Table 3.1: Product table of the rigid motions of △ restricted to rotations about •.

△	ι	σ	σ^2	τ	μ	λ
ι	ι	σ	σ^2	τ	μ	λ
σ	σ	σ^2	ι	μ	λ	τ
σ^2	σ^2	ι	σ	λ	τ	μ
τ	τ	λ	μ	ι	σ^2	σ
μ	μ	τ	λ	σ	ι	σ^2
λ	λ	μ	τ	σ^2	σ	ι

3.3 SUBGROUPS

In example 3.2.8 we restricted the set of rigid motions for an equilateral triangle to contain only rotations about its geometric centre, and discovered that this set was closed under the same operation — that of composition — as for the original group of rigid motions of △.

3.3.1 DEFINITION. For a group $\mathfrak{G} = (G, *)$, let $H \subseteq G$. Then $\mathfrak{H} = (H, *)$ is said to be a *subgroup of* $\mathfrak{G}$ if $\mathfrak{H}$ is itself a group, that is if H is a group under $*$ — the binary operation induced by $\mathfrak{G}$. ◆

Remark. A group $\mathfrak{H}$ being a subgroup of $\mathfrak{G}$ is written $\mathfrak{H} \leqslant \mathfrak{G}$. If $H \subset G$ (H is a proper subset of G), then $\mathfrak{H} < \mathfrak{G}$ ($\mathfrak{H}$ is a proper subgroup of $\mathfrak{G}$).

3.3.1 EXAMPLE. $(\mathbb{Z}, +) < (\mathbb{Q}, +) < (\mathbb{R}, +) < (\mathbb{C}, +)$, $(\mathbb{Q}^\times, \cdot) < (\mathbb{R}^\times, \cdot) < (\mathbb{C}^\times, \cdot)$, and $(m\mathbb{Z}_n, +) < (k\mathbb{Z}_n, +)$ if m is a multiple of k. ◇

3.3.2 EXAMPLE. $(\mathfrak{C}_n, \circ) < (\mathfrak{D}_n, \circ) < (\mathfrak{S}_n, \circ)$, and $(\langle i \rangle, \cdot) < (\mathbb{C}, \cdot)$ where $\langle i \rangle = \{i, -1, -i, 1\}$. ◇

3.3.3 EXAMPLE. $\mathbf{SL}_n(\mathbb{F}) < \mathbf{GL}_n(\mathbb{F})$ where $\forall M \in \mathbf{SL}_n(\mathbb{F}) : \det M = 1$. ◇

3.3.4 Example. Consider $O = \left\{ \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} : 0 \leq \varphi < 2\pi \right\}$. Since

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \cos(\varphi+\theta) & -\sin(\varphi+\theta) \\ \sin(\varphi+\theta) & \cos(\varphi+\theta) \end{pmatrix}$$

for any two matrices in O — due to standard trigonometric identities — O is closed under matrix multiplication. Moreover, for any matrices $A, B, C \in O$, $A(BC) = (AB)C$ since matrix multiplication is associative. For $\varphi = 0$:

$$\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence the unit matrix is contained in O. Lastly we have that $\det\begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix} = 1$, again due to trigonometry, regardless of φ. This is enough to verify that any matrix in O is invertible, and tells us that $O \subset \mathbf{SL}_2(\mathbb{R})$. Most importantly O satisfies all of the group axioms, and is a subgroup of $\mathbf{SL}_2(\mathbb{R})$. ◇

Remark. The group considered is called *the rotation group for* $\mathbb{R}^2$, or *the special orthogonal group for* $\mathbb{R}^2$, often denoted $\mathbf{SO}_2$. By rotations about the origin, it acts on vectors in $\mathbb{R}^2$.

3.3.5 Example. Consider $\mathbf{GL}_n(\mathbb{F}_p)$ — *the general linear group* — over a finite base field of order p. There's a bijection between an invertible matrix M in $\mathbf{GL}_n(\mathbb{F}_p)$ and a unique basis consisting of the columns of M, which spans $V(\mathbb{F}_p)$, since they are linearly independent due to M being invertible. This confronts us with the task of finding the number of bases for $V(\mathbb{F}_p)$. We achieve this by counting the number of basis vectors which can be chosen. The first basis vector $\mathbf{v}_1$ allows for any of the p elements of $\mathbb{F}_p$ in all of the n coordinates, except for an occurance of the zero vector. Thus $p^n - 1$ is the number of ways to construct the first basis vector. The second one, $\mathbf{v}_2$, is similarly constructed — except for any of the p linear combinations of $\mathbf{v}_1$. Thus $p^n - p$ is the number of ways to construct $\mathbf{v}_2$. There are p^2 linear combinations of $\mathbf{v}_1$, and $\mathbf{v}_2$. Hence there are $p^n - p^2$ ways to construct $\mathbf{v}_3$. Generally, there are p^k linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ and so there are $p^n - p^k$ ways to build the $k+1$:th vector. By the rule of product:

$$\left|\mathbf{GL}_n(\mathbb{F}_p)\right| = \prod_{k=0}^{n-1}\left(p^n - p^k\right). \quad ◇$$

3.3.6 Example. The factor group $\mathbb{F}_n^\times = \left(\mathbb{Z}/n\mathbb{Z}\right)^\times$ contains the invertible elements of $\mathbb{F}_n = \mathbb{Z}/n\mathbb{Z}$. An element $[x] \in \mathbb{Z}/n\mathbb{Z}$ has an inverse if, and only if $\gcd(x, n) = 1$. The order of $\left(\mathbb{Z}/n\mathbb{Z}\right)^\times$ must therefore equal the number of integers k, where $1 \leq k < n$ such that $\gcd(k, n) = 1$. This is the definition of Euler's totient function $\varphi(n)$. Hence

$$\left|\left(\mathbb{Z}/n\mathbb{Z}\right)^\times\right| = \varphi(n). \quad ◇$$

Remark. For a prime number $n = p$, $\varphi(p) = p - 1$, so that $\left|\left(\mathbb{Z}/p\mathbb{Z}\right)^\times\right| = p - 1$. In the list of every positive integer from 1 to p^n, there are p^{n-1} multiples of p, hence

$$\left|\left(\mathbb{Z}/p^n\mathbb{Z}\right)^\times\right| = p^n - p^{n-1}.$$

3.3.2 DEFINITION. For $\mathfrak{H}$ a subgroup of $\mathfrak{G}$, where $a \in \mathfrak{G}$, the set

$$a\mathfrak{H} = \{x \in \mathfrak{G} : x = ah, \text{ for some } h \in \mathfrak{H}\}$$

is called the *left coset of* $\mathfrak{H}$ *in* $\mathfrak{G}$ determined by a. The *right coset of* $\mathfrak{H}$ *in* $\mathfrak{G}$ determined by a is the set

$$\mathfrak{H}a = \{x \in \mathfrak{G} : x = ha, \text{ for some } h \in \mathfrak{H}\}.$$ ◆

3.3.1 LEMMA. *For* $\mathfrak{H}$ *a subgroup of* $\mathfrak{G}$, *and* $a, b \in \mathfrak{G}$, *either* $a\mathfrak{H} = b\mathfrak{H}$ *or* $a\mathfrak{H} \cap b\mathfrak{H} = \varnothing$.

Proof. Suppose that $x \in a\mathfrak{H} \cap b\mathfrak{H}$, then $x \overset{(1)}{=} ah_1$ and $x \overset{(2)}{=} bh_2$. Now let $y \in a\mathfrak{H}$, viz. $y = ah$ for some $h \in \mathfrak{H}$. We wish to show that $a\mathfrak{H} \subseteq b\mathfrak{H}$, and $b\mathfrak{H} \subseteq a\mathfrak{H}$. By (1), y can be written as $y = \left(xh_1^{-1}\right)h$ which, by associativity, is equivalent to $y = x\left(h_1^{-1}h\right)$. By (2), $y = (bh_2)\left(h_1^{-1}h\right)$ so that $y = b\left(h_2h_1^{-1}h\right)$, where $h_2h_1^{-1}h \in \mathfrak{H}$, hence y is an element in $b\mathfrak{H}$. Therefore $a\mathfrak{H} \subseteq b\mathfrak{H}$. To show that $b\mathfrak{H} \subseteq a\mathfrak{H}$ a similar argument applies, and we conclude that $a\mathfrak{H} = b\mathfrak{H}$. ■

3.3.2 LAGRANGE'S THEOREM. *If* $\mathfrak{G}$ *is a finite group and* $\mathfrak{H}$ *is a subgroup of* $\mathfrak{G}$, *then the order of* $\mathfrak{H}$ *divides the order of* $\mathfrak{G}$.

Proof. Each left coset of $\mathfrak{H}$ has the same cardinality as $\mathfrak{H}$, and by lemma 3.3.1 each left coset is distinct. Hence the left cosets of $\mathfrak{H}$ partition $\mathfrak{G}$, so that $|\mathfrak{G}| = k\,|\mathfrak{H}|$ where k equals the number of left cosets of $\mathfrak{H}$ in $\mathfrak{G}$. ■

3.3.3 DEFINITION. The number of left cosets of $\mathfrak{H}$ in $\mathfrak{G}$ is written $[\mathfrak{G} : \mathfrak{H}]$. ◆

3.3.7 EXAMPLE. The general linear group $\mathbf{GL}_n(\mathbb{F}_p)$ over a finite base field $\mathbb{F}_p$ is a group of finite order, where $\mathbf{SL}_n(\mathbb{F}_p) < \mathbf{GL}_n(\mathbb{F}_p)$. By Lagrange's Theorem

$$\left|\mathbf{GL}_n(\mathbb{F}_p)\right| = \left[\mathbf{GL}_n(\mathbb{F}_p) : \mathbf{SL}_n(\mathbb{F}_p)\right] \cdot \left|\mathbf{SL}_n(\mathbb{F}_p)\right|.$$

The set of left cosets of $\mathbf{SL}_n(\mathbb{F}_p)$ in $\mathbf{GL}_n(\mathbb{F}_p)$ is written $\mathbf{GL}_n(\mathbb{F}_p) / \mathbf{SL}_n(\mathbb{F}_p)$, where

$$\left|\mathbf{GL}_n(\mathbb{F}_p) / \mathbf{SL}_n(\mathbb{F}_p)\right| = \left[\mathbf{GL}_n(\mathbb{F}_p) : \mathbf{SL}_n(\mathbb{F}_p)\right]. \tag{3.1}$$

The elements in $\mathbf{GL}_n(\mathbb{F}_p) / \mathbf{SL}_n(\mathbb{F}_p)$ are equivalence classes, each containing matrices whose determinants are equal. We can therefore establish a bijection between equivalence classes and $\mathbb{F}_p^\times$. Thus $\left|\mathbf{GL}_n(\mathbb{F}_p) / \mathbf{SL}_n(\mathbb{F}_p)\right| = |\mathbb{F}_p^\times|$, where $|\mathbb{F}_p^\times| = p - 1$. Hence $\left[\mathbf{GL}_n(\mathbb{F}_p) : \mathbf{SL}_n(\mathbb{F}_p)\right] \underset{(3.2)}{=} p - 1$ by which we can compute that

$$\left|\mathbf{SL}_n(\mathbb{F}_p)\right| \underset{(3.1)}{=} \frac{\prod_{k=0}^{n-1}\left(p^n - p^k\right)}{p-1}.$$ ◇

3.4 Isomorphisms

Later on we will make use of *Cayley's Theorem* for which some essential terminology is required. We have previously hinted that two groups differing in appearance while displaying the same essential properties are not to be distinguished. One says that two such groups are *isomorphic* — essentially the same.

3.4.1 Definition. For two groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$, a *group isomorphism* is a bijective mapping

$$\phi : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2,$$

such that

$$\phi(g_1 g_2) = \phi(g_1)\phi(g_2),$$

for all $g_1 \in \mathfrak{G}_1$ and for all $g_2 \in \mathfrak{G}_2$. For the product $g_1 g_2$ the underlying binary operation is $*_1$ in $\mathfrak{G}_1$, while for $\phi(g_1)\phi(g_2)$ it is $*_2$. If an isomorphism exists between $\mathfrak{G}_1$ and $\mathfrak{G}_2$ the groups are isomorphic, which we write

$$\mathfrak{G}_1 \cong \mathfrak{G}_2. \qquad \blacklozenge$$

As a direct consequence of definition 3.4.1 it can easily be shown that for $e_1 \in \mathfrak{G}_1$, $\phi(e_1) = e_2 \in \mathfrak{G}_2$, and that for all $g \in \mathfrak{G}_1$ $\phi\left(g^{-1}\right) = \phi(g)^{-1}$.

3.4.1 Example. For two groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ their product $\mathfrak{G}_1 \times \mathfrak{G}_2$ also constitutes a group, called *the direct product of* $\mathfrak{G}_1$ *and* $\mathfrak{G}_2$ — where $(a_1, a_2) * (b_1, b_2) = (a_1 *_1 b_1, a_2 *_2 b_2)$, for $*_1 \in \mathfrak{G}_1$ and $*_2 \in \mathfrak{G}_2$. Moreover $\mathfrak{G}_1 \times \mathfrak{G}_2 \cong \mathfrak{G}_2 \times \mathfrak{G}_1$. The mapping

$$\phi : \mathfrak{G}_1 \times \mathfrak{G}_2 \longrightarrow \mathfrak{G}_2 \times \mathfrak{G}_1$$

by $(g_1, g_2) \longmapsto (g_2, g_1)$ is a bijection, which is easily verified. $\diamond$

3.4.2 Example. $(\mathbb{C}, +) \ncong \left(\mathbb{C}^\times, \cdot\right)$. In $\left(\mathbb{C}^\times, \cdot\right)$, the element i has order 4 while there exists no element in $(\mathbb{C}, +)$ of order 4. $\diamond$

3.4.3 Example. The set $F = \left\{f_{a,b} : \mathbb{R} \longrightarrow \mathbb{R} : f(x)_{a,b} = ax + b, \text{ where } a \neq 0\right\}$ is a group under composition of functions, and the set $U = \left\{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \neq 0\right\}$ is a subgroup of $\mathbf{GL}_2(\mathbb{R})$. The mapping

$$f_{a,b} \underset{\phi}{\longmapsto} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

is one-to-one by $\phi\left(f_{a,b}\right) = \phi\left(f_{c,d}\right) \iff a = c$ and $b = d \implies f_{a,b} = f_{c,d}$. For any $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in U$ there clearly exists $f_{a,b} \in F$ so that $\phi\left(f_{a,b}\right) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, so ϕ is onto. Lastly ϕ preserves group products, viz. $\phi\left(f_{a,b} \circ f_{c,d}\right) = \phi\left(f_{ac,ad+b}\right) = \begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix}$, where $\begin{pmatrix} ac & ad+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix} = \phi\left(f_{a,b}\right)\phi\left(f_{x,y}\right)$. Hence

$$(F, \circ) \cong (U, \cdot). \qquad \diamond$$

3.4.4 EXAMPLE. For a group $\mathfrak{G}$ and a fixed element $a \in \mathfrak{G}$, the mapping $\phi_a : \mathfrak{G} \longrightarrow \mathfrak{G}$ by $g \longmapsto aga^{-1}$ is an isomorphism. Assuming that $\phi(x) = \phi(y) \iff axa^{-1} = aya^{-1}$ cancellation immediately yields that $x = y$. Surjectivity is verified by picking the element $y \in \mathfrak{G}$, and since a is in $\mathfrak{G}$, a^{-1} too must be in $\mathfrak{G}$. Hence $a^{-1}ya$ is in $\mathfrak{G}$, and $\phi\left(a^{-1}ya\right) = aa^{-1}yaa^{-1} = y$, so φ is onto. Lastly we verify the conservation of products by

$$\phi(xy) = axya^{-1} = axa^{-1}aya^{-1} = \left(axa^{-1}\right)\left(aya^{-1}\right) = \phi(x)\phi(y). \qquad \diamond$$

3.5 HOMOMORPHISMS

Abandoning the requirements of bijectivity, while keeping the requirements for a mapping between groups to conserve products, we end up with a *group homomorphism.*

3.5.1 DEFINITION. A mapping ϕ between the groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is a *homomorphism* if

$$\phi(xy) = \phi(x)\phi(y),$$

for all $x, y \in \mathfrak{G}_1$. ◆

3.5.1 EXAMPLE. Returning to the group $\mathbf{GL}_n\left(\mathbb{F}_p\right)$ of example 3.3.5, we define the mapping $\phi : \mathbf{GL}_n\left(\mathbb{F}_p\right) \longrightarrow \mathbb{F}_p^\times$, by $M \longmapsto \det M$. Since $\mathbb{F}_p^\times$ is a group under multiplication, and since $\det XY = \det X \det Y$ for matrices $X, Y \in \mathbf{GL}_n\left(\mathbb{F}_p\right)$, we have established that ϕ is a homomorphism. ◇

3.5.2 DEFINITION. The *kernel* of a homomorphism ϕ between the groups $\mathfrak{G}_1$ and $\mathfrak{G}_2$ is the set

$$\ker\phi = \{g \in \mathfrak{G}_1 : \phi(g) = e \in \mathfrak{G}_2\}. \qquad ◆$$

3.5.2 EXAMPLE. As established in example 3.5.1, the mapping $\phi : \mathbf{GL}_n\left(\mathbb{F}_p\right) \longrightarrow \mathbb{F}_p^\times$ is a homomorphism, and $\ker\phi = \mathbf{SL}_n\left(\mathbb{F}_p\right)$. ◇

3.5.3 DEFINITION. A subgroup $\mathfrak{H}$ of the group $\mathfrak{G}$ is called *normal* if $ghg^{-1} \in \mathfrak{H}$ for all $h \in \mathfrak{H}$ and $g \in \mathfrak{G}$. For $\mathfrak{N}$ a normal subgroup of $\mathfrak{G}$, one writes

$$\mathfrak{N} \triangleleft \mathfrak{G}. \qquad ◆$$

3.5.1 PROP. *For $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ it holds that $ghg^{-1} \in \mathfrak{H}$ for all $h \in \mathfrak{H}$ and $g \in \mathfrak{G}$ if, and only if $g\mathfrak{H} = \mathfrak{H}g$ for all $g \in \mathfrak{G}$.*

Proof. Assume that $ghg^{-1} \in \mathfrak{H}$ for all $h \in \mathfrak{H}$ and $g \in \mathfrak{G}$. We need to show that $g\mathfrak{H} = \mathfrak{H}g$ for all $g \in \mathfrak{G}$, which holds if $g\mathfrak{H} \subseteq \mathfrak{H}g$ and $\mathfrak{H}g \subseteq g\mathfrak{H}$. Let h be an arbitrary element in $\mathfrak{H}$, then $ghg^{-1} \in \mathfrak{H}$ by the assumption that $\mathfrak{H}$ is normal. Hence $ghg^{-1} = h'$ for some $h' \in \mathfrak{H}$, so that $gh = h'g$ which entails that $gh \in \mathfrak{H}g$ since h was chosen arbitrarily. The other entailment is analogous. Now, assume that $g\mathfrak{H} = \mathfrak{H}g$ for all $g \in \mathfrak{G}$, and let $gh \in g\mathfrak{H}$. Then $gh = h'g$ by our assumption, hence $ghg^{-1} = h' \in \mathfrak{H}$. ■

3.5.4 DEFINITION. $\mathfrak{H}$ a subgroup of $\mathfrak{G}$ is called *normal* if $g\mathfrak{H} = \mathfrak{H}g$ for all $g \in \mathfrak{G}$. ◆

3.5.3 Example. For a group $\mathfrak{G}$, the set $\mathbf{Z}(\mathfrak{G}) = \{g \in \mathfrak{G} : ag = ga, \text{ for all } a \in \mathfrak{G}\}$ is called *the center of* $\mathfrak{G}$. Furthermore, $\mathbf{Z}(\mathfrak{G}) \lhd \mathfrak{G}$. For any $a, b \in \mathbf{Z}(\mathfrak{G})$ and $g \in \mathfrak{G}$ we have that $(ab)g = a(bg) = a(gb) = (ag)b = g(ab)$ so that ab commutes with every $g \in \mathfrak{G}$, hence $ab \in \mathbf{Z}(\mathfrak{G})$. Associativity is inherited, and $e \in \mathbf{Z}(\mathfrak{G})$ since it commutes with every other element in $\mathfrak{G}$. For any $a \in \mathbf{Z}(\mathfrak{G})$ we have that $ag = ga \iff ga^{-1} = a^{-1}g$ for all $g \in \mathfrak{G}$, hence $a^{-1} \in \mathbf{Z}(\mathfrak{G})$. Normality follows immediately from the definition of $\mathbf{Z}(\mathfrak{G})$, since for $a \in \mathbf{Z}(\mathfrak{G})$ we have that $ag = ga \iff gag^{-1} = a \in \mathbf{Z}(\mathfrak{G})$ for all $g \in \mathfrak{G}$. ◇

3.5.4 Example. For a homomorphism $\phi : \mathfrak{G}_1 \longrightarrow \mathfrak{G}_2$, $\ker\phi \lhd \mathfrak{G}_1$. For any two $a, b \in \ker\phi$ we have that $\phi(ab) = \phi(a)\phi(b) = e$ since ϕ is a homomorphism, hence $\ker\phi$ is closed. Associativity is inherited, and $\phi(e) = e$, so that $e \in \ker\phi$. Furthermore, $\phi(a)\phi\left(a^{-1}\right) = e \iff \phi\left(a^{-1}\right) = \phi(a)^{-1} = e$, hence $a^{-1} \in \ker\phi$. ◇

3.5.5 Example. Let $\mathfrak{G}$ be a finite group and let $\mathfrak{H}$ be a subgroup of $\mathfrak{G}$. Furthermore, let $[\mathfrak{G} : \mathfrak{H}] = 2$. Then $\mathfrak{H}$ has two left cosets in $\mathfrak{G}$, the first one being $x\mathfrak{H} = x\mathfrak{H}$ for all $x \in \mathfrak{H}$, and the second one being $x\mathfrak{H} = \mathfrak{G} \setminus \mathfrak{H}$ for all $x \notin \mathfrak{H}$. The right cosets of $\mathfrak{H}$ are $\mathfrak{H}x = \mathfrak{H}$ for all $x \in \mathfrak{H}$, and $\mathfrak{H}x = \mathfrak{G} \setminus \mathfrak{H}$ for all $x \notin \mathfrak{H}$. Thus, $x\mathfrak{H} = \mathfrak{H}x$ for all $x \in \mathfrak{H}$. Since the cosets partition $\mathfrak{G}$ into $\mathfrak{H}$ and $\mathfrak{G} \setminus \mathfrak{H}$, while $x\mathfrak{H} = \mathfrak{H}x$, it follows that $x\mathfrak{H} = \mathfrak{H}x$ for all $x \notin \mathfrak{H}$. Therefore $x\mathfrak{H} = \mathfrak{H}x$, both for $x \in \mathfrak{H}$ and $x \notin \mathfrak{H}$, i.e. for all $x \in \mathfrak{G}$. This is the definition of a normal subgroup, hence $\mathfrak{H} \lhd \mathfrak{G}$. ◇

3.5.6 Example. The *quaternion group* $\mathbf{Q}_8 = (Q, \cdot)$, where $Q = \{1, -1, i, -i, j, -j, k, -k\}$, is given by

$$\left\langle i, j, k \;\middle|\; \begin{matrix} i^2 = j^2 = k^2 = -1 \\ ij = k, jk = i, ki = j \\ ji = -k, kj = -i, ik = -j \end{matrix} \right\rangle.$$

Since $o(\pm i) = o(\pm j) = o(\pm k) = 4$ the only element of order 2 is -1, and so $\langle -1 \rangle = \{1, -1\} \leqslant \mathbf{Q}_8$ is the only subgroup of order 2. Looking at the above stated identities we observe that -1 and 1 also happens to be the only elements which commute with every other element of $\mathbf{Q}_8$. So $\mathbf{Z}(\mathbf{Q}_8) = \langle -1 \rangle$, hence $\langle -1 \rangle \lhd \mathbf{Q}_8$. For each remaining, non-trivial subgroup, we have that $[\mathbf{Q}_8 : \langle i \rangle] = [\mathbf{Q}_8 : \langle j \rangle] = [\mathbf{Q}_8 : \langle k \rangle] = 2$ so that $\langle i \rangle, \langle j \rangle, \langle k \rangle \lhd \mathbf{Q}_8$ by the fact that any subgroup with index 2 is normal as seen in example 3.5.5. ◇

4

Polytopes

In this brief interlude we look at the symmetries of some of the regular polygons and polyhedrons that we'll be dealing with later. We introduce a numerical labelling of the vertices, by means of which we express the group of symmetries — mappings of an object into itself — in a more familiar way, namely as a collection of permutations of a set of numbers.

4.1 Gons & Hedrons

4.1.1 Definition. A group $\mathfrak{G}$ can be written in terms of its *generating set* . Much like the idea of a linear hull — a set of basis vectors — spanning a vector space, a generating set is a set of group elements such that every element of $\mathfrak{G}$ can be expressed as a product of elements in the generating set. The generating set of $\mathfrak{G}$ is written

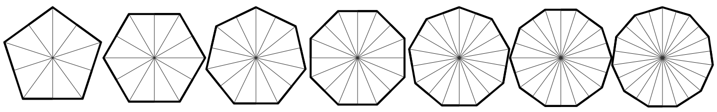
$$\langle g_1, g_2, \dots, g_n \in \mathfrak{G} : r_1(g_1), r_2(g_2), \dots, r_n(g_n) \rangle ,$$

where r_i $(1 \le i \le n)$ is some rule under which the *generator* g_i functions. ◆

4.1.1 Example. The dihedral group of order $2n$ is generated by a cycle containing $1, 2, \dots, n$, and a transposition, which is a cycle only containing two elements of $1, 2, \dots, n$, i.e. (12). Let $\sigma = (12 \dots n)$, $\tau = (12)$, and $(1) = e$, then

$$\mathfrak{D}_n = \left\langle \sigma, \tau : \sigma^n = e, \tau^2 = e, \tau\sigma = \sigma^{-1}\tau \right\rangle . \qquad \diamond$$

Figure 4.1: The n-gons for $5 \le n \le 11$.



4.1.2 EXAMPLE. The equilateral triangle △ has rotational symmetry about its geometric centre •. The axial symmetries are L, M and N. They are perpendicular to each edge, and passes through • (cf. figure 2.1). Label the vertices by 1, 2, and 3. Picture a clockwise rotation of △ about • by 120 degrees. Denote this transformation by (123). This transformation permutes the vertices: 1 is sent to 2, 2 to 3, and 3 to 1. The resulting triangle coincides with the initial one, and the transformation sends △ into itself. Applying (123) twice and three times to △ yields (132), and (1), where (1) is the identity permutation. Reflection in some axis can be pictured as a rotation by 180 degrees about the axis. Accounting for the axes L, M, and N yields the permutations (12), (23), and (13). We have previously mentioned that this is the group $\mathfrak{D}_3$, which is the same group as $\mathfrak{S}_3$. As per definition 4.1.1, we can write $\mathfrak{D}_3$ in terms of its generating set — for $\sigma = (123)$, $\tau = (12)$, and $e = (1)$ — as

$$\mathfrak{D}_3 = \left\langle \sigma, \tau : \sigma^3 = e, \tau^3 = e, \tau\sigma = \sigma^{-1}\tau \right\rangle . \qquad \diamond$$

Remark. See table 2.4.

4.1.3 EXAMPLE. A square with vertices 1, 2, 3, and 4 has rotational symmetry about its geometric centre • (cf. figure 2.2). The axial symmetries are K, L, M, and N. The transformations around • correspond to (1), (1234), $(13)(24)$, or (1432) respectively. Reflections in the axes K, L, M, and N correspond to $(12)(34)$, (24), $(14)(23)$, and (13) respectively. These are the group elements of $\mathfrak{D}_4$, a subgroup of $\mathfrak{S}_4$. ◇

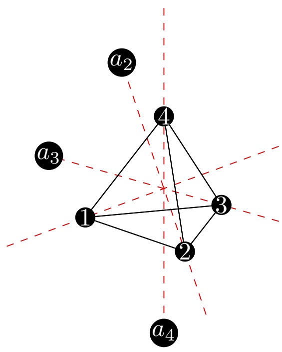
4.1.4 EXAMPLE. $\mathfrak{D}_4 = \left\langle \sigma, \tau : \sigma^4 = e, \tau^2 = e, \tau\sigma = \sigma^{-1}\tau \right\rangle$, where $\sigma = (1234)$, $\tau = (12)(34)$, and $\sigma^{-1} = \sigma^3$. These rules greatly simplifies the endeavour of drawing the product table of $\mathfrak{D}_4$. ◇

Table 4.1: Product table of $\mathfrak{D}_4$.

$\mathfrak{D}_4$	e	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
e	e	σ	σ^2	σ^3	τ	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$
σ	σ	σ^2	σ^3	e	$\sigma\tau$	$\sigma^2\tau$	$\sigma^3\tau$	τ
σ^2	σ^2	σ^3	e	σ	$\sigma^2\tau$	$\sigma^3\tau$	τ	$\sigma\tau$
σ^3	σ^3	e	σ	σ^2	$\sigma^3\tau$	τ	$\sigma\tau$	$\sigma^2\tau$
τ	τ	$\sigma^3\tau$	$\sigma^2\tau$	$\sigma\tau$	e	σ^3	σ^2	σ
$\sigma\tau$	$\sigma\tau$	τ	$\sigma^3\tau$	$\sigma^2\tau$	σ	e	σ^3	σ^2
$\sigma^2\tau$	$\sigma^2\tau$	$\sigma\tau$	τ	$\sigma^3\tau$	σ^2	σ	e	σ^3
$\sigma^3\tau$	$\sigma^3\tau$	$\sigma^2\tau$	$\sigma\tau$	τ	σ^3	σ^2	σ	e

4.1.5 Example. Among the polyhedra our first object of study is an ordinary tetrahedron, depicted in figure 4.2. A rigid motion is done with respect to one of its axial symmetries.

Figure 4.2: A tetrahedron and its symmetries.



Such a transformation maps the tetrahedron into itself. A rotation about a_4 corresponds to (123) or (132) while the product of, say, (123) and (124) yields $(124)(123) = (14)(23)$. In this sense the vertices are pairwise permutable. Accounting for all the symmetries and writing down the corresponding permutations of the vertices in cyclic notation, along with their respective types will suffice for this example. ◇

Table 4.2: Vertex permutations along with their cycle types, corresponding to the rigid motions of the tetrahedron.

$(1) : [1^4]$	$(12)(34) : [2^2]$	$(13)(24) : [2^2]$	$(14)(23) : [2^2]$
$(123) : [3^1]$	$(124) : [3^1]$	$(134) : [3^1]$	$(234) : [3^1]$
$(132) : [3^1]$	$(142) : [3^1]$	$(143) : [3^1]$	$(243) : [3^1]$

Remark. While writing down the table of products for a group might be a helpful exercise, it becomes too cumbersome and serves no real purpose as we progress to groups of greater order. For our purposes it is only necessary to know the order of a group and how its elements can be represented cyclically.

4.1.6 Example. Consider a cube with vertices 1 through 8, depicted in figure 4.3. Our task is to find its group of rigid motions — and represent it in terms of a collection of permutations of its vertices — hence we are interested in its symmetries. Figure 4.4 is an attempt to depict the symmetries of the cube, and a rigid motion is done with respect to one of its symmetries. Such a transformation maps the cube into itself.
Rotating the cube with respect to some $\mathbf{v}_n$ can be done by 120° or 240°. Rotation about some $\mathbf{f}_n$ can be done by 90°, 180°, or 270°. With respect to some $\mathbf{e}_n$ a rotation can be done by 180°. Accounting for the identity transformation, the sum total of all transformations is 24. This was to be expected however, as motivated by figure 4.5. ◇

Figure 4.3: A cube with vertices 1 through 8.

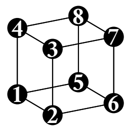
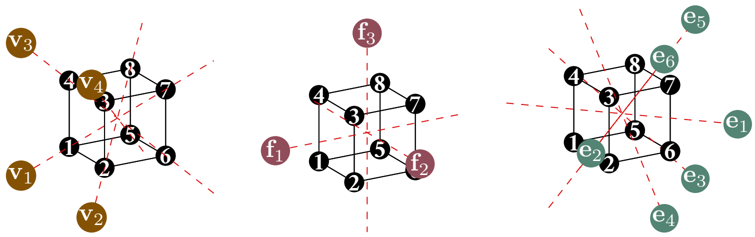
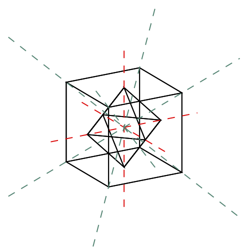


Figure 4.4: The cube and its symmetries.



There are 4 axes of symmetry, denoted $\mathbf{v_n}$*, going through opposite vertices. The axes of symmetry going through the centers of opposite faces are denoted* $\mathbf{f_n}$ *and those connecting the midpoints of two opposite edges are denoted* $\mathbf{e_n}$*. Observe that these symmetries are the same as those in figure 1.6.*

Figure 4.5: The cube and the octahedron are dual. The axes connecting the midpoints of two opposite edges have been omitted, since this would obscure the figure.



Looking upon a face of the cube — and shrinking it to a point — we regard it instead as a vertex. The subsequent graph so obtained, by connecting the "face-vertices", is an octahedron. The underlying group which acts on each solid, with respect to their respective symmetries, is the same — since the symmetries are the same. The difference in how we choose to represent this group is merely illustrative, but still important (cf. Chapter 8).

5

Group Actions

Soon we are done with the preliminaries on group theory. What remains to explain is the idea of letting a group act on a set. In section 1.1 a graph on three vertices $1, 2$, and 3 was given, with the edge set $\{\{1,2\},\{2,3\}\}$. We saw that that $\mathfrak{S}_3$ acted on the graph in such a way that it permuted the edges, while the number of edges in the resulting graphs remained constant under the repeated action of $\mathfrak{S}_3$. We shall begin this chapter with the basic notions and examples, after which we will finally arrive at one of the main ideas in this text.

5.1 Actions

5.1.1 Definition. Let $\mathfrak{G}$ be a group and let S be a set. The mapping $\varphi : \mathfrak{G} \times S \longrightarrow S$, by

$$(g,s) \underset{\varphi}{\longmapsto} gs$$

is called a *group action* of $\mathfrak{G}$ on S if for all $x \in S$ we have that $a(bx) = (ab)x$ for all $a, b \in \mathfrak{G}$, and $ex = x$ for the identity element $e \in \mathfrak{G}$. ◆

5.1.1 Example. Let $\mathfrak{G} = (\mathbb{Z}, +)$, and $S = \mathbb{R}$. Then $\mathfrak{G}$ acts on $\mathbb{R}$ by translation, via $\varphi(n,x) = n + x$. Viz. $\varphi(m, \varphi(n,x)) = \varphi(m, n+x) = m + (n+x) = (m+n)+x = \varphi(m+n, x)$, and $\varphi(0,x) = 0 + x = x$. ◇

5.1.2 Example. Let $\mathfrak{G} = (G, \cdot)$, where $G = \left\{e^{i\theta} : 0 \le \theta < 2\pi\right\}$, and let $S = \mathbb{C}$. Then $\mathfrak{G}$ acts on $\mathbb{C}$ by rotation, via $\varphi\left(e^{i\theta}, z\right) = e^{i\theta}\,|z|\,e^{i\alpha} = |z|\,e^{i(\theta+\alpha)}$, where $z = |z|\,e^{i\alpha}$ and $\alpha = \arg z$. ◇

5.1.3 Example. Among the axioms of a vector field there's *compatibility of scalar multiplication with field multiplication* . Let $\mathbf{V}(\mathbb{F})$ be a vector space. As we've seen, $\mathbb{F}^{\times}$ is a group under standard multiplication, so that $\left(\mathbb{F}^{\times}, \cdot\right)$ acts on $\mathbf{V}(\mathbb{F})$, via $\varphi\left(a, \vec{v}\right) = a\vec{v}$. ◇

5.1.4 Example. The group $\mathbf{GL}_n(\mathbb{F})$ acts by ordinary matrix multiplication on the vectors of $\mathbb{F}^n$. ◇

5.1.2 DEFINITION. A group $\mathfrak{G}$ is said to act transitively on a set S if for all elements $x, y \in S$ there exists $g \in \mathfrak{G}$ such that $gx = y$. ◆

5.1.5 EXAMPLE. The symmetric group $\mathfrak{S}_n$ acts transitively on $N = \{1, 2, 3, \ldots, n\}$ since for every $k \in N$ we can get to every other element $m \in N$ by applying the 2-cycle to k: $(km)(k) = m$. ◇

5.1.6 EXAMPLE. The group $\mathbf{GL}_n(\mathbb{R})$ acts transitively on $V = \{\mathbf{v} \in \mathbb{R}^n : \mathbf{v} \neq \mathbf{0}\}$. ◇

5.1.3 DEFINITION. Let $\mathfrak{G}$ be a group acting on a set S. For an element $s \in S$, the *orbit* of s is the set

$$\mathfrak{G}s = \{x \in S : s = gx, \text{ for some } g \in \mathfrak{G}\}.$$ ◆

5.1.4 DEFINITION. Let $\mathfrak{G}$ be a group acting on a set S. For an element $s \in S$, the *stabilizer* of s under $\mathfrak{G}$ is the set

$$\mathfrak{G}_s = \{g \in \mathfrak{G} : gs = s\}.$$ ◆

5.1.5 DEFINITION. Let $\mathfrak{G}$ be a group acting on a set S. The *subset of S fixed by* $\mathfrak{G}$ is the set

$$S^{\mathfrak{G}} = \{x \in S : gx = x, \text{ for all } g \in \mathfrak{G}\}.$$ ◆

Remark. Oftentimes one denotes $\mathfrak{G}s$, $\mathfrak{G}_s$, and $S^{\mathfrak{G}}$ by writing $\text{Orb}_{\mathfrak{G}}(s)$, $\text{Stab}_{\mathfrak{G}}(s)$, and $\text{Fix}_{\mathfrak{G}}(S)$ respectively.

5.1.7 EXAMPLE. Let $\mathfrak{G}$ be a group, and $\mathfrak{H} \leqslant \mathfrak{G}$ a subgroup. Define $\varphi : \mathfrak{H} \times \mathfrak{G} \longrightarrow \mathfrak{G}$ by

$$(h, g) \underset{\varphi}{\longmapsto} hg.$$

This is a group action, where $\mathfrak{H}$ acts on the group elements of $\mathfrak{G}$, since $\varphi(k, \varphi(k, g)) = h(kg) = (hk)g = \varphi(hk, g)$, and since $\varphi(e, g) = g$. The orbit of $g \in \mathfrak{G}$ is the right coset $\mathfrak{H}g = \{x \in \mathfrak{G} : x = hg, \text{ for some } h \in \mathfrak{H}\}$. The stabilizer of $g \in \mathfrak{G}$ is $\mathfrak{H}_g = \{h \in \mathfrak{H} : hg = g\} = \{e\}$, while $\mathfrak{G}^{\mathfrak{H}} = \{x \in \mathfrak{G} : hx = x, \text{ for all } h \in \mathfrak{H}\} = \varnothing$ is the subset of $\mathfrak{G}$ fixed by $\mathfrak{H}$ in the case where $\mathfrak{H}$ is non-trivial. ◇

5.1.8 EXAMPLE. It is easily veriefied that $\mathfrak{S}_n$ acts on the set $N = \{1, 2, 3, \ldots, n\}$. The set $\text{Stab}_{\mathfrak{S}_n}(k) = \{\sigma \in \mathfrak{S}_n : \sigma(k) = k\}$ is the stabilizer of $k \in N$, and it is a subgroup of $\mathfrak{S}_n$. Let $\sigma, \tau \in \text{Stab}_{\mathfrak{S}_n}(k)$, then $\sigma(\tau(k)) = \sigma(k) = k$, and so $\sigma\tau \in \text{Stab}_{\mathfrak{S}_n}(k)$, which verifies the closedness property. It is an inherited property from $\mathfrak{S}_n$ that $\sigma(\tau\mu) = (\sigma\tau)\mu$, for all $\sigma, \tau, \mu \in \text{Stab}_{\mathfrak{S}_n}(k)$. It is indeed the case that $e(k) = k$, for the identity permutation $e \in \mathfrak{S}_n$, hence $e \in \text{Stab}_{\mathfrak{S}_n}(k)$. Lastly, for $\sigma \in \text{Stab}_{\mathfrak{S}_n}(k)$ we have that $\sigma(k) = k \iff \sigma^{-1}(\sigma(k)) = \sigma^{-1}(k) \iff k = \sigma^{-1}(k)$, and so $\sigma^{-1} \in \text{Stab}_{\mathfrak{S}_n}(k)$. Moreover we can define a mapping $\varphi : \text{Stab}_{\mathfrak{S}_n}(k) \times N \setminus \{k\} \longrightarrow N \setminus \{k\}$, with the new group $\text{Stab}_{\mathfrak{S}_n}(k) \leqslant \mathfrak{S}_n$, which is the group action of $\text{Stab}_{\mathfrak{S}_n}(k)$ on $N \setminus \{k\}$. ◇

5.1.9 EXAMPLE. Generally, for a group $\mathfrak{G}$ acting on a set X, the stabilizer of $y \in X$ is a subgroup of $\mathfrak{G}$ which acts on $X \setminus \{y\}$. ◇

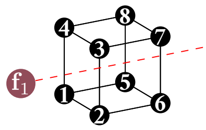
5.1.10 EXAMPLE. One can extend on the idea of transitivity in Definition 5.1.2 to the notion of double-transitivity.

> *Let $\mathfrak{G}$ be a group actiong on a set X. If, for all $(x_1, y_1), (x_2, y_2) \in X \times X$, there is a group element g such that $g(x_1, y_1) = (x_2, y_2)$, then the group action is called doubly transitive.*

The property that $\mathfrak{G}$ acts doubly-transitively on X, where $|X| > 2$, is equivalent to $\text{Stab}_{\mathfrak{G}}(y)$ acting transitively on $X \setminus \{y\}$. Let $\mathfrak{G}$ be a group acting on the set X, where $|X| > 2$, and assume that it acts doubly-transitively. Let $x, z \in X \setminus \{y\}$ be two distinct elements. This is possible since $|X| > 2$. Then $(x, y), (z, y) \in X \times X$ are tuples of distinct elements of X. By our assumption that $\mathfrak{G}$ acts doubly-transitively we therefore have that $g(x, y) = (z, y)$, which suggests that $g \in \text{Stab}_{\mathfrak{G}}(y)$. The elements $x, z \in X \setminus \{y\}$ being distinct, together with $gx = z$, entails that $\text{Stab}_{\mathfrak{G}}(y)$ acts transitively on $X \setminus \{y\}$. Moreover, suppose that $\text{Stab}_{\mathfrak{G}}(y)$ acts transitively on $X \setminus \{y\}$ for all $y \in X$. Let $(x_1, z_1), (x_2, z_2) \in X \times X$ be distinct tuples of elements in X so that $g_1 z_1 = z_2$, and $g_2 x_1 = x_2$, for $g_1 \in \text{Stab}_{\mathfrak{G}}(x_1)$, and $g_2 \in \text{Stab}_{\mathfrak{G}}(z_2)$. Then $g_2 g_1(x_1, z_1) = g_2(x_1, z_2) = (x_2, z_2)$. Hence $\mathfrak{G}$ acts doubly-transitively on X, and we conclude that $\mathfrak{G}$ acts doubly-transitively on X if, and only if $\text{Stab}_{\mathfrak{G}}(y)$ acts transitively on $X \setminus \{y\}$. ◇

5.1.11 EXAMPLE. Consider the cube from Example 4.1.6, and its rigid motions restricted to only one of the axial symmetries, in this case $\mathbf{f_1}$. This gives rise to the subgroup $\mathfrak{H} = \big\langle (1234)(5678) : [(1234)(5678)]^4 = (1) \big\rangle$. It is clear that $\mathfrak{H}$ acts on the set of

Figure 5.1: The cube with respect to $\mathbf{f}_1$.



vertices, $V = \{1, 2, 3, 4, 5, 6, 7, 8\}$, of the cube by permuting them. Pick a vertex, say 1, then $\text{Orb}_{\mathfrak{H}}(1) = \{1, 2, 3, 4\}$, and $\text{Stab}_{\mathfrak{H}}(1) = \{(1) \in \mathfrak{H}\}$, while $\text{Fix}_{\mathfrak{H}}(V) = \varnothing$. ◇

5.2 THE ORBIT-STABILIZER THEOREM

For a group $\mathfrak{G}$ acting on a non-empty set X, it is natural to define the relation $\sim$ on X by the rule that, for $x, y \in X$, $x \sim y \iff gx = y$ for some $g \in \mathfrak{G}$. Since $ex = x$, we have that $x \sim x$. If $x \sim y$, we have that $y \sim x$ since $gx = y \iff g^{-1}y = x$. If $x \sim y$, and $y \sim z$, for $x, y, z \in X$, we have that $x \sim z$, since $gx = y$, and $hy = z$ implies that $hgx = z$. Hence the properties of reflexivity, symmetry, and transitivity are satisfied and so $\sim$ is an equivalence relation on X. It follows that $\sim$ partitions X into equivalence classes — the orbits of each and every element of X — and we denote this partition by $X/_{\mathfrak{G}}$.

5.2.1 Theorem. *Let $\mathfrak{G}$ be a group of finite order, and X a finite set subjected to the action of $\mathfrak{G}$. There exists a bijection*

$$\Psi : \mathfrak{G}/_{Stab_{\mathfrak{G}}(x)} \longrightarrow Orb_{\mathfrak{G}}(x), \text{ by}$$

$$\Psi\left(gStab_{\mathfrak{G}}(x)\right) = gx,$$

for elements in the orbit $Orb_{\mathfrak{G}}(x)$ and the left cosets of $Stab_{\mathfrak{G}}(x)$ in $\mathfrak{G}$.

Proof. Initially, it has to be verified that Ψ satisfies the very definition of a function. Thus we need to show that Ψ takes the elements g_1, g_2 of some left coset $g\text{Stab}_{\mathfrak{G}}(x)$ to one and the same element in $\text{Orb}_{\mathfrak{G}}(x)$. Since $g_1, g_2 \in g\text{Stab}_{\mathfrak{G}}(x)$ we have that $g_1 = gh_1$, and $g_2 = gh_2$, for $h_1, h_2 \in \text{Stab}_{\mathfrak{G}}(x)$. Hence $g = g_1h_1^{-1}$, and $g = g_2h_2^{-1} \implies g_1h_1^{-1} = g_2h_2^{-1} \implies g_1 = g_2\left(h_2^{-1}h_1\right)$, where $h_2^{-1}h_1 \in \text{Stab}_{\mathfrak{G}}(x)$. Let $h_2^{-1}h_1 = h$, since $h \in \text{Stab}_{\mathfrak{G}}(x)$ we have that $g_2x = g_2(hx) = (g_2h)x = g_1x$, which is what we wanted to verify. Moreover, it has to be verified that Ψ is a bijection. Let $\Psi\left(g_1\text{Stab}_{\mathfrak{G}}(x)\right) = \Psi\left(g_2\text{Stab}_{\mathfrak{G}}(x)\right)$, i.e. $g_1x = g_2x$, then $g_1^{-1}g_2x = x \implies g_1^{-1}g_2 \in \text{Stab}_{\mathfrak{G}}(x)$, which entails that $g_1, g_2 \in g_1\text{Stab}_{\mathfrak{G}}(x)$ so that g_1, and g_2 belongs to the same coset wherefore $g_1\text{Stab}_{\mathfrak{G}}(x) = g_2\text{Stab}_{\mathfrak{G}}(x)$. Since Ψ is clearly surjective our proof is complete. ■

5.2.2 Prop. *Take $\mathfrak{G}$ and X as in Theorem 5.2.1, where $x \in X$. Then*

$$|\mathfrak{G}| = |Orb_{\mathfrak{G}}(x)|\,|Stab_{\mathfrak{G}}(x)|\,.$$

Proof. By Lagrange's Theorem (Theorem 3.3.2) we have, $\text{Stab}_{\mathfrak{G}}(x)$ being a subgroup of $\mathfrak{G}$, that

$$\left|\mathfrak{G}/_{\text{Stab}_{\mathfrak{G}}(x)}\right| = \left[\mathfrak{G} : \text{Stab}_{\mathfrak{G}}(x)\right] \iff \frac{|\mathfrak{G}|}{|\text{Stab}_{\mathfrak{G}}(x)|} = \left[\mathfrak{G} : \text{Stab}_{\mathfrak{G}}(x)\right].$$

In theorem 5.2.1 we saw that Ψ was a bijection between $\mathfrak{G}/_{\text{Stab}_{\mathfrak{G}}(x)}$, and $\text{Orb}_{\mathfrak{G}}(x)$, hence $|\text{Orb}_{\mathfrak{G}}(x)| = \left[\mathfrak{G} : \text{Stab}_{\mathfrak{G}}(x)\right]$ and so

$$|\mathfrak{G}| = |\text{Orb}_{\mathfrak{G}}(x)|\,|\text{Stab}_{\mathfrak{G}}(x)|\,. \qquad ■$$

5.2.1 Example. In example 5.1.10 we saw that a group $\mathfrak{G}$ acts doubly transitively on a set X if, and only if $\text{Stab}_{\mathfrak{G}}(y)$ acts transitively on $X \setminus \{y\}$. We can show that if $\mathfrak{G}$ acts doubly transitively on X, where $|X| \geq 2$, then $n(n-1)\big|\,|\mathfrak{G}|$. Assume therefore that $\mathfrak{G}$ acts doubly transitively on X, viz. for all $(x_1, y_1), (x_2, y_2) \in X \times X$ there exists $g \in \mathfrak{G}$ such that $g(x_1, y_2) = (x_2, y_2)$. In particular, there is $g \in \mathfrak{G}$ such that $g(x_1, x_1) = (y_2, y_2)$ and so $\mathfrak{G}$ acts transitively on X. Since $\mathfrak{G}$ acts transitively on X we have that $\text{Orb}_{\mathfrak{G}}(y) = X$, and by Proposition 5.2.2 that $|\text{Orb}_{\mathfrak{G}}(y)|\,|\text{Stab}_{\mathfrak{G}}(y)| = |\mathfrak{G}|$, hence $n\,|\text{Stab}_{\mathfrak{G}}(y)| = |\mathfrak{G}|$ since $|\text{Orb}_{\mathfrak{G}}(y)| = |X| = n$. Again, by Proposition 5.2.2, we have that $|\text{Orb}_{\mathfrak{G}}(z)|\,|\text{Stab}_{\mathfrak{G}}(z)| = |\text{Stab}_{\mathfrak{G}}(y)|$ for an element $z \in X \setminus \{y\}$. Since $\mathfrak{G}$ acts doubly transitively on X we have that $\text{Stab}_{\mathfrak{G}}(y)$ acts transitively on $X \setminus \{y\}$, for all $y \in X$. Hence $\text{Orb}_{\mathfrak{G}}(z) = X \setminus \{y\}$ and so $|\text{Orb}_{\mathfrak{G}}(z)| = |X \ \{y\}| = n - 1$. Hence $(n-1)\,|\text{Stab}_{\mathfrak{G}}(z)| = |\text{Stab}_{\mathfrak{G}}(y)|$, and so $|\mathfrak{G}| = n(n-1)\,|\text{Stab}_{\mathfrak{G}}(z)|$. We conclude that $n(n-1) \mid |\mathfrak{G}|$. ◇

5.2.2 Example. Consider the tetrahedral group from Example 4.1.5. The task of finding the number of transformations in it is greatly simplified by Proposition 5.2.2. Looking at the vertex 4 we realize that it is possible to map it to any of the vertices 1,2,3 — and, of course, to itself — by a suitable rigid motion of the tetrahedron about one of its axial symmetries. Hence $\text{Orb}_{\mathfrak{G}}(4) = \{1, 2, 3, 4\}$. There are two transformations which permutes all vertices but vertex 4, namely (123), and (132). The identity transformation, too, fixes vertex 4. Hence $\text{Stab}_{\mathfrak{G}}(4) = \{(1), (123), (132)\}$, and so the tetrahedral group has order $|\text{Orb}_{\mathfrak{G}}(4)|\,|\text{Stab}_{\mathfrak{G}}(4)| = 4 \cdot 3 = 12$. ◇

5.2.3 Example. Consider the cube group, from Example 4.1.6. By a suitable rigid motion about any of its axial symmetries, a vertex v can be mapped to any of the others — including v, by the identity transformation. Hence $\text{Orb}_{\mathfrak{G}}(v) = \{1, 2, 3, 4, 5, 6, 7, 8\} = V$, viz $|textOrb_{\mathfrak{G}}(v)| = 8$. For any vertex of the cube there's three transformations which fixes it — the identity, and the two rotations about the axis of symmetry connectiong v and its diagonal opposite — and so $|\text{Stab}_{\mathfrak{G}}(v)| = 3$. By the Orbit Stabilizer Theorem we get that the cube group has order $|\text{Orb}_{\mathfrak{G}}(v)|\,|\text{Stab}_{\mathfrak{G}}(v)| = 8 \cdot 3 = 24$. ◇

6

Counting Orbits

For a group $\mathfrak{G}$ acting on a non-empty set X, it is natural to define the relation ~ on X by the rule that, for $x, y \in X$, $x \sim y \iff gx = y$ for some $g \in \mathfrak{G}$. Since $ex = x$, we have that $x \sim x$. If $x \sim y$, we have that $y \sim x$ since $gx = y \iff g^{-1}y = x$. If $x \sim y$, and $y \sim z$, for $x, y, z \in X$, we have that $x \sim z$, since $gx = y$, and $hy = z$ implies that $hgx = z$. Hence the properties of reflexivity, symmetry, and transitivity are satisfied and so ~ is an equivalence relation on X. It follows that ~ partitions X into equivalence classes — the orbits of each and every element of X — and we denote this partition by $X/\mathfrak{G}$. The question which obviously comes to mind is how to determine the cardinality of this set. In his book [3], Burnside stated and proved a very famous theorem which he attributed to Frobenius, while the theorem was also known to Cauchy. For this reason it sometimes (jokingly) goes under the name Not Burnside's Lemma . Here we will abide to convention and refer to it as *Burnside's lemma* , and in the following pages we will present it together with a few examples.

6.1 Burnside's Lemma

Perhaps we should remind ourselves of what this text is about. As mentioned in the very first chapter, we're interested in counting a set of combinatorial configurations, whatever they may be, while realizing that some of them may not be distinguishable. In section 1.2 in chapter 1 we saw that the set of graphs with three vertices had size 8, but that there were only 4 graphs which were esentially distinguishable. We realized this by letting the group $\mathfrak{S}_3$ act on the vertices of the graphs, and we came to the conclusion that the set of graphs partitioned into orbits — each orbit representing a distinct isomporhism class. The following theorem helps our understanding in this regard, and in later parts we will elaborate on it.

6.1.1 Theorem (Burnside's Lemma). *Let $\mathfrak{G}$ be a group of finite order, and X a finite set subjected to the action of $\mathfrak{G}$. The total number of orbits is*

$$\left|{}^{X}\!\big/_{\mathfrak{G}}\right| = \frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}} |Fix(g)|\,,$$

where summation is taken over all $g \in \mathfrak{G}$ and, for an arbitrary $g \in \mathfrak{G}$, $Fix(g) = \{x \in X : gx = x\}$.

Proof. Consider the set $S = \{(g,x) \in \mathfrak{G} \times X : gx = x\}$. We shall count $|S|$ in two ways.

I: Fix an element $g \in \mathfrak{G}$. We have that $|\text{Fix}(g)| = |\{x \in X : gx = x\}|$ is the number of elements in X which are fixed by g, hence

$$|S| = \sum_{g\in\mathfrak{g}} |\text{Fix}(g)|\,; \tag{6.1}$$

II: Fix an element $x \in X$. We have that $|\text{Stab}_{\mathfrak{G}}(x)| = |\{g \in \mathfrak{G} : gx = x\}|$ is the number of elements of $\mathfrak{G}$ under which x is invariant, hence

$$|S| = \sum_{x\in X} |\text{Stab}_{\mathfrak{G}}(x)|\,. \tag{6.2}$$

By (6.1) and (6.2) we have that $\sum_{g\in\mathfrak{g}} |\text{Fix}(g)| = \sum_{x\in X} |\text{Stab}_{\mathfrak{G}}(x)|$, and by Proposition 5.2.2 we have that $|\text{Stab}_{\mathfrak{G}}(x)| = |\mathfrak{G}| \,/\, |\text{Orb}_{\mathfrak{G}}(x)|$. Therefore

$$\sum_{g\in\mathfrak{G}} |\text{Fix}(g)| = \sum_{x\in X} \frac{|\mathfrak{G}|}{|\text{Orb}_{\mathfrak{G}}(x)|} \iff \frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}} |\text{Fix}(g)| = \sum_{x\in X} \frac{1}{|\text{Orb}_{\mathfrak{G}}(x)|}.$$

Since X is the disjoint union of all of its orbits in ${}^{X}\!\big/_{\mathfrak{G}}$ the sum over X — $\sum_{x\in X}$ — can be broken up into sums over each individual orbit. Assuming there are k orbits, and denoting each orbit by O_i, we have that

$$\sum_{x\in X} \frac{1}{|\text{Orb}_{\mathfrak{G}}(x)|} = \sum_{i=1}^{k} \sum_{x\in O_i} \frac{1}{|O_i|}.$$

But $\sum_{x\in O_i} \frac{1}{|O_i|} = 1$, for $1 \le i \le k$, so that $\sum_{x\in X} \frac{1}{|\text{Orb}_{\mathfrak{G}}(x)|} = \sum_{i=1}^{k} 1 = k$, hence

$$\left|{}^{X}\!\big/_{\mathfrak{G}}\right| = k = \frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}} |\text{Fix}(g)|\,.$$

■

6.1.1 Example. Suppose that we are to put seven beads on a necklace, the beads evenly distributed, and the necklace the shape of a circle. Furthermore, four beads are to be black, and three beads are to be white. Our task is to determine how many such necklaces we can make. We can think of the seven beads being placed at the corners of a regular 7-gon, and realize that such a necklace, when made, can be transformed in a manner corresponding to the dihedral group

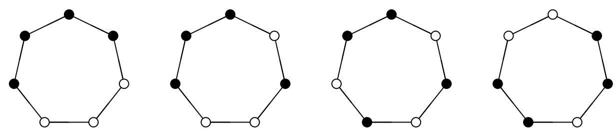
$$\mathfrak{D}_7 = \left\langle (1234567), (1)(27)(36)(45) : (1234567)^7 = [1(27)(36)(45)]^2 = (1) \right\rangle .$$

Not all necklaces, then, are distinguishable since one could be obtained by a rotation, or reflection, of some other necklace. Our task is reduced to that of counting the number of orbits when $\mathfrak{D}_7$ acts on X, where X is the set containing all necklaces that are possible to make regardless of rotational or reflective symmetries. The cardinality of X is $\binom{7}{4} = 35$, since we can choose to put the four black beads in seven places, where the internal order of the black beads lacks importance since they are indistinguishable. Placing the four black beads anywhere completely determines where the three white beads go. The cardinality of $\mathfrak{D}_4$ is 14, it consists of 7 rotations about the geometric centre, and 7 reflections in the axes though a vertex and its opposite side. None of the 35 necklaces is fixed in the same position under the 6 proper rotations, while all of them remain fixed under the under (1). For each reflection in $\mathfrak{D}_4$, the number of necklaces which are kept unchanged is 3. Using Burnside's lemma, and that there are 7 reflective symmetries, we conclude that the number of distinguishable necklaces is

$$\frac{1}{|\mathfrak{D}_7|} \sum_{g \in \mathfrak{D}_7} |\mathrm{Fix}\,(g)| = \frac{35 + 7 \cdot 3}{14} = \frac{56}{14} = 4,$$

as illustrated in the figure below. ◇

Figure 6.1: The four up to isomorphism distinguishable necklaces.



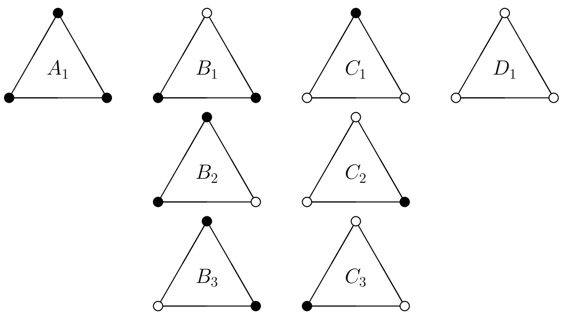
In the above example it must be observed that the set X does not consist of the vertices $\{1, 2, 3, 4, 5, 6, 7\}$, but of the 35 *configurations* — the colorings — of the 7-gon, by using white and black beads. Hence the group $\mathfrak{D}_7$ acts on a set of configurations, and so a bit of confusion arises in how to denote its group elements.

6.1.2 Example. Similar to the 7-gon in the previous example, we think of the number of ways to construct a necklace using black beads and white beads, 3 in total, and evenly distributed. Determining the number of possible configurations we arrive at

$$\binom{3}{0} + \binom{3}{1} + \binom{3}{2} + \binom{3}{3} = 8,$$

in terms of choosing, for instance, the number of black beads to be placed in the necklace. The task of determining each up to ismorphism distinguishable coloring of the vertices of an equilateral triangle isn't too painful, and in figure 6.2 we list them, along with the 8 configurations. The group acting on the set $X = \{A_1, B_1, B_2, B_3, C_1, C_2, C_3, D_1\}$ of the 8 configurations is $\mathfrak{D}_3$. and it is worthwhile examining how such an action is represented.

Figure 6.2: Configurations of necklaces with three beads — black, and/or white.



The letters A, B, C, and D denotes the isomorphism class of a necklace, while the index counts its cardinality.

Any group element in $\mathfrak{D}_3$ can either be represented in standard notation — as a permutation of the vertices of a square— or in terms of elements in X. For our purposes, the latter notation is both illustrative and convenient. For instance, pick $\tau = (132)$. Letting τ act on the configurations, as displayed in figure 6.3, we can instead represent it by

$$\tau^* = \begin{pmatrix} A_1 & B_1 & B_2 & B_3 & C_1 & C_2 & C_3 & D_1 \\ A_1 & B_3 & B_1 & B_2 & C_3 & C_1 & C_2 & D_1 \end{pmatrix},$$

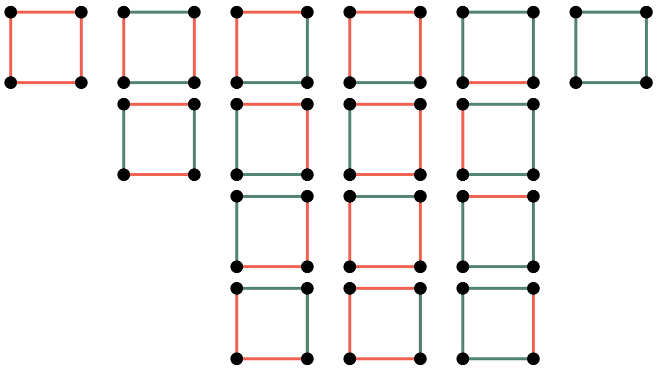
in terms of configurations. Even more conveniently, we can write τ^* in cycle decomposition, as $\tau^* = (A_1)(B_1B_3B_2)(C_1C_3C_2)(D_1)$. Such a representation is beneficial since, by the disjoint cycles, it explicitly states the equivalence classes — the orbits — of X, which are four.

$$\left| {}^{X}\!/_{\mathfrak{D}_3} \right| = \frac{1}{6} \sum_{g \in \mathfrak{D}_3} |\text{Fix}(g)| = \frac{8 + 3 \cdot 4 + 2 + 2}{6} = 4. \qquad \diamond$$

6.1.3 Example. Our next example deals with colorings of the edges of a square. Using a set S of n different colors, and starting with a set of two colors, $S = \{R, B\}$, red and blue, we are allowed to color the square in any way. The set of configurations X has cardinality 16, as there are 16 mappings from the set of edges of the square, $E = \{e_1, e_2, e_3, e_4\}$, to the set of colors $S = \{R, B\}$. Indeed, we can just as well write $X = \{f_1, f_2, \dots, f_{16}\}$ in terms of the different mappings from E to S. Letting $\mathfrak{D}_4$ act on E, two configurations $f_i, f_j \in X$ are equivalent if $f_i\sigma = f_j$, for some $\sigma \in \mathfrak{D}_4$. The equivalence classes are precisely ${}^{X}\!/_{\mathfrak{D}_4}$, and so Burnside's lemma (Theorem 6.1.1) applies. In $\mathfrak{D}_4$, the identity element (1) fixes all of X, a rotation around the horizontal axis, $(14)(23)$, fixes 8 elements, as does a rotation, $(12)(34)$, around the vertical axis. A flip about a diagonal axis, either (13) or (24), fixes 4 elements. The permutation (1234) fixes only 2 elements, and the same goes for (1432), while 4 elements of X are invariant under $(13)(24)$. In figure 6.3 we list the elements of X. The cardinality of ${}^{X}\!/_{\mathfrak{D}_4}$ — the equivalence classes, i.e. orbits — is

$$\frac{1}{|\mathfrak{D}_4|}\sum_{g\in\mathfrak{D}_4}|\mathrm{Fix}(g)| = \frac{16+2\cdot 8+2\cdot 4+2\cdot 2+4}{8} = \frac{48}{8} = 6.$$

Figure 6.3: The color configurations of X, with $S = \{R, B\}$.



We turn now to the case where S is a set containing n different colors. Here, a slight alteration of our reasoning must take place. The set of configurations still consists of every mapping from E to S, and has cardinality n^4, but counting the number of configurations fixed under some permutation gets a bit more involved. All color configurations are obsiously invariant under the identity permutation. For $(12)(34)$, a flip/rotation (or reflection) in the vertical axis of symmetry keeps n^3 color configurations fixed, and the same goes for $(14)(23)$. The transformations (13), and (24), about a diagonal axis, each fixes n^2 colorings. A rotation about the geometric centre of the square, either by (1234), or (1432), fixes n colourings. The permutation $(13)(24)$ corresponds to a 180° rotation about the geometric centre, which fixes n^2 colorings. We get that

$$\left|{}^{X}\!/_{\mathfrak{D}_4}\right| = \frac{1}{|\mathfrak{D}_4|}\sum_{g\in\mathfrak{D}_4}|X^g| = \frac{n^4+2n^3+3n^2+2n}{8},$$

is the number of equivalence classes of the color configurations, when acted on by $\mathfrak{D}_4$. ◇

Remark. The color configurations of the vertices of a square when acted on by $\mathfrak{D}_4$ has the same number of equivalence classes. Subdivide a square, so that each edge gets a vertex in its midpoint, and join those vertices in a cycle — a new, tilted square is so obtained. Apply the same reasoning as above.

6.1.4 Example. Our final task is to color the faces of a cube, using n colors, and to determine the number of distinguishable colorings. We denote the set of color configurations with X, and as per the standard argument its cardinality is n^6 — the number of mappings from the set of edges to the set of colors. To our aid we look at the rotational symmetries of a cube, depicted in figure 4.4. A permutation with respect to an axis $\mathbf{v}_k$ going through a vertex and its diagonal opposite leaves n^2 color configurations fixed, and there are 8 such permutations — for each of the 4 axes $\mathbf{v}_k$ we can go 120° or 240° — and together they sum up to $8n^2$ invariant colorings. A proper permutation with respect to an orthogonal axis can be done by 90°, 180°, or 270° — a 90° rotation leaves n^3 coloring unchanged, as does a rotation by 270°, while n^4 configurations are invariant under a 180° rotation.

There are three orthogonal axes, accounting for all of them sums up to $3n^4 + 6n^3$ invariant colorings. A permutation with respect to an axis going through the midpoints of two opposing edges leaves n^3 colorings unchanged, and there are 6 such permutations, which accounts for $6n^3$ invariant colorings. The sum total of fixed configurations acted on by the rigid motions of a cube is therefore

$$n^6 + 3n^4 + 12n^3 + 8n^2.$$

Applying Burnside's lemma, we get that the number of distinguishable colorings of the faces of a cube — lets denote it, here, by $\mathfrak{Q}$ — is

$$\frac{1}{|\mathfrak{Q}|}\sum_{g\in\mathfrak{Q}} |X^g| = \frac{n^6 + 3n^4 + 12n^3 + 8n^2}{24}. \qquad \diamond$$

Remark. The color configurations of the vertices of an octahedron when acted on by the rigid motions of a cube has the same number of equivalence classes. The reason being, as mentioned earlier, that the cube and the octahedron are dual.

7

Generating Functions

7.1 Introduction As we extend on the Orbit Counting Lemma there's a few basic notions from the theory of generating functions we can borrow to help our understanding. First, an observation can be made about the interplay of addition and exclusive disjunction.

7.1.1 Example. Take for example the task of flipping a coin. The possible number of outcomes is two — heads or tails, with no simultaneous occurance of the two — and through the use of the polynomial $x + y$ we have modelled the situation accordingly. Here, the coefficient before x is the one possible outcome heads, while the coefficient before y is the one possible outcome tails. We can further develop on this idea, to describe a situtation where a coin is to be flipped n times, now using the polynomial $(x + y)^n = (x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k = 1 + nx + \binom{n}{2} x^{n-2} y^2 + \ldots + nxy^{n-1}$. This polynomial enumerates every possible outcome, i.e. there are $\binom{n}{k}$ ways to toss a coin to get $n - k$ heads and k tails. ◇

7.2 Generating Functions A different problem which also comes to mind is, for example, that of finding the number of positive integer solutions to the equation $x_1 + x_2 + x_3 = 10$. Each term x_i can take on 8 different values as, for instance, $x_1 = 8$ determines that $x_{2,3} = 1$. Hence each term can be written as a polynomial $x_i = x^1 + x^2 + \ldots + x^8$, i.e. x_i takes on the values 1, *or* 2, *or* 3, . . ., *or* 8, where the coefficient before each term x^k simply states that there is one possible way to pick $x_i = k$. The interplay of addition and exclusive disjunction does not relate to our original equation $x_1 + x_2 + x_3 = 10$, as x_1, x_2, and x_3 are independent variables, only to $x^1 + x^2 + \ldots + x^8$. By the rule of product we investigate $\left(x^1 + x^2 \ldots + x^8\right)^3$, where the coefficient before x^{10} is 36, and so we have computed the answer to our problem. This can be verified with the classical idea of using *stars and bars* : $\binom{10-1}{3-1} = \binom{9}{2} = 36$.

For our purposes there's no need for a thorough exposition. In this chapter we will only provide the most basic ideas, and present results in keeping with the theme of this text.

7.2.1 EXAMPLE. Lets return to example 6.1.3: colored edges of a square. We can assign to each edge $e_i \in E = \{e_1, e_2, e_3, e_4\}$ the polynomial $r + b$, so that $(r + b)^4 = r^4 + 4r^3b + 6r^2b^2 + 4rb^3 + b^4$, which generates all of the 16 configurations. ◇

7.2.2 EXAMPLE. The colorings of the faces of a cube, such as that in example 6.1.4, has n^6 configurations for a set of $S\{c_1, c_2, \ldots, c_n\}$ of n colors. As in example 7.1.1, we can assign to each face the polynomial $c_1 + c_2 + \ldots + c_n$, where the sum of the coefficients in the expansion of $(c_1 + c_2 + \ldots + c_n)^6$ equals n^6. ◇

7.2.1 DEFINITION. Let $a_0, a_1, a_3, \ldots$ be a sequence of integers. The formal power series

$$A(x) = \sum_{i=0}^{\infty} a_i x_i$$

is called the *generating function*, or the *generating formal power series*, of $a_0, a_1, a_3, \ldots$ ◆

Remark. It must be observed that in this text a generating function is not to be evaluated for any specific x. We do not concern ourselves about the meaning of x or its powers, they serve only as symbols to designate the position of a coefficient.

7.2.3 EXAMPLE. For any positive integer $n \in \mathbb{Z}^+$ we have that

$$(1 + y)^n = \sum_{k=0}^{n} \binom{n}{k} y^k,$$

which is the generating function for $1, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n}$. By letting $y = ax$, for any $a \in \mathbb{R}$, we have that

$$(1 + ax)^n = \sum_{k=0}^{n} \binom{n}{k} (ax)^k,$$

is the generating function for the sequence $1, a\binom{n}{1}, a^2\binom{n}{2}, \ldots, a^n\binom{n}{n}$. ◇

7.2.4 EXAMPLE. We have that $(1-y)(1+y+y^2+y^3 \ldots) = 1$. Hence the infinite sequence $(1)_{k=0}^{\infty} = 1, 1, 1, \ldots$ has as its generating function

$$\frac{1}{1-y} = \sum_{k=0}^{\infty} y^k.$$

By letting $y = ax$, for any $a \in \mathbb{R}$, we have that

$$\frac{1}{1-ax} = \sum_{k=0}^{\infty} (ax)^k$$

is the generating function for the infinite sequence $\left(a^k\right)_{k=0}^{\infty} = 1, a, a^2, \ldots$ ◇

7.2.1 Prop. *For* $A(x) = \sum_{k=0}^{\infty} a_k x^k$, *and* $B(x) = \sum_{k=0}^{\infty} b_k x^k$,

$$A(x)B(x) = \sum_{k=0}^{\infty} a_k x^k \sum_{k=0}^{\infty} b_k x^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} a_i b_{k-i} x^k$$

Remark. This product of power series is called *discrete convolution*, or *the Cauchy product* of two infinite series.

In the table below we collect a few of the standard identities relating sequences to their generating functions.

Table 7.1: Some standard identities.

$m, n \in \mathbb{Z}^+$,	$a \in \mathbb{R}$.
I.	$(1+x)^n = \binom{n}{0} + \binom{n}{1}x^1 + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n.$
II.	$(1+ax)^n = \binom{n}{0} + \binom{n}{1}ax + \binom{n}{2}(ax)^2 + \ldots + \binom{n}{n}(ax)^n.$
III.	$(1+x^m)^n = \binom{n}{0} + \binom{n}{1}x^m + \binom{n}{2}x^{2m} + \ldots + \binom{n}{n}x^{nm}.$
IV.	$\frac{1-x^{n+1}}{1-x} = 1 + x + x^2 + \ldots + x^n.$
V.	$\frac{1}{1-x} = 1 + x + x^2 + \ldots = \sum_{k=0}^{\infty} x^k.$
VI.	$\frac{1}{1-ax} = 1 + ax + (ax)^2 + \ldots = \sum_{k=0}^{\infty}(ax)^k.$
VII.	$\frac{1}{(1+x)^n} = \sum_{k=0}^{\infty} (-1)^k \binom{n+k-1}{k} x^k.$
VIII.	$\frac{1}{(1-x)^n} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} x^k.$

7.2.5 Example. We can use a generating function to determine the number of ways to pick k objects from a set $S = \{o_1, o_2, \ldots, o_n\}$ of n distinct objects, where an object can be picked repeatedly. The power series $1 + x + x^2 + x^3 + \ldots$ represent the possible choices for an object. There are n distinct objects, and so by the rule of product our generating function is

$$\left(1 + x + x^2 + x^3 + \ldots\right)^n = (1-x)^{-n},$$

where $(1-x)^{-n} = \sum_{i=0}^{\infty} \binom{n+i-1}{k} x^k$. Hence we seek the coefficient before x^k in the expansion of $\sum_{i=0}^{\infty} \binom{n+i-1}{k} x^i$, which is $\binom{n+k-1}{k}$. Again, this can be verified by the method of stars and bars. ◇

7.2.6 Example. We find the coefficient before x^{50} in $\left(x^7 + x^8 + x^9 + \ldots\right)^6$, first by factorizing it into

$$\left[x^7\left(1 + x + x^2 + \ldots\right)\right]^6 = x^{42}\left(1 + x + x^2 + \ldots\right)^6 = x^{42}(1-x)^{-6},$$

and then by determining the coefficient before x^8 in the expansion of $(1-x)^{-6} = \sum_{k=0}^{\infty} \binom{6+k-1}{k}$, which is $\binom{13}{8} = 1287$. ◇

7.2.7 Example. The six outcomes of rolling a die can be represented by the polynomial $x^1 + x^2 + x^3 + x^4 + x^5 + x^6$, while rolling it 10 times corresponds to $\left(x^1 + x^2 + \ldots + x^6\right)^{10}$. We wish to know the likelihood of obtaining the sum 40 after 10 rolls. Initially, we must calculate the coefficient before x^{40} in $\left(x^1 + x^2 + \ldots + x^6\right)^{10}$, by factorizing it as $\left[x(1 + x + \ldots + x^5)\right]^{10} = x^{10}\left[(1 - x^6)(1 - x)^{-1}\right]^{10} = x^{10}\left(1 - x^6\right)^{10}(1 - x)^{-10}$. Hence we seek the coefficient before x^{30} in

$$\left(1 - x^6\right)^{10}(1 - x)^{-10} = \sum_{k=0}^{10}\binom{10}{k}\left(-x^6\right)^k \sum_{k=0}^{\infty}\binom{10 + k - 1}{k}x^k,$$

which, by a tedious computation, is equal to

$$\binom{39}{30} - \binom{10}{1}\binom{33}{24} + \binom{10}{2}\binom{27}{18} - \binom{10}{3}\binom{21}{12} + \binom{10}{4}\binom{15}{6} - \binom{10}{5} = 2930455.$$

The size of the sample space is 6^{10}, and so the likelihood of 10 rolls of a die summing up to 40 is

$$\frac{2930455}{6^{10}}.$$

◇

8

The Cycle Index

8.1 INTRODUCTION We recall the *type* of a permutation. It's a compact way to encode its shape, i.e. how many cycles of each length there are in a permutation $\sigma \in \mathfrak{S}_n$, as illustrated in example 2.2.1. For instance, the permutation $\sigma = (1327)(45)(68)(9)$ has type $\left[1^1, 2^2, 4^1\right]$. In this chapter we are going to associate a polynomial to a group of permutations — a generating function in several variables called the *cycle index*— in which the terms are related to the cycle structure of each group element. Our aim is to simplify the computations necessary to determine the equivalence classes of n-colorings of an object.

8.2 THE CYCLE INDEX Lets review The Orbit Counting Lemma, and once again consider Example 6.1.4. We reached the result that for n colors there existed $\frac{n^6+3n^4+12n^3+8n^2}{24}$ distinguishable n-colorings of a cube. We try now to use a different approach, which is based on the type of each group element of $\mathfrak{Q}$, when expressed as permutations. For this, we need to define the *cycle structure representation* of a permutation.

8.2.1 DEFINITION. A cycle structure representation is analogous to the *type* of a permutation. If the type of a permutation $\sigma \in \mathfrak{S}_n$ is written $\left[1^{\alpha_1}, 2^{\alpha_2}, 3^{\alpha_3}, \ldots, n^{\alpha_n}\right]$, then the cycle structure representation of σ is written $x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_n^{\alpha_n}$ (cf. Section 1.2). We associate with each element $g \in \mathfrak{G} \leqslant \mathfrak{S}_n$ its cycle structure representation, the monomial

$$\zeta_g\left(x_1, x_2, x_3, \ldots, x_n\right) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3} \ldots x_n^{\alpha_n}. \qquad \blacklozenge$$

8.2.1 EXAMPLE. For $\sigma = (1234)(5678) \in \mathfrak{Q}$, $\zeta_\sigma\left(x_1, x_2, \ldots, x_8\right) = x_4^2$. ◇

8.2.2 DEFINITION. The *cycle index* of a finite group of permutations $\mathfrak{G}$ is the formal sum

$$\zeta_{\mathfrak{G}}\left(x_1, x_2, x_3, \ldots, x_n\right) = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \zeta_g\left(x_1, x_2, x_3, \ldots, x_n\right). \qquad \blacklozenge$$

8.2.2 EXAMPLE. For the group $\mathfrak{Q}$ — the group of rigid motions of the cube — we have already determined that there are 24 group elements. Labelling each vertex in an arbitrary manner, for instance such as in Example 4.1.6, each permutation corresponding to a rigid motion has a related cycle structure representation. We list them in the table below.

Table 8.1: Vertex permutations of a cube, their types, and their cycle structure representations.

$g \in \mathfrak{Q}_3$	*Type*	ζ_g
(1)(2)(3)(4)(5)(6)(7)(8)	$[1^8]$	x_1^8
(1234)(5678)	$[4^2]$	x_4^2
(1485)(2376)	$[4^2]$	x_4^2
(1265)(4378)	$[4^2]$	x_4^2
(1432)(5876)	$[4^2]$	x_4^2
(1584)(2673)	$[4^2]$	x_4^2
(1562)(4873)	$[4^2]$	x_4^2
(18)(27)(36)(45)	$[2^4]$	x_2^4
(13)(24)(57)(68)	$[2^4]$	x_2^4
(16)(25)(38)(47)	$[2^4]$	x_2^4
(14)(28)(35)(67)	$[2^4]$	x_2^4
(17)(23)(46)(58)	$[2^4]$	x_2^4
(17)(28)(34)(56)	$[2^4]$	x_2^4
(12)(35)(46)(78)	$[2^4]$	x_2^4
(15)(28)(37)(46)	$[2^4]$	x_2^4
(17)(26)(35)(48)	$[2^4]$	x_2^4
(1)(7)(245)(386)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(1)(7)(254)(368)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(2)(8)(136)(475)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(2)(8)(163)(457)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(3)(5)(186)(247)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(3)(5)(168)(274)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(4)(6)(138)(275)	$[1^2, 3^2]$	$x_1^2 x_3^2$
(4)(6)(183)(257)	$[1^2, 3^2]$	$x_1^2 x_3^2$

There is only one element of $\mathfrak{Q}$ for which $\zeta_g(x_1, x_2, \ldots, x_8) = x_1^8$, namely the identity permutation. There are six for which $\zeta_g(x_1, x_2, \ldots, x_8) = x_4^2$, nine for which $\zeta_g(x_1, x_2, \ldots, x_8) = x_2^4$, and eight for which $\zeta_g(x_1, x_2, \ldots, x_8) = x_1^2 x_3^2$. And so, by Definition 8.2.2,

$$\zeta_{\mathfrak{Q}}(x_1, x_2, \ldots, x_8) = \frac{x_1^8 + 6x_4^2 + 9x_2^4 + 8x_1^2 x_3^2}{24}. \qquad \diamond$$

For a cube, however, it makes more sense to color the faces rather than the vertices. As we've already discussed the duality between the octahedron and the cube, we know now that the cycle index of a cube with respect to a group of permutations of its faces must therefore be the same as for vertex permutations of octahedron.

8.2.3 Example. With respect to its group of vertex permutations, the cycle index for an octahedron (cf. Section 1.2) is

$$\frac{x_1^6 + 8x_3^2 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3}{24},$$

which is also the cycle index for a cube with respect to face permutations.

8.2.4 Example. The group $\mathfrak{T}$ of rigid motions of a tetrahedron induces a permutation group on the 4 vertices. We have the following table.

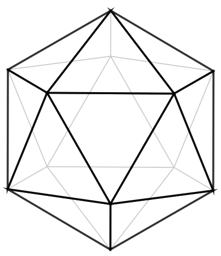
Table 8.2: The types, and cycle structure representations of the vertex permutations of a tetrahedron.

g (Example)	*Type*	ζ_g	#	*Sum*
$(1)(234)$	$\left[1^1, 3^1\right]$	x_1x_3	8	$8x_1x_3$
$(12)(34)$	$\left[2^2\right]$	x_2^2	3	$3x_2^2$
$(1)(2)(3)(4)$	$\left[1^4\right]$	x_1^4	1	x_1^4

And so $\zeta_{\mathfrak{T}}(x_1, x_2, x_3, x_4) = \frac{x_1^4 + 8x_1x_3 + 3x_2}{12}$. The tetrahedron is a *self-dual* platonic solid. Therefore, the cycle index of the group of vertex permutations is the same as for edge permutations. ◇

8.2.5 Example. Our next example concerns yet another platonic solid, called *the icosahedron.*This object is a bit more awkward to deal with when determining what permutations are induced by its group of rigid motions.

Figure 8.1: Icosahedron.



By the Orbit Stabilizer Theorem, we can quickly determine the order of its group of vertex permutations. Each vertex can be sent to every other, hence the orbit of any vertex is the set of all vertices, and so $\left|\mathrm{Orb}_{\mathfrak{G}(x)}\right| = 12$. Except for the five rotations about an axis going through two opposing vertices there are no other transformations which fixes a vertex, hence $\left|\mathrm{Stab}_{\mathfrak{G}}(x)\right| = 5$ so that $|\mathfrak{G}| = \left|\mathrm{Orb}_{\mathfrak{G}(x)}\right| \left|\mathrm{Stab}_{\mathfrak{G}}(x)\right| = 12 \cdot 5 = 60$. We conclude that the group of rigid motions yields a permutation group of order 60.

With the aid of figure 8.1, and by straining our spatial faculties (alternatively, referring to [2]), we produce the following table.

Table 8.3: The types, and cycle structure representations of the (vertex) permutation group, induced by the rigid motions of an icosahedron.

g (Shape)	*Type*	ζ_g	#	*Sum*
$(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)(\cdot)$	$\left[1^{12}\right]$	x_1^{12}	1	x_1^{12}
$(\cdot)(\cdot)(\cdot\cdot\cdot\cdot\cdot)(\cdot\cdot\cdot\cdot\cdot)$	$\left[1^2, 5^2\right]$	$x_1^2 x_5^2$	24	$24x_1^2x_5^2$
$(\cdot\cdot\cdot)(\cdot\cdot\cdot)(\cdot\cdot\cdot)(\cdot\cdot\cdot)$	$\left[3^4\right]$	x_3^4	20	$20x_3^4$
$(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot\cdot)$	$\left[2^6\right]$	x_2^6	15	$15x_2^6$

Thus we have that the cycle index of the icosahedron is

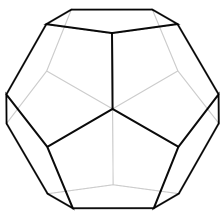
$$\frac{1}{60}\left(x_1^{12} + 24x_1^2x_5^2 + 20x_3^4 + 15x_2^6\right). \qquad \diamond$$

8.2.6 Example. The final platonic solid to deal with is the dodecahedron, the dual of the icosahedron. The permutation group induced by rigid motions acting on the vertices of the dodecahedron yields the cycle index

$$\frac{1}{60}\left(x_1^{20} + 20x_1^2x_3^6 + 15x_2^{10} + 24x_5^4\right),$$

which is also the cycle index for the permutation group of the faces of the icosahedron. ⋄

Figure 8.2: Dodecahedron



8.2.1 Prop. *The cyclic group* $\mathfrak{C}_n = \langle g : g^n = e \rangle$ *has the cycle index*

$$\zeta_{\mathfrak{C}_n}(x_1, x_2, \ldots, x_n) = \frac{1}{n}\sum_{d|n} \varphi(d) x_d^{n/d}.$$

Proof. See [2]. ■

8.2.2 Prop. *The cycle index of* $\mathfrak{D}_n$ *is*

$$\frac{1}{2}\zeta_{\mathfrak{C}_n}(x_1, x_2, \ldots, x_n) + \begin{cases} \frac{1}{4}\left(x_2^{n/2} + x_1^2 x_2^{n/2-1}\right) & \text{if } n \text{ is even} \\ \frac{1}{2}x_1 x_2^{(n-1)/2} & \text{if } n \text{ is odd} \end{cases}.$$

Proof. See [2]. ■

8.2.7 Example. Consider the problem of coloring a bracelet with 6 beads, which are evenly distributed around it. If we are permitted only to rotate the bracelet around its center, and the positions of a bead is clockwise labelled with the letters $1, 2, 3, 4, 5, 6$, we are dealing with the cyclic group $\mathfrak{C}_6$. By Proposition 8.2.1 we have that

$$\zeta_{\mathfrak{C}_6}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{6}\sum_{d|6} \varphi(d) x_d^{n/d} = \frac{1}{6}\left(x_1^6 + x_2^3 + 2x_3^2 + 2x_6\right). \quad \diamond$$

8.2.8 Example. If we consider the same bracelet as in Example 8.2.7, only this time we are also allowed to flip it about some axis, then we are dealing with the dihedral of a 6-gon — $\mathfrak{D}_6$. By Proposition 8.2.2 we have that

$$\zeta_{\mathfrak{D}_6}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{12}\left(x_1^6 + x_2^3 + 2x_3^2 + 2x_6\right) + \frac{1}{4}\left(x_2^3 + x_1^2 x_2^2\right). \quad \diamond$$

8.2.9 Example. Continuing our discussion on the cube, and the relation between the Orbit Counting Lemma and the cycle index of a permutation group, we consider the permutations of its faces. Let $S = \{c_1, c_2\}$ be a set of 2 colors. Each face can be assigned any color, with no simultaneous occurance of the two on one and the same face, and the polynomial $c_1 + c_2$ models the situation of coloring a face. The identity permutation consists of 6 disjoint 1-cycles, $(f_1)(f_2)(f_3)(f_4)(f_5)(f_6)$, each containing the label of a face. Each face can be assigned $c_1 + c_2$ (c_1 *or* c_2) independently, and so $(c_1 + c_2)^6$ generates the 64 colorings — all invariant under the identity permutation. For one of the 6 permutations with the shape $(f_1)(f_2)(f_3 f_4 f_5 f_6)$ we ask how to color the faces so that they remain invariant. As the cycles are disjoint, we can regard each by itself, and assign $c_1 + c_2$ to the 1-cycles. In the 4-cycle every face has to be the same color, and there are two colors to choose from, hence we assign $c_1^4 + c_2^4$ to it. Our permutation thus becomes the polynomial $(c_1 + c_2)^2\left(c_1^4 + c_2^4\right) = c_1^6 + 2c_1^5 c_2 + c_1^4 c_2^2 + c_1^2 c_2^4 + 2c_1 c_2^5 + c_2^6$. We can repeat this process of thought using the cycle structure representation of a permutation of the faces of a cube, since they are analogous. The proper group to consider would be the permutation group of vertices of an octahedron, and summing up each polynomial, acquired in this way, for every permutation would then give us

$$24c_1^6 + 24c_1^5c_2 + 48c_1^4c_2^2 + 48c_1^3c_2^3 + 48c_1^2c_2^4 + 24c_1c_2^5 + 24c_2^6. \tag{8.1}$$

The polynomial in (8.1) is peculiar since dividing it by 24 yields the expression

$$c_1^6 + c_1^5c_2 + 2c_1^4c_2^2 + 2c_1^3c_2^3 + 2c_1^2c_2^4 + c_1c_2^5 + c_2^6, \tag{8.2}$$

which generates the distinct colorings of the cube. From it we see, for instance, that there are 2 ways to color the cube so that two sides has color c_1 while four sides has color c_2. By letting $c_1 = c_2 = 1$, and substituting it into (8.2) we get 10, which is the number of inequivalent colorings. The same result (from Example 6.1.4) would have been reached using the Orbit Counting Lemma, where in $\frac{n^6+3n^4+12n^3+8n^2}{24}$ we put 2 instead of n, obtaining

$$\frac{2^6 + 3 \cdot 2^4 + 12 \cdot 2^3 + 8 \cdot 2^2}{24} = 10.$$

Likewise, we could just as well have put 2 instead of x_i in the cycle index of Example 8.2.3 (the octahedron), i.e.

$$\frac{2^6 + 8 \cdot 2^2 + 6 \cdot 2^2 \cdot 2 + 3 \cdot 2^2 \cdot 2^2 + 6 \cdot 2^3}{24} = 10.$$

This is what we're going to look at in the chapter that follows. ◇

9

The Pattern Inventory

9.1 Patterns It is time to introduce a consistent terminology and to refurbish some of the theory which we've earlier gone through. Let X and Y be finite sets, and consider mappings from the domain X to the range Y. The set of all such mappings is, as usual, denoted by Y^X. In previous examples the set Y consisted of colors, while X contained parts of some geometrical structure. We chose to name Y^X *the set of colorings of* X, *the set of color configurations of* X, or simply a *set of configurations*. A coloring, or configuration, is a mapping $f : X \longrightarrow Y$. In discussing the Orbit Counting Lemma, in Chapter 6, an ambiguity seemed to arise in how to denote an element in the underlying group acting on a set. If, for instance, we are to determine distinguishable colorings of the vertices of a square using n colors, where $X = \{v_1, v_2, v_3, v_4\}$, and $Y = \{c_1, c_2, \ldots, c_n\}$, then Y^X is the set of configurations while n^4 is its cardinality. The permutation group of X is $\mathfrak{D}_4$, and so it is the underlying group acting on Y^X. An element of $\mathfrak{D}_4$, it seems, can be written in terms of the vertices v_1, v_2, v_3, v_4 of a square, where $\mathfrak{D}_4$ acts on X, and in terms of configurations in Y^X, since $\mathfrak{D}_4$ acts on Y^X — we say that an element g of $\mathfrak{D}_4$ induces a permutation $\widehat{g}$ of Y^X, and we call the induced group $\widehat{\mathfrak{D}}_4$. Generally, for a permutation group $\mathfrak{G}$ of a finite set X, where $\widehat{\mathfrak{G}}$ is the induced group acting on Y^X for finite Y, we introduce a relation on Y^X, and say that f_1 and f_2 are equivalent — $f_1 \sim f_2$ — if there is an element $g \in \mathfrak{G}$ so that $f_1(gx) = f_2(x)$, for $x \in X$, and $g \in \mathfrak{G}$. We establish quickly that $\sim$ is an equivalence relation.

I. Since $e \in \mathfrak{G}$ we have $f(x) = f(ex) = f(x)$, and so $f \sim f$.

II. If $g \in \mathfrak{G}$, then $g^{-1} \in \mathfrak{G}$, and so $f_1(gx) = f_2(x) \Longrightarrow f_2\left(g^{-1}x\right) = f_1(x)$. Hence $f_1 \sim f_2 \Longrightarrow f_2 \sim f_1$.

III. If $g, h \in \mathfrak{G}$, then $gh \in \mathfrak{G}$, so that if $f_1(gx) = f_2(x)$, and $f_2(hx) = f_3(x)$ then $f_1(hgx) = f_2(hx) = f_3(x)$. Hence $f_1 \sim f_3$.

9.1.1 Definition. Let X and Y be finite sets, and G a group of permutations of X. The equivalence relation introduced by $f_1 \sim f_2$ if $f_1(gx) = f_2(x)$, for some $g \in \mathfrak{G}$, splits Y^X into equivalence classes called *patterns* . ◆

Naturally, we wish to determine the number of patterns in $\left|Y^X/\widehat{\mathfrak{G}}\right|$. As one might suspect it is equal to $\left|Y^X/\mathfrak{G}\right|$. The Orbit Counting Lemma (Theorem 6.1.1) comes into play.

9.1.2 Definition. ' Let X and Y be finite sets, and G a group of permutations of X. The element $g \in \mathfrak{G}$ induces a permutation of Y^X in the following way: $(\widehat{g}f)(x) = f\left(g^{-1}(x)\right)$, viz. $\widehat{g}f = fg^{-1}$, for all colorings $f \in Y^X$. The set of all such induced permutations is the group $\widehat{\mathfrak{G}}$, which satisfies the axioms of a group action, and so it acts on Y^X.

I. For $e \in \mathfrak{G}, \widehat{e}f = fe^{-1} = f$.

II. For $g_1, g_2 \in \mathfrak{G}, \widehat{g_1}(\widehat{g_2}f) = \widehat{g_1}\left(fg_2^{-1}\right) = fg_2^{-1}g_1^{-1} = f(g_1g_2)^{-1} = \widehat{g_1g_2}f$. ◆

9.1.1 Lemma. *The group $\widehat{\mathfrak{G}}$, as defined in 9.1.2, is isomorphic to $\mathfrak{G}$.*

Proof. For $g_1, g_2 \in \mathfrak{G}$ we have that $\widehat{g_1g_2}f = f(g_1g_2)^{-1} = fg_2^{-1}g_1^{-1} = \widehat{g_2}fg_1^{-1} = \widehat{g_1}\widehat{g_2}f$, and so the homomorphy condition is satisfied. Next we need to establish that $\widehat{g_1} = \widehat{g_2} \iff g_1 = g_2$, so that the function φ taking g to $\widehat{g}$ is a bijection. Therefore, suppose that $\widehat{g_1} = \widehat{g_2}$, so that $(\widehat{g_1}f)(x) = (\widehat{g_2}f)(x) \iff f\left(g_1^{-1}(x)\right) = f\left(g_2^{-1}(x)\right)$. By our assumption $f\left(g_1^{-1}(x)\right) = f\left(g_2^{-1}(x)\right)$ holds for all $f \in Y^X$, and is true in particular for the coloring f which assigns a specified color to $g^{-1}(x)$, and another color to every other member of X. In this case, the equation $f\left(g_1^{-1}(x)\right) = f\left(g_2^{-1}(x)\right)$ implies that $g_1^{-1}(x) = g_2^{-1}(x)$, and because the same argument works for each $x \in X$, we conclude that $g_1 = g_2$. Hence $\varphi(g) = \widehat{g}$ is a bijection, and

$$\mathfrak{G} \cong \widehat{\mathfrak{G}}.$$ ■

9.1.2 Theorem. *For $Y^X = \{f : f \text{ a mapping from} X \text{ to } Y\}$ a set of configurations, where $\mathfrak{G}$ is a permutation group which acts on X, the number of patterns is*

$$\left|Y^X \diagup \mathfrak{G}\right| = \frac{1}{|\mathfrak{G}|}\sum_{g \in \mathfrak{G}} |Fix(g)|$$

where $Fix(g) = \left\{f \in Y^X : f(g(x)) = f(x), \forall x \in X\right\}$, the set of colorings fixed by g.

Proof. The underlying group $\mathfrak{G}$ acts on X, and the induced group $\widehat{\mathfrak{G}}$ acts on Y^X. Let O denote an orbit in $Y^X/\widehat{\mathfrak{G}}$, and let P denote a pattern in $Y^X/\mathfrak{G}$. If $f_1, f_2 \in O$, we have that $\widehat{g}f_1 = f_2$, for some $\widehat{g} \in \widehat{G}$. This implies that $f_1 g^{-1} = f_2$, so that $f_1, f_2 \in P$. Hence $f_1, f_2 \in \mathcal{O}$ if, and only if, $f_1, f_2 \in P$. By the Orbit Counting Lemma we have that

$$\left| Y^X/\widehat{\mathfrak{G}} \right| = \frac{1}{\left|\widehat{\mathfrak{G}}\right|} \sum_{\widehat{g} \in \widehat{\mathfrak{G}}} |\mathrm{Fix}(\widehat{g})| ,$$

where $\mathrm{Fix}(\widehat{g}) = \left\{ f \in Y^X : (\widehat{g}f)(x) = f(x), \forall x \in X \right\}$. Now if $f \in \mathrm{Fix}(\widehat{g})$, then $\widehat{g}f = f$ which implies that $fg^{-1} = f \iff fg = f$, by Definition 9.1.2. Hence $\widehat{g}f = f \iff fg = f$, and so $|\mathrm{Fix}(\widehat{g})| = |\mathrm{Fix}(g)|$. Since $\mathfrak{G} \cong \widehat{\mathfrak{G}} \implies |\mathfrak{G}| = \left|\widehat{\mathfrak{G}}\right|$, by Lemma 9.1.1, we see that

$$\frac{1}{\left|\widehat{\mathfrak{G}}\right|} \sum_{\widehat{g} \in \widehat{\mathfrak{G}}} |\mathrm{Fix}(\widehat{g})| = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} |\mathrm{Fix}(g)| ,$$

and we conclude that

$$\left| Y^X/\widehat{\mathfrak{G}} \right| = \frac{1}{\mathfrak{G}} \sum_{g \in \mathfrak{G}} \mathrm{Fix}(g) = \left| Y^X/\mathfrak{G} \right| . \qquad \blacksquare$$

9.2 WEIGHTS

In Example 8.2.9 we touched on the idea of assigning a *weight* to a color configuration. For sets X, Y, with mappings Y^X, the initial idea is to assign an element of a commutative ring to each member of $y \in Y$, called the weight of y. In doing so, we can form sums, products and rational multiples of weights (provided that the ring R contains the rational numbers), which satisfies the usual axioms.

9.2.1 DEFINITION. For a finite set Y, and a commutative ring R containing $\mathbb{Q}$, the function $w : Y \longrightarrow R$ assigns to each member $y \in Y$ its *weight* $w(y) \in R$. ◆

9.2.2 DEFINITION. For X, Y, and the set of mappings Y^X, the function $W : Y^X \longrightarrow R$ assigns to each configuration $f \in Y^X$ its weight $W(f) \in R$, where

$$W(f) = \prod_{x \in X} w(f(x)) . \qquad ◆$$

9.2.1 LEMMA. *If $f_1, f_2 \in P \subseteq Y^X$, where P is a pattern (orbit) in Y^X, then*

$$W(f_1) = W(f_2) .$$

Proof. If $f_1, f_2 \in P \subseteq Y^X$, then $f_1 \sim f_2$, by Definition 9.1.1, viz. $f_1(gx) = f_2(x)$ where $g \in \mathfrak{G}$. The products $\prod_{x \in X} w(f_1(x))$, and $\prod_{x \in X} w(f_1(gx))$ have the same factors, only in a different order, since g only permutes the index set, which is all of X. Hence $\prod_{x \in X} w(f_1(x)) = \prod_{x \in X} w(f_1(gx)) = \prod_{x \in X} w(f_2(x)) = W(f_2)$. ■

9.2.3 DEFINITION. The weight of a pattern $P \subseteq Y^X$ is denoted $W(P)$. All configurations $f \in P$ have the same weight, and $W(P) = W(f)$, for some $f \in P$. ◆

9.2.1 EXAMPLE. Let $X = \{e_1, e_2, e_3, e_4\}$ be the edges of a square, and let $Y = \{r, b\}$ contain the colors r, and b. Let $\mathbb{Q}[x, y]$ be the polynomial ring in two variables x, and y, with rational coefficients. Assign the weight $w(r) = x$ to r, and $w(b) = y$ to b. For the 2-colorings of a square with respect to its edges, we determined in Example 6.1.3 that there are 16 configurations, and 6 patterns in all.

P_1: All edges in X are mapped to $r \in Y$ so that, for $f \in P_1$, $W(f) = x^4 = W(P_1)$;

P_2: Three edges in X are mapped to $r \in Y$, one is mapped to $b \in Y$. Therefore $W(P_2) = x^3y$;

P_3: Two adjacent edges are mapped to $r \in Y$, and two adjacent edges are mapped to $b \in Y$. Hence $W(P_3) = x^2y^2$;

P_4: Two opposite edges are mapped to r, and to opposite edges are mapped to b. Thus $W(P_4) = x^2y^2$;

P_5: Three edges are mapped to b, one is mapped to r. Thus $W(P_5) = xy^3$;

P_6: All edges are mapped to b, so that $W(P_6) = y^4$.

◇

9.2.2 EXAMPLE. Let $X = \{a_1, a_2, a_3, a_4, a_5, a_6\}$ be the faces of a cube, and let $Y = \{r, b\}$ contain the colors r, and b. Let $\mathbb{Q}[x, y]$ be the polynomial ring in two variables x, and y, with rational coefficients. Assign the weight $w(r) = x$ to r, and $w(b) = y$ to b. For the 2-colorings of a cube with respect to its faces, we determined in Example 8.2.9 that there are 64 configurations, and 10 patterns in all.

P_1: All faces in X are mapped to $r \in Y$ so that, for $f \in P_1$, $W(f) = x^6 = W(P_1)$;

P_2: All faces, but one, are mapped to $r \in Y$. One face is mapped to $b \in Y$. We have that $W(P_2) = x^5y$;

P_3: Four faces are mapped to r, and two adjacent faces are mapped to b. And so $W(P_3) = x^4y^2$;

P_4: Four faces are mapped to r, and two opposite faces are mapped to b. Thus $W(P_4) = x^4y^2$;

P_5: Three faces meeting at a vertex are mapped to r, and three faces (also meeting at a vertex) are mapped to b. We have that $W(P_5) = x^3y^3$;

P_6: Two opposite faces, and one more face, are mapped to r. Three faces are mapped to b. Therefore $W(P_6) = x^3y^3$;

P_7: Two adjacent faces are mapped to r, and four faces are mapped to b. We have that $W(P_7) = x^2y^4$;

P_8: Two opposite faces are mapped to r, and four faces are mapped to b. We have that $W(P_8) = x^2y^4$;

P_9: All faces, but one, are mapped to b. Thus $W(P_9) = xy^5$;

P_{10}: All faces in X are mapped to $b \in Y$, hence $W(P_{10}) = y^6$.

◇

Remark. It is worthwile to observe that

$$\sum_{i=1}^{10} W(P_i) = x^6 + x^5y + 2x^4y^2 + 2x^3y^3 + 2x^2y^4 + xy^5 + y^6.$$

Compare this sum with the expression (8.2) found in Example 8.2.9. This is the *pattern inventory* of 2-colorings of a cube moving freely in space.

9.3 The Pattern Inventory

We are to present a famous theorem due to Pólya. In short, our task is to establish a function — called the pattern inventory — which generates and/or enumerates the patterns of a set Y^X of configurations. To help us in this regard we shall extend on the ideas of weights of configurations, and weights of patterns.

9.3.1 Definition. For a finite set Y, where each element $y \in Y$ has been assigned the weight $w(y)$, we say that

$$I(Y) = \sum_{y \in Y} w(y)$$

is the *inventory of* Y. ◆

9.3.2 Definition. Given the finite sets X, Y, and where $W(f) = \prod_{x \in X} w(f(x))$ (definition 9.2.2), then the inventory of Y^X is given by

$$I\left(Y^X\right) = \sum_{f \in Y^X} W(f).$$ ◆

9.3.1 Lemma.

$$I\left(Y^X\right) = (I(Y))^{|X|}.$$

Proof. See [4]. ■

9.3.2 Lemma. *Let X be a disjoint union $\bigcup_{i=1}^{n} X_i$ of finite sets X_i, so that (by the rule of sum) $|X| = |X_1| + \ldots + |X_n|$. Let $Y = \{y_1, y_2, \ldots, y_m\}$, and let $S \subseteq Y^X$ where $S = \left\{f \in Y^X : f \text{ is constant on each } X_i\right\}$. Then*

$$I(S) = \sum_{f \in S} W(f) = \prod_{i=i}^{n} \sum_{y \in Y} [w(y)]^{|X_i|}$$

Proof. See [4]. ■

9.3.3 THEOREM (BABY PÓLYA). *For finite sets X, and Y, where $\mathfrak{G}$ is a group of permutations of X, the number of patterns in Y^X is given by the following formula:*

$$\left| Y^X \diagup \mathfrak{G} \right| = \frac{1}{\mathfrak{G}} \sum_{g \in \mathfrak{G}} |Y|^{c(g)},$$

where $c(g)$ is the number of cycles in g, as expressed in the decomposition of X under the action of g.

Proof. By Theorem 9.1.1 we have that $\left| Y^X \diagup \mathfrak{G} \right| = \frac{1}{\mathfrak{G}} \sum_{g \in \mathfrak{G}} |\mathrm{Fix}(g)|$, where $\mathrm{Fix}(g) = \left\{ f \in Y^X : f(g(x)) = f(x), \forall x \in X \right\}$. We must therefore show that $|\mathrm{Fix}(g)| = |Y|^{c(g)}$. We make the observation that $g \in \mathfrak{G}$ is a permutation of X, and that each g splits X into a disjoint union of $c(g)$ cycles $X_1, X_2, \ldots, X_{c(g)}$, where each cycle is cyclically permuted by g. If $f \in \mathrm{Fix}(g)$, then $f = fg = fg^2 = \ldots$, hence f is constant on each cycle X_i. Conversely, if f is constant on each X_i, then $f = fg$ since $g(x) \in X_i$, for $x \in X_i$, and so $f \in \mathrm{Fix}(g)$. Thus $f \in \mathrm{Fix}(g)$ if, and only if, f is constant on each cycle X_i. Therefore, all of the elements in a cycle X_i are mapped to one and the same member of Y. There are $c(g)$ cycles, and for each cycle X_i there are $|Y|$ possible elements to which one can map all members of X_i. By the rule of product we get that $|\mathrm{Fix}(g)| = |Y|^{c(g)}$, and so

$$\left| Y^X \diagup \mathfrak{G} \right| = \frac{1}{\mathfrak{G}} \sum_{g \in \mathfrak{G}} |\mathrm{Fix}(g)| = \frac{1}{\mathfrak{G}} \sum_{g \in \mathfrak{G}} |Y|^{c(g)}. \qquad \blacksquare$$

9.3.3 DEFINITION. Let $\mathcal{P} = \{P_1, \ldots, P_k\}$ be the set of all patterns in Y^X. We recall from definition 9.2.3 that for the weight of some pattern P: $W(P) = W(f)$, where $f \in P$. The *pattern inventory* , also called the *pattern generating function* (PGF for short) is defined

$$I(\mathcal{P}) = \sum_{P \in \mathcal{P}} W(P). \qquad \blacklozenge$$

9.3.4 PÓLYA'S ENUMERATION THEOREM. *Let X, and Y be finite sets where $|X| = n$, and let $\mathfrak{G}$ be a group of permutations of X. We recall that the cycle index of $\mathfrak{G}$ is the formal sum*

$$\zeta_{\mathfrak{G}}\left(x_1, x_2, x_3, \ldots, x_n\right) = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \zeta_g\left(x_1, x_2, x_3, \ldots, x_n\right).$$

The pattern inventory (definition 9.3.3) is given by

$$\zeta_{\mathfrak{G}}\left\{\sum_{y \in Y} w(y), \sum_{y \in Y}[w(y)]^2, \ldots, \sum_{y \in Y}[w(y)]^n\right\}.$$

If all weights are chosen to be equal to 1 we get the number of patterns, viz. $\left|Y^X / \mathfrak{G}\right|$.

Proof. Let ω be some value that the weight of a function may have. For $f_1, f_2 \in Y^X$, and $g \in \mathfrak{G}$, if $f_1 = f_2 g$ then $W(f_1) = W(f_2)$ (Lemma 9.2.1). We take $S_\omega \subseteq Y^X$, where $\left\{f \in Y^X : W(f) = \omega\right\}$, hence if $f_1 = f_2 g \in S_\omega$ then $f_1 g^{-1} \in S_\omega$. Let $\mathrm{Fix}_\omega(g) = \left\{f \in Y^X : fg = f, \text{ and } W(f) = \omega\right\}$. The number of patterns (Theorem 9.1.1) contained in S_ω is

$$\frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} |\mathrm{Fix}_\omega(g)| \,. \tag{9.1}$$

The patterns contained in S_ω have the same weight ω, and if we multiply (9.1) by ω, and sum over all possible values of ω, we obtain the pattern inventory

$$I(\mathcal{P}) = \sum_{P \in \mathcal{P}} W(P) = \frac{1}{|\mathfrak{G}|} \sum_{\omega} \sum_{g \in \mathfrak{G}} |\mathrm{Fix}_\omega(g)| \,\omega. \tag{9.2}$$

If we let $\mathrm{Fix}(g) = \left\{f \in Y^X : fg = f\right\}$, then we have that

$$\sum_{\omega} |\mathrm{Fix}_\omega(g)| \,\omega = \sum_{f \in \mathrm{Fix}(g)} W(f) \,,$$

and since the indices in (9.2) are finite we can exchange the order of summation. The right hand side in (9.2) becomes

$$\frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \sum_{\omega} |\mathrm{Fix}_\omega(g)| \,\omega = \frac{1}{|\mathfrak{G}|} \sum_{g \in \mathfrak{G}} \sum_{f \in \mathrm{Fix}(g)} W(f) \,. \tag{9.3}$$

It remains to evaluate the sum $\sum_{f \in \mathrm{Fix}(g)} W(f)$ in (9.3). We make the observation that $g \in \mathfrak{G}$ is a permutation of X, and each g splits X into a disjoint union of m cycles $X_1, X_2, \ldots, X_m$, where $m \leq n$. Each cycle is cyclically permuted by g. If $f \in \mathrm{Fix}(g)$, then $f = fg = fg^2 = \ldots$, hence f is constant on each cycle X_i. Conversely, if f is constant on each X_i, then $f = fg$ since $g(x) \in X_i$, for $x \in X_i$, and so $f \in \mathrm{Fix}(g)$. We can therefore apply Lemma 9.3.2:

$$\sum_{f \in \mathrm{Fix}(g)} W(f) = \prod_{i=1}^{m} \sum_{y \in Y} [w(y)]^{|X_i|} \,. \tag{9.4}$$

Proof, continuation. Expanding on the right hand side of (9.4), we get

$$\left(\sum_{y\in Y}[w(y)]^{|X_1|}\right)\left(\sum_{y\in Y}[w(y)]^{|X_2|}\right)\cdots\left(\sum_{y\in Y}[w(y)]^{|X_m|}\right) \tag{9.5}$$

Let $\left[1^{\alpha_1}, 2^{\alpha_2}, \ldots, n^{\alpha_n}\right]$ be the type of g. This means that among the numbers $|X_1|$, $|X_2|, \ldots, |X_n|$, 1 occurs α_1 times, 2 occurs α_2 times, and so on. We can therefore write (9.5) as

$$\sum_{f\in\mathrm{Fix}(g)} W(f) = \left(\sum_{y\in Y}[w(y)]\right)^{\alpha_1}\left(\sum_{y\in Y}[w(y)]^2\right)^{\alpha_2}\cdots\left(\sum_{y\in Y}[w(y)]^n\right)^{\alpha_n}. \tag{9.6}$$

Hence the right hand side in (9.3) becomes

$$\frac{1}{|\mathfrak{G}|}\sum_{g\in\mathfrak{G}}\left\{\left(\sum_{y\in Y}[w(y)]\right)^{\alpha_1}\left(\sum_{y\in Y}[w(y)]^2\right)^{\alpha_2}\cdots\left(\sum_{y\in Y}[w(y)]^n\right)^{\alpha_n}\right\}. \tag{9.7}$$

We note that $\left(\sum_{y\in Y}[w(y)]\right)^{\alpha_1}\left(\sum_{y\in Y}[w(y)]^2\right)^{\alpha_2}\cdots\left(\sum_{y\in Y}[w(y)]^n\right)^{\alpha_n}$ is precisely what is obtained by substitution of

$$x_1 = \sum_{y\in Y} w(y), \quad x_2 = \sum_{y\in Y}[w(y)]^2, \ldots, \quad x_n = \sum_{y\in Y}[w(y)]^n$$

into the cycle structure representation $\zeta_g(x_1, x_2, \ldots, x_n)$ of g. We can therefore conclude that (9.7) is the cycle index

$$\zeta_{\mathfrak{G}}\left\{\sum_{y\in Y} w(y), \sum_{y\in Y}[w(y)]^2, \ldots, \sum_{y\in Y}[w(y)]^n\right\}. \qquad \blacksquare$$

9.4 Gons & Hedrons

Pólya's Enumeration Theorem reduces the problem of finding equivalence classes — patterns — of a set of configurations to that of finding a cycle index $\zeta_{\mathfrak{G}}$ — the result being a generating function: the pattern inventory. Here we give a brief exposition on the colorings of the geometrical objects we've encountered.

9.4.1 Example. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$ be the positionings of 5 beads in a necklace, and let Y be a set of 3 colors. Consider the problem of making a bracelet with 5 colored beads, which are evenly distributed around it, while we are only allowed to rotate the necklace about the centre. The group in question which acts on X is $\mathfrak{C}_5 = \left\langle g : g^5 = e \right\rangle$, and has the cycle index (Theorem 8.2.1)

$$\zeta_{\mathfrak{C}_7}\left(x_1, x_2, x_3, x_4, x_5\right) = \frac{1}{5}\sum_{d|5}\varphi(d)x_d^{5/d} = \frac{1}{5}\left(x_1^5 + 4x_5\right).$$

We can choose to assign some weight x, y, z to each color in Y. Applying Theorem 9.3.4 we get that

$$\zeta_{\mathfrak{C}_5}\left\{\sum_{y\in Y} w(y), \ldots, \sum_{y\in Y}\left[w(y)\right]^5\right\} = \frac{1}{5}\left((x+y+z)^5 + 4\left(x^5 + y^5 + z^5\right)\right). \tag{9.8}$$

The expression on the right hand side in (9.8) is, when expanded, equal to:

$$x^5 + x^4y + x^4z + 2x^3y^2 + 4x^3yz + 2x^3z^2 + 2x^2y^3 + 6x^2y^2z + 6x^2yz^2 + 2x^2z^3 + xy^4 + 4xy^3z + 6xy^2z^2 + 4xyz^3 + xz^4 + y^5 + y^4z + 2y^3z^2 + 2y^2z^3 + yz^4 + z^5,$$

which is the pattern inventory of the 3-colorings of a 5-beaded necklace. Among the 21 terms we see, for instance, that there are 6 distinct colorings where there are two of the first, one of the second, and two of the third color. ◇

9.4.2 Example. We consider the same 5-beaded necklace, again using three colors, with the extra condition that we allow for reflections (that is, we are allowed to flip it about some axis of symmetry). The group in question which acts on X is $\mathfrak{D}_5 = \left\langle g : g^5 = h^2 = e \right\rangle$, which has the cycle index (Theorem 8.2.2)

$$\frac{1}{2}\zeta_{\mathfrak{C}_5}\left(x_1, x_2, x_3, x_4, x_5\right) + \frac{1}{2}x_1x_2^{(5-1)/2} = \frac{1}{10}\left(x_1^5 + 4x_5 + 5x_1x_2^2\right).$$

We can choose to assign some weight x, y, z to each color in Y. Applying Theorem 9.3.4 we get the pattern inventory

$$\frac{1}{10}\left((x+y+z)^5 + 4\left(x^5 + y^5 + z^5\right) + 5(x+y+z)(x^2+y^2+z^2)^2\right). \tag{9.9}$$

By expanding (9.9) we obtain

$$x^5 + x^4y + x^4z + 2x^3y^2 + 2x^3yz + 2x^3z^2 + 2x^2y^3 + 4x^2y^2z + 4x^2yz^2 + 2x^2z^3 + xy^4 + 2xy^3z + 4xy^2z^2 + 2xyz^3 + xz^4 + y^5 + y^4z + 2y^3z^2 + 2y^2z^3 + yz^4 + z^5.$$

Comparing the terms in this expression to the corresponding ones in the previous example, we remark that the coefficients are smaller this time. Naturally — since two colorings which are distinct with respect only to rotations might this time be equivalent under reflections. ◇

9.4.3 Example. From example 8.2.3, we have that the cycle index of the group $\mathfrak{G}$ acting on the faces of a cube is $\frac{x_1^6+8x_3^2+6x_1^2x_4+3x_1^2x_2^2+6x_2^3}{24}$. Lets determine the 3-colorings of the faces of a cube, using the colors red, blue and yellow. Assign to each color the weights r, b, y. By Theorem 9.3.4, we get that the pattern inventory is

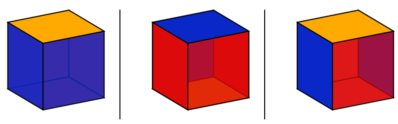
$$\frac{(r+b+y)^6+8(r^3+b^3+y^3)^2+6(r+b+y)^2(r^4+b^4+y^4)+3(r+b+y)^2(r^2+b^2+y^2)^2+6(r^2+b^2+y^2)^3}{24}$$

which, when expanded, is equal to

$$r^6+r^5b+r^5y+2r^4b^2+2r^4by+2r^4y^2+2r^3b^3+\underline{3r^3b^2y}+3r^3by^2+2r^3y^3+$$
$$2r^2b^4+3r^2b^3y+6r^2b^2y^2+3r^2by^3+2r^2y^4+rb^5+2rb^4y+3rb^3y^2+3rb^2y^3+$$
$$2rby^4+ry^5+b^6+b^5y+2b^4y^2+2b^3y^3+2b^2y^4+by^5+y^6.$$

We remark that there are 3 distinct ways of coloring the cube so that there are three red faces, two blue faces, and one yellow face. This was also the answer to the question posed in section 1.2. Naturally — since the cube and the octahedron are dual solids. We make an attempt to interpret this in figure 9.1. ◇

Figure 9.1: The distinct colorings corresponding to $3r^3b^2y$.



9.4.4 Example. We have that the cycle index of the group $\mathfrak{G}$ acting on the vertices of an icosahedron (example 8.2.5) is

$$\frac{1}{60}\left(x_1^{12}+24x_1^2x_5^2+20x_3^4+15x_2^6\right).$$

Considering a set of two colors, for instance black, and white — with weights b, and w — yields the pattern inventory

$$b^{12}+b^{11}w+3b^{10}w^2+5b^9w^3+12b^8w^4+14b^7w^5+24b^6w^6+14b^5w^7+$$
$$12b^4w^8+5b^3w^9+3b^2w^{10}+bw^{11}+w^{12}.$$

Of course, the same inventory goes for the 2-colorings of the faces of a dodecahedron since it is the dual of the icosahedron. ◇

9.4.5 Example. The group which acts on the set of vertices of the dodecahedron has the (example 8.2.6) cycle index

$$\frac{1}{60}\left(x_1^{20}+20x_1^2x_3^6+15x_2^{10}+24x_5^4\right).$$

Again we consider a 2-coloring black, and white — with weights b, and w. The pattern inventory is

$b^{20} + b^{19}w + 6b^{18}w^2 + 21b^{17}w^3 + 96b^{16}w^4 + 262b^{15}w^5 + 681b^{14}w^6 + 1302b^{13}w^7 + 2157b^{12}w^8 + 2806b^{11}w^9 + 3158b^{10}w^{10} + 2806b^9w^{11} + 2157b^8w^{12} + 1302b^7w^{13} + 681b^6w^{14} + 262b^5w^{15} + 96b^4w^{16} + 21b^3w^{17} + 6b^2w^{18} + bw^{19} + w^{20}.$

◇

10

Graphical Enumeration

It would be slightly inappropriate to omit a discussion on graphical enumeration, since we actually began this text with such an example (Section 1.1). Lets resume this discussion now. We review, first, the basic definitions in graph theory, and transition shortly to the application of Pólya's Enumeration Theorem.

10.1 Simple Graphs

10.1.1 Definition. Let V be a finite, non-empty, set — we will typically have that $V = \{1, 2, \ldots, n\}$. Let $E \subseteq \binom{V}{2}$, where $\binom{V}{2} = \{\{i, j\} : i, j \in V, i \neq j\}$. We call V the vertex set, E the edge set, and $G = (V, E)$ the *undirected, loop-free (and labelled) simple graph* with vertex set V, and edge set E. We will denote $\binom{V}{2}$ by $V^{(2)}$, and we have that

$$\left|V^{(2)}\right| = \left|\binom{V}{2}\right| = \binom{|V|}{2}.$$

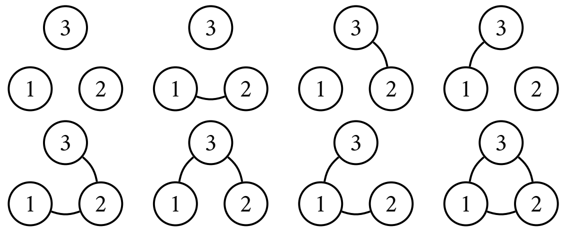
◆

10.1.1 Example. Let $V = \{1, 2, 3\}$. Then $\left|V^{(2)}\right| = \binom{3}{2} = 3$. As per the discussion in Section 1.2 we can consider all possible graphs. As shown in figure 10.1, there's only one possible graph with three labelled vertices and zero edges, three possible graphs with one and two edges respectively, and finally there is only one graph with three edges. Accounting for all possibilities — and summing them up,

$$\binom{\binom{3}{2}}{0} + \binom{\binom{3}{2}}{1} + \binom{\binom{3}{2}}{2} + \binom{\binom{3}{2}}{3} = 2^{\binom{3}{2}},$$

so provides us with 8 distinct, labelled, graphs where $V = \{1, 2, 3\}$. ◇

Figure 10.1: Every possible graph with the three labelled vertices 1, 2, *and* 3.



10.1.1 Prop. *There are* $2^{\left|V^{(2)}\right|} = 2^{\binom{|V|}{2}}$ *labelled graphs with* $|V|$ *vertices.*

Proof. It is sufficient to observe that for each edge $e \in \binom{V}{2}$ we can choose either to join it with E, or not. This can be illustrated with the binomial theorem:

$$\binom{\binom{|V|}{2}}{0} + \binom{\binom{|V|}{2}}{1} + \ldots + \binom{\binom{|V|}{2}}{\binom{|V|}{2}} = 2^{\binom{|V|}{2}}. \qquad \blacksquare$$

10.1.2 Definition. Let $G_1 = (V_1, E_1)$, and $G_2 = (V_2, E_2)$ be two undirected graphs. We say that G_1 and G_2 are isomorphic if there exists a bijection $\sigma : V_1 \longrightarrow V_2$, such that, for all $i, j \in V_1$, $\{i, j\} \in E_1$ if, and only if, $\{\sigma(i), \sigma(j)\} \in E_2$. If such a σ exists it is called an isomorphism of graphs, and we write

$$G_1 \cong G_2. \qquad \blacklozenge$$

10.1.2 Example. We have that

$$G_1 = (\{1, 2, 3\}, \{\{1, 2\}, \{2, 3\}\}) \cong G_2 = (\{1, 2, 3\}, \{\{1, 2\}, \{1, 3\}\}),$$

since $\sigma = (12)$ is a bijection, between V_1 and V_2, satisfying the sought after properties of definition 10.1.2.

10.1.3 Definition. Let $V = \{1, 2, \ldots, v\}$ be the vertex-set of a graph G. The natural choice to consider, as a permutation group acting on the vertices of G, is the symmetric group $\mathfrak{S}_v$. We define *the symmetric pair group* $\mathfrak{S}_v^{(2)}$ as the group induced by the permutations in $\mathfrak{S}_n$. A *pair-permutation* $\sigma^{(2)}$ is an element of $\mathfrak{S}_v^{(2)}$, induced by σ, which permutes $V^{(2)}$ by

$$\sigma^{(2)}\{i, j\} = \{\sigma(i), \sigma(j)\}, \text{ for each } \{i, j\} \in V^{(2)}. \qquad \blacklozenge$$

Remark. The pair group $\mathfrak{S}_n^{(2)}$ acts on $V^{(2)}$, since:

I. $e^{(2)}\{i, j\} = \{e(i), e(j)\} = \{i, j\}$, for all $\{i, j\} \in V^{(2)}$;

II. $g^{(2)}\left(h^{(2)}\{i, j\}\right) = g^{(2)}\{h(i), h(j)\} = \{gh(i), gh(j)\} = (gh)^{(2)}\{i, j\}$, for all $\{i, j\} \in V^{(2)}$.

10.1.3 Example. For $V = \{1,2,3\}$ we have that $V^{(2)} = \{\{1,2\},\{1,3\},\{2,3\}\}$. We establish, here, the induced group $\mathfrak{S}_3^{(2)}$.

I: To $e = (1)(2)(3) \in \mathfrak{S}_3$ corresponds the element $e^{(2)} = (12)(13)(23) \in \mathfrak{S}_3^{(2)}$. $(\{1,2\})(\{1,3\})(\{2,3\})$ would, of course, be the correct way to denote $e^{(2)}$, but we choose to omit the curly brackets and commas in order to obtain a more convenient notation;

II: For $(123) \in \mathfrak{S}_3$, we have that $(123)^{(2)} = (12,23,13) \in \mathfrak{S}_3^{(2)}$;

III: For $(132) \in \mathfrak{S}_3$, we have that $(132)^{(2)} = (12,13,23) \in \mathfrak{S}_3^{(2)}$;

IV: For $(3)(12) \in \mathfrak{S}_3$, we have that $(12)^{(2)} = (12)(13,23) \in \mathfrak{S}_3^{(2)}$;

V: For $(2)(13) \in \mathfrak{S}_3$, we have that $(13)^{(2)} = (13)(12,23) \in \mathfrak{S}_3^{(2)}$;

VI: For $(1)(23) \in \mathfrak{S}_3$, we have that $(23)^{(2)} = (23)(12,13) \in \mathfrak{S}_3^{(2)}$. ◇

We need a way to establish the isomorphism classes of labelled graphs and we will do so using the same approach as before — by finding the cycle structure monomials $\zeta_{g^{(2)}}(x_1, x_2, \ldots)$ for each $\sigma^{(2)} \in \mathfrak{S}_v^{(2)}$ in order to determine the cycle index

$$\zeta_{\mathfrak{S}_v^{(2)}}(x_1, x_2, \ldots)$$

of $\mathfrak{S}_v^{(2)}$, and thereafter by applying Pólya's Enumeration Theorem. Introducing the set of colors $Y = \{A, B\}$, "absent" and "present", we make use of the observation that K_v — the complete graph on v vertices — contains all possible $e = \{i,j\} \in V^{(2)}$, i.e. $|K_n| = \left|V^{(2)}\right|$, so that our current predicament can be adressed as a problem of coloring the edge set of K_n, where $w(A) = x^0 = 1$ (the weight of an edge which is absent), and $w(B) = x$ (a present edge). The pattern inventory determines the isomorphism classes of $Y^{V^{(2)}} / \mathfrak{S}_n^{(2)}$ — by enumerating the distinct unlabelled graphs, given a number of edges — and will be expressed as a polynomial

$$\sum_{e=0}^{\binom{v}{2}} g_{v,e} x^e.$$

10.1.4 Example. A quick glance at figure 10.1, together with our findings in example 10.1.3, determines that

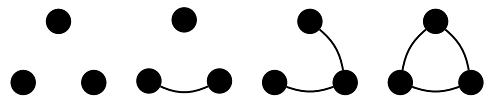
$$\zeta_{\mathfrak{S}_3^{(2)}}(x_1, x_2, x_3) = \frac{1}{6}\left(x_1^3 + 2x_3 + 3x_1x_2\right).$$

Applying PET, we get that

$$\zeta_{\mathfrak{S}_3^{(2)}}\left(1+x, 1+x^2, 1+x^3\right) = \frac{1}{6}\left((1+x)^3 + 2(1+x^3) + 3(1+x)(1+x^2)\right),$$

which equals $1 + x + x^2 + x^3$. In other words we have that for three unlabelled vertices, there's one distinct graph with 0 edges, one distinct graph with 1 edge, one distinct graph with 2 edges, and one distinct graph with 3 edges, as depicted in figure 10.2. ◇

Figure 10.2: Isomorphism classes of graphs with three vertices.



10.1.5 EXAMPLE. Table 10.1 contains the elements of $\mathfrak{S}_4^{(2)}$ — the group of pair-permutations on $V^{(2)} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3\},\{2,4\},\{3,4\}\}$.

Table 10.1: A list of group elements in $\mathfrak{S}_4^{(2)}$.

$(12,13,14)(23,34,24)$	$(12,23,24)(13,34,14)$	$(12,24,14)(13,23,34)$
$(12,14,13)(23,24,34)$	$(12,24,23)(13,14,34)$	$(12,14,24)(13,34,23)$
$(12,23,13)(14,24,34)$	$(12,13,23)(14,34,24)$	$(12)(34)(13,24)(14,23)$
$(13)(24)(12,34)(14,23)$	$(14)(23)(12,34)(13,24)$	$(12)(34)(13,14)(23,24)$
$(13)(24)(12,14)(23,34)$	$(23)(14)(12,24)(13,34)$	$(13)(24)(12,23)(14,34)$
$(14)(23)(12,13)(24,34)$	$(12)(34)(13,23)(14,24)$	$(13,24)(12,23,34,14)$
$(14,23)(12,24,34,13)$	$(12,34)(13,23,24,14)$	$(14,23)(12,13,34,24)$
$(12,34)(13,14,24,23)$	$(13,24)(12,14,34,23)$	$(12)(13)(14)(23)(24)(34)$

Table 10.2: Types, and cycle structure monomials of elements in $\mathfrak{S}_4^{(2)}$.

Type	$\zeta_{\sigma^{(2)}}$	#	*Sum*
$[1^6]$	x_1^6	1	x_1^6
$[1^2, 2^2]$	$x_1^2x_2^2$	9	$9x_1^2x_2^2$
$[2^1, 4^1]$	x_2x_4	6	$6x_2x_4$
$[3^2]$	x_3^2	8	$8x_3^2$

From table 10.2 we can determine the cycle index, which is

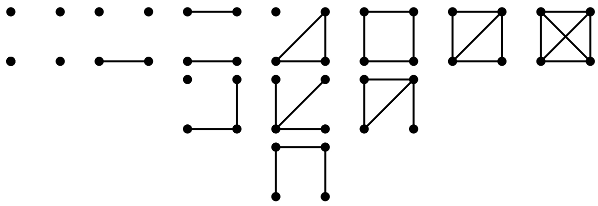
$$\zeta_{\mathfrak{S}_4^{(2)}}(x_1,x_2,x_3,x_4,x_5,x_6) = \frac{1}{24}\left(x_1^6 + 9x_1^2x_2^2 + 6x_2x_4 + 8x_3^2\right).$$

Applying PET, we get that

$$\zeta_{\mathfrak{S}_4^{(2)}} = \frac{1}{24}\left((1+x)^6 + 9(1+x)^2(1+x^2)^2 + 6(1+x^2)(1+x^4) + 8(1+x^3)^2\right),$$

which equals $1 + x + 2x^2 + 3x^3 + 2x^4 + x^5 + x^6$. Hence with four unlabelled vertices we have one distinct graph with 0 edges, one with 1 edge, two with 2 edges, three with 3 edges, two with 4 edges, one with 5 edges, and one with 6 edges (viz. K_4). Figure 10.3 depicts them. ◇

Figure 10.3: Isomorphism classes of graphs with 4 vertices.



Before congratulating ourselves we should consider that the computations involved in determining $\zeta_{\mathfrak{S}_v^{(2)}}$ has, for far, been rather laborious. Needless to say, for larger v the computational load increases drastically. The explicit formula for computing $\zeta_{\mathfrak{S}_v^{(2)}}$ is established in [7], by Harary and Palmer.

10.1.2 THEOREM.

$$\zeta_{\mathfrak{S}_v^{(2)}} = \frac{1}{\cancel{v}!}\sum_{(j)}\left\{\frac{\cancel{v}!}{\prod k^{j_k} j_k!}\prod_k x_{2k+1}^{kj_{2k+1}}\prod_k\left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}}\prod_{r<t} x_{[r,t]}^{(r,t)j_r j_t}\right\},$$

where $(r,t) = gcd(r,t)$, *and* $[r,t] = lcm(r,t)$. *Summation is taken over partitions* $(j) = \left[1^{j_1}, 2^{j_2}, \ldots, k^{j_k}, \ldots, v^{j_v}\right]$ *of* v *— where the notation* $(j_1, j_2, \ldots, j_v)$ *is used.*

Proof. See []. ■

10.1.6 EXAMPLE. We end this section with an application of Theorem 10.1.2 by computing $\zeta_{\mathfrak{S}_5^{(2)}}$, and determining the pattern inventory. There are seven partitions of 5: $\left[1^5\right]$, $\left[1^3, 2^1\right]$, $\left[1^2, 3^1\right]$, $\left[1^1, 4^1\right]$, $\left[1^1, 2^2\right]$, $\left[2^1, 3^1\right]$, and $\left[5^1\right]$. We go through them, case by case, using the $(j_1, j_2, \ldots, j_v)$ notation. For brevity we cannot display complete calculations, only their results.

I: For $(5,0,0,0,0)$, we have that

$$\begin{array}{lcccr}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^5 5! 2^0 0! \ldots 5^0 0!} & = & \frac{1}{1^5 5!}; \\
\prod_k x_{2k+1}^{kj_{2k+1}} & = & x_3^{1\cdot 0}\ldots x_{11}^{5\cdot 0} & = & 1; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & x_1^{10}\ldots\left(x_5 x_{10}^{4}\right)^0 x_5^{5\cdot\binom{0}{2}} & = & x_1^{10}; \\
\prod_{r<t} x_{[r,t]}^{(r,t)j_r j_t} & = & x_2^0 x_3^0 \ldots x_{20}^0 & = & 1.
\end{array}$$

II: $(3,1,0,0,0)$, we have that

$$\begin{array}{lcccr}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^3 3! 2^1 1! 3^0 0! \ldots 5^0 0!} & = & \frac{1}{1^3 3! 2^1 1!}; \\
\prod_k x_{2k+1}^{kj_{2k+1}} & = & x_3^{1\cdot 0} x_5^{2\cdot 0}\ldots x_{11}^{5\cdot 0} & = & 1; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & x_1^{4}\ldots\left(x_5 x_{10}^{4}\right)^0 x_5^{0} & = & x_1^{4}; \\
\prod_{r<t} x_{[r,t]}^{(r,t)j_r j_t} & = & x_2^3 x_3^0 \ldots x_{20}^0 & = & x_2^3.
\end{array}$$

III: $(2,0,1,0,0)$, we have that

$$\begin{array}{lcccl}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^2 2! 2^0 0! 3^1 1! \ldots 5^0 0!} & = & \frac{1}{6}; \\
\prod_k x_{2k+1}^{k j_{2k+1}} & = & x_3^1 x_5^0 \ldots x_{11}^0 & = & x_3; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & x_1^1 \left(x_2 x_4^1\right)^0 x_2^0 \ldots \left(x_5 x_{10}^4\right)^0 x_5^0 & = & x_1; \\
\prod_{r<t} x_{[r,t]}^{(r,t) j_r j_t} & = & x_2^0 x_3^2 \ldots x_{20}^0 & = & x_3^2.
\end{array}$$

IV: $(1,0,0,1,0)$, we have that

$$\begin{array}{lcccl}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^1 1! 2^0 0! 3^0 0! 4^1 1! 5^0 0!} & = & \frac{1}{4}; \\
\prod_k x_{2k+1}^{k j_{2k+1}} & = & x_3^0 x_5^0 \ldots x_{11}^0 & = & 1; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & \left(x_1 x_2^0\right)^0 x_1^0 \left(x_2 x_4^1\right)^1 x_2^0 \ldots & = & x_2 x_4; \\
\prod_{r<t} x_{[r,t]}^{(r,t) j_r j_t} & = & x_2^0 x_3^0 x_4^1 x_5^0 \ldots x_{20}^0 & = & x_4.
\end{array}$$

V: $(1,2,0,0,0)$, we have that

$$\begin{array}{lcccl}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^1 1! 2^2 2! 3^0 0! 4^0 0! 5^0 0!} & = & \frac{1}{8}; \\
\prod_k x_{2k+1}^{k j_{2k+1}} & = & x_3^0 x_5^0 \ldots x_{11}^0 & = & 1; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & \left(x_1 x_2^0\right)^2 x_1^0 \left(x_2 x_4^1\right)^0 x_2^{2\cdot\binom{2}{2}} \ldots & = & x_1^2 x_2^2; \\
\prod_{r<t} x_{[r,t]}^{(r,t) j_r j_t} & = & x_2^{1\cdot 2} x_3^0 \ldots x_{20}^0 & = & x_2^2.
\end{array}$$

VI: $(0,1,1,0,0)$, we have that

$$\begin{array}{lcccl}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^0 0! 2^1 1! 3^1 1! 4^0 0! 5^0 0!} & = & \frac{1}{6}; \\
\prod_k x_{2k+1}^{k j_{2k+1}} & = & x_3^1 x_5^0 \ldots x_{11}^0 & = & x_3; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & \left(x_1 x_2^0\right)^1 x_1^0 \left(x_2 x_4^1\right)^0 x_2^0 \ldots & = & x_1; \\
\prod_{r<t} x_{[r,t]}^{(r,t) j_r j_t} & = & x_2^0 x_3^0 \ldots x_6^1 \ldots x_{20}^0 & = & x_6.
\end{array}$$

VII: $(0,0,0,0,1)$, we have that

$$\begin{array}{lcccl}
\frac{1}{\prod k^{j_k} j_k!} & = & \frac{1}{1^0 0! \ldots 5^1 1!} & = & \frac{1}{5}; \\
\prod_k x_{2k+1}^{k j_{2k+1}} & = & x_3^0 x_5^2 \ldots x_{11}^0 & = & x_5^2; \\
\prod_k \left(x_k x_{2k}^{k-1}\right)^{j_{2k}} x_k^{k\binom{j_k}{2}} & = & \left(x_1 x_2^0\right)^0 x_1^0 \ldots \left(x_5 x_{10}^4\right)^0 x_5^0 & = & 1; \\
\prod_{r<t} x_{[r,t]}^{(r,t) j_r j_t} & = & x_2^0 x_3^0 \ldots x_{20}^0 & = & 1.
\end{array}$$

Multiplying the terms in each of (I) to (VII) respectively, and summing up all of the resulting products, so provides us with

$$\frac{x_1^{10}}{120} + \frac{x_1^4 x_2^3}{12} + \frac{x_1 x_3^3}{6} + \frac{x_2 x_4^2}{4} + \frac{x_1^2 x_2^4}{8} + \frac{x_1 x_3 x_6}{6} + \frac{x_5^2}{5},$$

so that

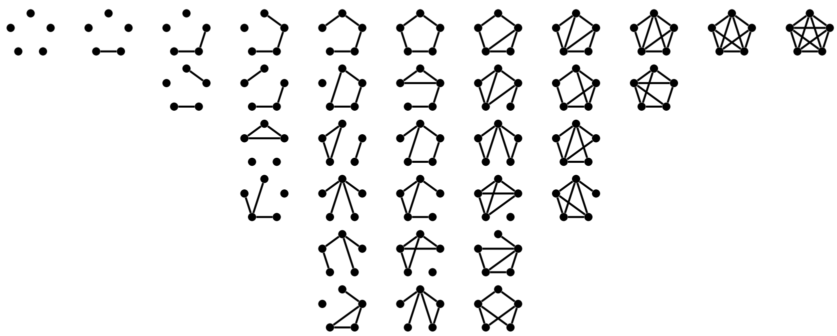
$$\zeta_{\mathfrak{S}_v^{(2)}} = \frac{1}{120}\left(x_1^{10} + 10x_1^4x_2^3 + 20x_1x_3^3 + 30x_2x_4^2 + 15x_1^2x_2^4 + 20x_1x_3x_6 + 24x_5^2\right).$$

Applying Pólya's Enumeration Theorem — viz. substituting $1 + x^k$ in a x_k-representative, and cleaning up the result — yields the polynomial

$$x^{10} + x^9 + 2x^8 + 4x^7 + 6x^6 + 6x^5 + 6x^4 + 4x^3 + 2x^2 + x + 1,$$

which is the pattern inventory of $(5, e)$-graphs. In figure 10.4 we draw each isomorphism class. ◇

Figure 10.4: Isomorphism classes of graphs with 5 *vertices.*



10.2 MULTIGRAPHS Generalizing on the idea of 2-colorings of the complete graph K_v is a natural next step which provides a method of classifying non-isomorphic undirected multigraphs on a set of v vertices. Instead of the colors "absent", and "present", we introduce the set $Y = \{0, 1, 2, \dots, m\}$ of multiplicities of an edge $e \in V^{(2)}$, with weights $w(i) = w_i$. The weight of a mapping $f \in Y^{V^{(2)}}$, i.e. for $e = \{v_1, v_2\} \in V^{(2)}$, simply states how many edges there are between v_1, and v_2:

$$w\left(f(\{v_1, v_2\})\right) = w(k) = w_k,$$

indicating that there are k edges between v_1, and v_2.

10.2.1 EXAMPLE. Let $V^{(2)} = \{\{1,2\}, \{1,3\}, \{2,3\}\}$ be as in example 10.1.3, and let $Y = \{0, 1, 2\}$ be the set of edge multiplicities. We saw that

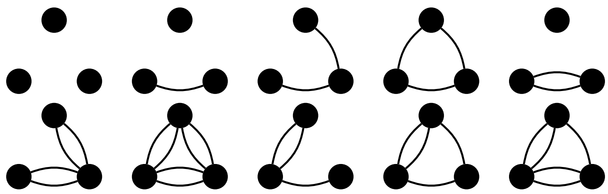
$$\zeta_{\mathfrak{S}_3^{(2)}}(x_1, x_2, x_3) = \frac{1}{6}\left(x_1^3 + 2x_3 + 3x_1x_2\right).$$

By applying Pólyas Enumeration Theorem we get the polynomial

$$w_0^3 + w_0^2w_1 + w_0^2w_2 + w_0w_1^2 + w_0w_1w_2 + w_0w_2^2 + w_1^3 + w_1^2w_2 + w_1w_2^2 + w_2^3,$$

which generates all possible non-isomorphic multigraphs with three vertices and a maximum edge multiplicity of 2. Note that by letting $w_0 = w_1 = 1$, and $w_2 = 0$, we get the number of non-isomorphic simple graphs of three vertices (figure 10.2), which is 4. ◇

Figure 10.5: Non-isomorphic multigraphs with three vertices and a maximum edge multiplicity of 2.



10.2.2 EXAMPLE. Let $V = \{1, 2, 3, 4, 5\}$, and consider multigraphs of 5 vertices. We still deal with a maximum edge multiplicity of 2. It was seen in example 10.1.6 that

$$\zeta_{\mathfrak{S}_5^{(2)}} = \frac{1}{120}\left(x_1^{10} + 10x_1^4x_2^3 + 20x_1x_3^3 + 30x_2x_4^2 + 15x_1^2x_2^4 + 20x_1x_3x_6 + 24x_5^2\right).$$

As in example 10.2.1 we apply Pólyas Enumeration Theorem, through substituting $w_0^i + w_1^i + w_2^i$ into a x_i-representative, by which we obtain

$$\begin{aligned}
&w_0^{10} + w_0^9w_1 + 2w_0^8w_1^2 + 4w_0^7w_1^3 + 6w_0^6w_1^4 + 6w_0^5w_1^5 + 6w_0^4w_1^6 + 4w_0^3w_1^7 + \\
&2w_0^2w_1^8 + w_0w_1^9 + w_1^{10} + w_0^9w_2 + 2w_0^8w_1w_2 + 6w_0^7w_1^2w_2 + 12w_0^6w_1^3w_2 + \\
&16w_0^5w_1^4w_2 + 16w_0^4w_1^5w_2 + 12w_0^3w_1^6w_2 + 6w_0^2w_1^7w_2 + 2w_0w_1^8w_2 + w_1^9w_2 + \\
&2w_0^8w_2^2 + 6w_0^7w_1w_2^2 + 17w_0^6w_1^2w_2^2 + 30w_0^5w_1^3w_2^2 + 37w_0^4w_1^4w_2^2 + 30w_0^3w_1^5w_2^2 + \\
&17w_0^2w_1^6w_2^2 + 6w_0w_1^7w_2^2 + 2w_1^8w_2^2 + 4w_0^7w_2^3 + 12w_0^6w_1w_2^3 + 30w_0^5w_1^2w_2^3 + \\
&47w_0^4w_1^3w_2^3 + 47w_0^3w_1^4w_2^3 + 30w_0^2w_1^5w_2^3 + 12w_0w_1^6w_2^3 + 4w_1^7w_2^3 + 6w_0^6w_2^4 + \\
&16w_0^5w_1w_2^4 + 37w_0^4w_1^2w_2^4 + 47w_0^3w_1^3w_2^4 + 37w_0^2w_1^4w_2^4 + 16w_0w_1^5w_2^4 + \\
&6w_1^6w_2^4 + 6w_0^5w_2^5 + 16w_0^4w_1w_2^5 + 30w_0^3w_1^2w_2^5 + 30w_0^2w_1^3w_2^5 + 16w_0w_1^4w_2^5 + \\
&6w_1^5w_2^5 + 6w_0^4w_2^6 + 12w_0^3w_1w_2^6 + 17w_0^2w_1^2w_2^6 + 12w_0w_1^3w_2^6 + 6w_1^4w_2^6 + \\
&4w_0^3w_2^7 + 6w_0^2w_1w_2^7 + 6w_0w_1^2w_2^7 + 4w_1^3w_2^7 + 2w_0^2w_2^8 + 2w_0w_1w_2^8 + 2w_1^2w_2^8 + \\
&w_0w_2^9 + w_1w_2^9 + w_2^{10}.
\end{aligned}$$

It is an unwieldy expression, but it provides plenty of information. If we let $w_0 = w_1 = 1$, and $w_2 = 0$, it sums up to 34 — the number of non-isomorphic simple graphs presented in figure 10.4. If we let $w_0 = w_1 = w_2 = 1$ it sums up to the number of non-isomorphic multigraphs of five unlabelled vertices — there are 792 such graphs. Moreover, letting $w_0 = 0$, $w_1 = 1$, and $w_2 = 2$ counts the total number — 3275 — of edges which are present in the list of all non-isomorphic multigraphs of five unlabelled vertices. ◇

11

Chemical Enumeration

Pólya's original article dealt in ways of enumerating chemical isomers. Such an example will be presented in the end of this chapter, and it will be the very same one as given in [8].

11.1 Direct Products

We begin our discussion with some ideas related to group actions.

11.1.1 Example. Let $X = \{x_1, x_2, \ldots, x_n\}$, and $Y = \{y_1, y_2, \ldots, y_m\}$ be two finite, and disjoint sets. We take $\mathfrak{G}$ to be a group of permutations of X, and we take $\mathfrak{H}$ to be a group of permutations of Y. We have that

$$\zeta_{\mathfrak{G}} = \frac{1}{|\mathfrak{G}|} \sum_{(i)} g_{i_1 i_2 \ldots i_n} x_1^{i_1} x_2^{i_2} \ldots x_n^{i_n};$$

$$\zeta_{\mathfrak{H}} = \frac{1}{|\mathfrak{G}|} \sum_{(j)} h_{i_1 i_2 \ldots i_m} x_1^{j_1} x_2^{j_2} \ldots x_n^{j_m}$$

are the cycle indices for $\mathfrak{G}$, and $\mathfrak{H}$, where $g_{i_1 i_2 \ldots i_n}$ (resp. $h_{i_1 i_2 \ldots i_n}$) are the number of permutations in $\mathfrak{G}$ (resp. $\mathfrak{H}$) of type $\left[1^{i_1}, 2^{i_2}, \ldots, n^{i_n}\right]$ $\left(\text{resp. } \left[1^{j_1}, 2^{j_2}, \ldots, m^{j_m}\right]\right)$. Summation is taken over all partitions (i), and (j), of n and m respectively. Let $U = X \cup Y$, then to each choice of $g \in \mathfrak{G}$ and $h \in \mathfrak{H}$ there corresponds a new permutation group — of U — which we define by

$$x \longmapsto gx, \text{ if } x \in X, \text{ and } y \longmapsto hy \text{ if } y \in Y.$$

We denote this permutation group by $\mathfrak{G} \times \mathfrak{H}$ — the direct product of $\mathfrak{G}$ and $\mathfrak{H}$ (cf. example 3.4.1). There are $|\mathfrak{G}|\,|\mathfrak{H}|$ permutations of the $n + m$ objects in U, and in Cauchy's notation each pair $g \times h$ correspond to

$$\begin{pmatrix} x_1 & x_2 & \ldots & x_n & y_1 & y_2 & \ldots & y_m \\ x_{1'} & x_{2'} & \ldots & x_{n'} & y_{1'} & y_{2'} & \ldots & y_{m'} \end{pmatrix}.$$

If $g \in \mathfrak{G}$ has type $\left[1^{i_1}, 2^{i_2}, \ldots n^{i_n}\right]$, and $h \in \mathfrak{H}$ has type $\left[1^{j_1}, 2^{j_2}, \ldots n^{j_m}\right]$, then the type of $g \times h$ is $\left[1^{i_1+j_i}, 2^{i_2+j_2}, \ldots n^{i_n+j_n}\right]$ — since each cycle in U lies either entirely in X, or entirely in Y. We therefore have that the cycle structure monomial $\zeta_{g\times h}$ — the term in the cycle index $\zeta_{\mathfrak{G}\times\mathfrak{H}}$ corresponding to the element $g \times h$ — must be equal to the product of the term in $\zeta_{\mathfrak{G}}$ corresponding to g, and the term in $\zeta_{\mathfrak{H}}$ corresponding to h, viz. $\zeta_{g\times h} = \zeta_g \zeta_h$. This applies to all terms of $\zeta_{\mathfrak{G}}$, and all terms of $\zeta_{\mathfrak{H}}$. Hence

$$\zeta_{\mathfrak{G}\times\mathfrak{H}} = \zeta_{\mathfrak{G}}\zeta_{\mathfrak{H}}. \qquad \diamond$$

11.1.2 Example. The preceeding example can be illustrated by an enumeration problem which concerns a traingle and a square. The groups of rigid motions $\mathfrak{D}_4$ and $\mathfrak{D}_3$ permutes the vertices $V_\square = \{x_1, x_2, x_3, x_4\}$ of $\square$, and the vertices $V_\triangle = \{y_1, y_2, y_3\}$ of $\triangle$ respectively. We have that

$$\zeta_{\mathfrak{D}_3} = \frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right);$$

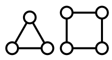
$$\zeta_{\mathfrak{D}_3} = \frac{1}{8}\left(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4\right).$$

The direct product $\mathfrak{D}_3 \times \mathfrak{D}_4$ is a permutation group of $V_\triangle \cup V_\square$, and so

$$\zeta_{\mathfrak{D}_3\times\mathfrak{D}_4} = \frac{1}{48}\left(x_1^3 + 3x_1x_2 + 2x_3\right)\left(x_1^4 + 2x_1^2x_2 + 3x_2^2 + 2x_4\right).$$

We examine the colorings of the triangle together with the square, i.e. we wish to color them both at the same time. The given colors are $C = \{\text{red}, \text{blue}\}$ with the respective weights $w(\text{red}) = r$, and $w(\text{blue}) = b$. We imagine the triangle and the square alongside eachother.

Figure 11.1



The number of color configurations is $2^7 = 128$, but not all of them are distinct. By a suitable transformation of $\triangle$, or $\square$ (inclusive disjunction), one configuration can be obtained from another. We seek the patterns of $C^{V_\triangle \cup V_\square}$, viz. we wish to determine the pattern inventory of

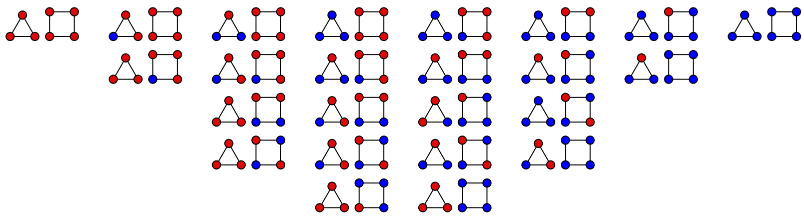
$$C^{V_\triangle \cup V_\square} \big/ \mathfrak{G} \times \mathfrak{H}.$$

Applying PET — substituting $r^k + b^k$ in a x_k-representative, and cleaning up the result — yields the polynomial

$$r^7 + 2r^6b + 4r^5b^2 + 5r^4b^3 + 5r^3b^4 + 4r^2b^5 + 2rb^6 + b^7.$$

We interpret this result in figure 11.2. $\diamond$

Figure 11.2



11.2 The Kranz Group Let $X = \{x_1, x_2, \ldots, x_n\}$, and $Y = \{y_1, y_2, \ldots, y_m\}$ be two disjoint and finite sets as before. We take $\mathfrak{G}$ to be a (finite) group of permutations of X, and we take $\mathfrak{H}$ to be a (finite) group of permutations of Y. Consider the cartesian product $X \times Y = \{(x, y) : x \in X, \text{ and } y \in Y\}$. We can construct a group of permutations of $X \times Y$ in which the group elements are defined in the following way:

> Choose an element in $g \in \mathfrak{G}$, and to each $x \in X$ choose an element $h_x \in \mathfrak{H}$. These elements determine a permutation of $X \times Y$ by
>
> $$(x, y) \longmapsto (gx, h_x y), \text{ where } x \in X, y \in Y.$$

There are $|\mathfrak{G}|\,|\mathfrak{H}|^n$ different choices of (g, h_x) — permutations of $\mathfrak{G} \times \mathfrak{H}$ — which together form a group called *the corona of* $\mathfrak{G}$ *with respect to* $\mathfrak{H}$, or the *Kranz* $\mathfrak{G}[\mathfrak{H}]$.

11.2.1 Theorem. *The cycle index of* $\mathfrak{G}[\mathfrak{H}]$ *is obtained by substituting*

$$y_k = \zeta_{\mathfrak{H}}(x_k, x_{2k}, x_{3k}, \ldots) \text{ into } \zeta_{\mathfrak{G}}(y_1, y_2, y_3, \ldots).$$

Viz.

$$\zeta_{\mathfrak{G}[\mathfrak{H}]}(x_1, x_2, \ldots) = \zeta_{\mathfrak{G}}\{\zeta_{\mathfrak{H}}(x_1, x_2, \ldots), \zeta_{\mathfrak{H}}(x_2, x_4, \ldots), \ldots\}.$$

Proof. See [4]. ■

11.2.1 Example. Lets consider three cubes. We want to color each one of them using $C = \{\text{red}, \text{blue}\}$. The set of cubes can be permuted, while each separate cube can also be rotated, and we wish to find the number of non-equivalent colorings under permutations and rotations. The group under consideration is $\mathfrak{S}_3[\mathfrak{G}]$, where $\mathfrak{G}$ is the group of permutations of the faces of a cube, and S_3 is the symmetric group of degree 3. We have that

$$\zeta_{\mathfrak{S}_3}(x_1, x_2, x_3) = \frac{1}{6}\left(x_1^3 + 3x_1x_2 + 2x_3\right);$$
$$\zeta_{\mathfrak{G}}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{24}\left(x_1^6 + 8x_3^2 + 6x_1^2x_4 + 3x_1^2x_2^2 + 6x_2^3\right).$$

Applying Theorem 11.2.1 we get that $\zeta_{\mathfrak{S}_3[\mathfrak{G}]}(x_1, x_2, \ldots)$ equals

$$\frac{1}{6}\left\{[\zeta_{\mathfrak{G}}(x_1, x_2, \ldots)]^3 + 3\zeta_{\mathfrak{G}}(x_1, x_2, \ldots)\zeta_{\mathfrak{G}}(x_2, x_4, \ldots) + 2\zeta_{\mathfrak{G}}(x_3, x_6, \ldots)\right\}. \tag{11.1}$$

It would be feasible to expand (11.1), but this would yield a cycle index too unwieldy. Instead we choose to give weights $w(\text{red}) = w(\text{blue}) = 1$ so that, when applying PET, each $\zeta_{\mathfrak{G}} = 10$. Thus

$$\zeta_{\mathfrak{S}_3[\mathfrak{G}]}(2, 2, \ldots) = \frac{1}{6}\left(10^3 + 3 \cdot 10 \cdot 10 + 2 \cdot 10\right) = 220.$$

We can therefore conclude that there are 220 non-equivalent colorings of the cubes. ◇

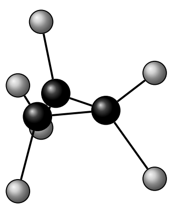
Table 11.1: Characteristics of $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{G} \times \mathfrak{H}$, and $\mathfrak{G}[\mathfrak{H}]$.

Group	$\mathfrak{G}$	$\mathfrak{H}$	$\mathfrak{G} \times \mathfrak{H}$	$\mathfrak{G}[\mathfrak{H}]$
Degree	n	m	$n + m$	nm
Order	$\lvert\mathfrak{G}\rvert$	$\lvert\mathfrak{H}\rvert$	$\lvert\mathfrak{G}\rvert\,\lvert\mathfrak{H}\rvert$	$\lvert\mathfrak{G}\rvert\,\lvert\mathfrak{H}\rvert^n$
Cycle index	$\zeta_{\mathfrak{G}}$	$\zeta_{\mathfrak{H}}$	$\zeta_{\mathfrak{G}}\zeta_{\mathfrak{H}}$	$\zeta_{\mathfrak{G}}\{\zeta_{\mathfrak{H}}(x_1, x_2, \ldots), \zeta_{\mathfrak{H}}(x_2, x_4, \ldots), \ldots\}$

11.3 Cyclopropane

A C-H graph represents a molecule formed by atoms of valences 1 and 4. An atom of valence 4 is identified with a vertex C, a carbon atom (a black ball in figure 11.3), and an atom of valence 1 is identified with H (hydrogen, a gray ball). We shall follow the discussion as presented by Pólya in [8], sections 56 – 57, to look at derivatives of cyclopropane and their isomers. The graph of cyclopropane consists of three carbon atoms and six hydrogen atoms. The three carbon atoms, of valence 4, are joined in a cyclic arrangement, and the hydrogen atoms are joined pairwise to each carbon atom. Our first concern is to determine in what ways the graph of cyclopropane can be mapped into itself. Three interpretations 1^{st}, 2^{nd}, and 3^{rd} (as given in [8]) must be considered.

Figure 11.3: Structure of a C_3H_6-cyclopropane molecule.



1^{st}: We can identify a group of rigid motions which leaves a right prism with an equilateral triangular base invariant under spatial rotations. The 6 endpoints in figure 11.3 are the six vertices of the prism. These points are subjected to a permutation group which Pólya called *the group of the stereoformula*. We denote this group by $\mathfrak{G}$, and we have that

$$\zeta_{\mathfrak{G}}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{6}\left(x_1^6 + 3x_2^3 + 2x_3^2\right);$$

2^{nd}: The triangle with vertices corresponding to carbon atoms (of valence 4) can be mapped into itself in 6 ways ($\mathfrak{D}_3$). There are three pairs, consisting of two endpoints connected to a vertex of valence 4. The vertices in each pair can be interchanged. Hence for each mapping of the triangle, the remaing vertices can be permuted in $2^3 = 8$ ways. We therefore obtain $6 \cdot 8 = 48$ topologically congruent selfmaps, as Pólya phrased it. These selfmaps constitute a group, which turns out to be the kranz $\mathfrak{S}_3[\mathfrak{S}_2]$, or — as Pólya called it — *the group of the structural formula*. By Theorem 11.2.1 we have that $\zeta_{\mathfrak{S}_3[\mathfrak{S}_2]}(x_1, x_2, x_3, x_4, x_5, x_6)$ equals

$$\frac{1}{48}(x_1^6 + 3x_1^4 x_2 + 9x_1^2 x_2^2 + 6x_1^2 x_4 + 7x_2^3 + 6x_2 x_4 + 8x_3^2 + 8x_6);$$

3rd: We look at the prism as described in the 1st case, this time as subjected also to reflections in addition to spatial rotations. Here, the six vertices of the prism are subjected to a permutation group of order 12 — *the extended group of the stereoformula*, as phrased by Pólya. We denote this group by $\mathfrak{E}$, and we have that

$$\zeta_{\mathfrak{E}}(x_1, x_2, x_3, x_4, x_5, x_6) = \frac{1}{12}\left(x_1^6 + 4x_2^3 + 2x_3^2 + 3x_1^2 x_2^2 + 2x_6\right).$$

A derivative of cyclopropane is acquired when hydrogen atoms are replaced by so-called monovalent radicals — a monovalent radical being a monovalent atom, or a molecule with a free bond. We imagine the six radicals at the endpoints of the cyclopropane graph in figure 11.3. They form a configuration, and each configuration provides a chemical formula for a cyclopropane derivative. If two configurations can be transformed into eachother — i.e. if they are equivalent with respect to the associated group — then they represent the same derivative. In the case of stereoisomers the relevant group is $\mathfrak{G}$, the group of the stereochemical formula. The group associated with structural isomers is $\mathfrak{S}_3[\mathfrak{S}_2]$, the group of the structural formula. In the extended group of the stereochemical formula, $\mathfrak{E}$, a derivative is equivalent to its enantiomer — its mirror image (an optical isomer) — in addition to the molecules to which it is equivalent under spatial rotations.

We seek to determine the number of isomeric substitutes of cyclopropane of the form

$$\mathrm{C}_3 X_k Y_l Z_m,$$

where $k + l + m = 6$, and X, Y, Z are so-called independent radicals. Independent means that if $X_k Y_l Z_m$ and $X_{k'}, Y_{l'}, Z_{m'}$ have the same molecular structure then $k = k'$, $l = l'$, and $m = m'$. For example the radicals H, CH_3, C_2H_5 are not independent of eachother — the radicals H_5 and C_2H_5 attached to C_3 have the same molecular structure as if we were to attach H_4 and $(CH_3)_2$ to C_3.

Assign to X, Y, Z the weights $w(X) = x$, $w(Y) = y$, and $w(Z) = z$. Applying Theorem 9.3.4 to $\zeta_{\mathfrak{G}}$, $\zeta_{\mathfrak{S}_3\mathfrak{S}_2}$, and $\zeta_{\mathfrak{E}}$ provides a pattern inventory with respect to each group.

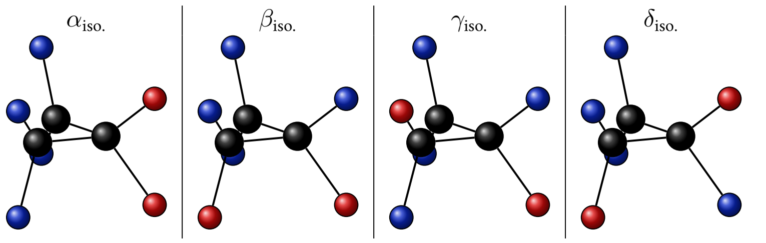
I: $\zeta_{\mathfrak{G}}(x + y + z, \ldots) = x^6 + x^5 y + x^5 z + \underline{4x^4 y^2} + 5x^4 yz + 4x^4 z^2 + 4x^3 y^3 + 10x^3 y^2 z + 10x^3 yz^2 + 4x^3 z^3 + 4x^2 y^4 + 10x^2 y^3 z + 18x^2 y^2 z^2 + 10x^2 yz^3 + 4x^2 z^4 + xy^5 + 5xy^4 z + 10xy^3 z^2 + 10xy^2 z^3 + 5xyz^4 + xz^5 + y^6 + y^5 z + 4y^4 z^2 + 4y^3 z^3 + 4y^2 z^4 + yz^5 + z^6$;

II: $$\zeta_{\mathfrak{S}_3\mathfrak{S}_2}(x+y+z,\ldots) = x^6 + x^5y + x^5z + \underline{2x^4y^2} + 2x^4yz + 2x^4z^2 + 2x^3y^3 + 3x^3y^2z + 3x^3yz^2 + 2x^3z^3 + 2x^2y^4 + 3x^2y^3z + 5x^2y^2z^2 + 3x^2yz^3 + 2x^2z^4 + xy^5 + 2xy^4z + 3xy^3z^2 + 3xy^2z^3 + 2xyz^4 + xz^5 + y^6 + y^5z + 2y^4z^2 + 2y^3z^3 + 2y^2z^4 + yz^5 + z^6;$$

III: $$\zeta_{\mathfrak{E}}(x+y+z,\ldots) = x^6 + x^5y + x^5z + \underline{3x^4y^2} + 3x^4yz + 3x^4z^2 + 3x^3y^3 + 6x^3y^2z + 6x^3yz^2 + 3x^3z^3 + 3x^2y^4 + 6x^2y^3z + 11x^2y^2z^2 + 6x^2yz^3 + 3x^2z^4 + xy^5 + 3xy^4z + 6xy^3z^2 + 6xy^2z^3 + 3xyz^4 + xz^5 + y^6 + y^5z + 3y^4z^2 + 3y^3z^3 + 3y^2z^4 + yz^5 + z^6.$$

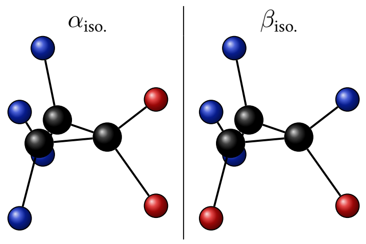
We interpret the result in the I^{st} expression by reading the coefficient before x^4y^2. Here, there are four non-equivalent stereoisomers of the cyclopropane derivative of the form $C_3X_4Y_2$. Two of these are enantiomers, which the coefficient before x^4y^2 in the III^{rd} expression indicates. In the III^{rd} expression the coefficient is 3, since optical isomers are equivalent — they are in the same orbit, when subjected to the extended stereochemical group. In the II^{nd} expression the coefficient before x^4y^2 is 2 — indicating that spatial arrangements are disregarded alltogether — i.e. there's only two non-equivalent structural isomers of the $C_3X_4Y_2$ cyclopropane derivative.

Figure 11.4: The four non-equivalent stereoisomers of the $C_3X_4Y_2$ cyclopropane derivative, when subjected to $\mathfrak{G}$ — the group of the stereoformula. (X = blue ball, and Y = red ball.)



The two isomers $\gamma_{iso.}$ and $\delta_{iso.}$ are enantiomers — mirror isomers — and therefore equivalent with respect to $\mathfrak{E}$, the extended group of the stereoformula.

Figure 11.5: The two non-equivalent structural isomers of the $C_3X_4Y_2$ cyclopropane derivative, when subjected to $\mathfrak{S}_3[\mathfrak{S}_2]$ — the group of the structural formula.



Bibliography

[1] Beachy, J. A., and Blair, W. D. *Abstract Algebra*. Waveland Press, 2006.

[2] Biggs, N. L. *Discrete Mathematics, 2/e*. Oxford University Press, 2002.

[3] Burnside, W. *Theory of Groups of Finite Order*. Cambridge Univerity Press, 1911.

[4] De Bruijn, N. G. *Polya's Theory of Counting. In E. F. Beckenbach (Ed.), Applied Combinatorical Mathematics*. John Wiley & Sons, 1965.

[5] Dummit, D. S., and Foote, R. M. *Abstract Algebra*. Wiley Hoboken, 2004.

[6] Grimaldi, R. P. *Discrete and Combinatorial Mathematics, 5/e*. Addison Wesley, 2006.

[7] Harary, F., and Palmer, E. M. *Graphical Enumeration*. Academic Press, 1973.

[8] Polya, G., and Read, R. C. *Combinatorial Enumeration of Groups, Graphs, and Chemical Compounds*. Springer-Verlag, 1987.