

Semisimple Lie algebras and the Cartan decomposition

Martin Nilsson

June 6, 2018

Abstract

Consider a set of matrices that is closed under both linear combinations and the "commutator" $AB - BA$ of any pair of matrices A, B of the set. This is what is known as a linear Lie algebra; these generalize to abstract Lie algebras, which possess a commutator-like operation but need not consist of matrices. We begin with a brief discussion on how Lie algebras arise, followed by an investigation of some basic properties of Lie algebras and what can be said in the general case. We then turn to semisimple Lie algebras—those that can be built up from "simple" ones—and study in depth their representations, or ways to inject them into linear Lie algebras in a structure-preserving fashion. After deriving a sufficient breadth of results, we then proceed with exploiting a certain representation and its properties in order to deconstruct any given semisimple algebra into its so-called Cartan decomposition. Finally, we show how any such decomposition can be understood in terms of its "root system", an associated geometric object embedded in some Euclidean space.