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Implications of Prosocial preferences in game theory

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Abstract

Games are a well-defined mathematical object for modeling dynamic decision processes under competition between players. The normal form is a way of describing a game and predicting the outcomes, using a payoff matrix based on random strategies. This is an introduction to the game theory, that includes simple definitions and examples of basic kinds of games and their solution concepts. Furthermore, we introduce the utility function and underline the effect of prosocial preferences that players may have for possible outcomes.

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Contents

1 Introduction

Game theory is a mathematical study of interaction and strategy. It enables universal mathematical techniques for analyzing games, which are strategic interactions, where individuals make decisions that will influence others' interests. The outcome is, in most interactions, not only determined by one single actor, but on external factors and all actors' behaviors, strategies, beliefs and expectations. Although game theory can be used to analyze parlour games, its applications are much broader and most research focuses on how groups of people interact. It involves conflicts of interest arising in fields as economics, sociology and political science, as well as in biology and computer science.

1.1 History and impact

The mathematical theory of games was invented by John von Neumann and Oscar Morgenstern and laid out in their book "The theory of Games and Economic Behavior", in 1944. This foundational work contains a method for finding mutually consistent solutions for two-person zero-sum games, which are games that involve only two players and where one player's gain is equivalent to the other player's loss. This work, which claims that any economic situation can be understood as a game between two competing actors, really launched the field of game theory. The precursors to this work was a sequence of papers by Emile Borel in the 1920s and von Neumann's paper on the maxmin theorem for zero-sum games. [1]

Game theory emerged from the analysis in competitive games in the fields of mathematics and economics. During World War II and the beginnings of the Cold War, it was mobilized to analyze politics and warfare. Game theory began essentially as an exercise to locate enemy submarines. [2]

The next major development occurred in the 1950s. The mathematician John Nash developed a definition of equilibrium, that has become the central solution for non-cooperative[1] game theory. If each player has chosen a strategy and no player can benefit by changing strategies while the other players strategies are unchanged, then the current set of strategy choices and the corresponding payoffs constitutes what would become known as Nash

[1]Games that study conflicts among players and focuses on which moves players should rationally make. The outcome of the game will depend on the acts of all players.

equilibrium. This powerful solution concept has reshaped the landscape of research in economics, as it has been used to reinvent the study of fields such as market competition around oligopoly theory and the auction theory, as well as strategic interactions. Today's understanding of consumer search, limit pricing, entry deterrence, strategic advertising etc. are all predicated on models that rely mostly on Nash equilibrium as the solution concept.

In 1967-1968, John Harsanyi constructed the theory of games of incomplete information[2] and showed that the Nash equilibrium could be generalized to this type of games. Reinhard Selten further introduced his solution concept of subgame perfect equilibria, in an article in 1965. This further refined the Nash equilibrium. In his article in 1975, Selten viewed rationality as the limit of bounded rationality. [1]

In the 1970s, game theory was extensively applied in biology. The concept of an Evolutionarily Stable Strategy was introduced to evolutionary game theory, by John Maynard Smith. A necessary condition for behaviors to be evolutionarily stable is that they constitute Nash equilibrium, why there are close ties between non-cooperative and evolutionary game theory.

Over the last forty years, advances in game theory were made in many areas of research, such as complete and incomplete information, repeated games, stochastic games, games with many players etc. It is a powerful tool that can be used for example in modeling of conflict among nations, competitions among firms and political campaigns. Despite that it has become a universally accepted paradigm in a wide diversity of areas, it is still a young and developing science.

[2]Strategic interactions where players do not know each others' preferences and/or strategy sets.

2 What are games?

The object of study in game theory is the game, which is a formal model of a strategic situation, with a set of rules that describes it. It typically involves several players. The formal definition lays out the players, their preferences, their information, the strategic actions available to them and how each decision influence the outcome. Depending on the model, various other requirements or assumptions may be necessary.

2.1 Normal-form games

The normal form game is a static model, that specifies a set of strategies for each player and the outcomes that result from each possible combination of choices. A strategy is a complete contingent plan of action and an outcome is represented by a separate payoff for each player, which is a real number that represents the player's preference rankings of the alternative plays of the game. The normal-form game specifies the pure strategies for each player. A strategy is called pure if it involves no randomization. A player may also randomize between his pure strategies. A strategy that involves some randomization, is called mixed. Mixed strategies will be further discussed in section 2.3. The normal forms are usually represented as bimatrices, in which the first player's pure strategies are listed as rows, the pure strategies of the second player as columns, and the entries are the associated payoff-pairs. [1]

Definition 1 *A normal-form game is a triplet $G = \langle I, S, \pi \rangle$, where $I = 1, 2, ..., n$ is the set of players, for some positive integer n. A vector $s = (s_1, s_2, ..., s_n) \in S$, where s_i is a pure strategy for player i, $s_i \in S_i$, is called a strategy profile. The Cartesian product, $S = \times_i S_i$, is the set S of strategy profiles in the game, with S_i denoting the strategy set of each player $i \in I$. For any strategy profile $s \in S$ and player $i \in I$, $\pi_i(s) \in \mathbb{R}$ is the associated payoff to player i when strategy profile s is played and $\pi : S \to \mathbb{R}^n$ is the payoff function.*

Suppose that the individual in player role $i \in I$ has preferences $\succeq$ over pure-strategy profiles $s \in S$. A utility function $u_i : S \to \mathbb{R}^n$ represents a preference relation $\succeq$ on S such that

$$u_i(s) \geq u_i(s') \quad \Leftrightarrow s \succeq_i s' \quad \forall s, s' \in S \tag{2.1}$$

if and only if $\succeq_i$ is complete and transitive.

The matrix below is a normal-form representation of a two-player game:

	a	b	c
A	4,3	1,1	0,0
B	0,0	3,4	1,2
A	1,3	0,2	3,0

(2.2)

In each cell, the first number represents the payoff to the row player and the second number represents the payoff to the column player. We will further denote the row player as player 1 and the column player as player 2. The players make their decisions simultaneously, where player 1 has a choice with strategies A, B and C and player 2 can choose between strategies a, b and c. If player 1 chooses A and player 2 chooses a, then player 1 receives payoff 4 and player 2 receives payoff 3.

Games can be divided into finite and infinite games. A finite normal-form game is a normal-form game $G = \langle I, S, \pi \rangle$ where S is finite. Then $I = 1, 2, ..., n$ is the finite set of players and for each player $i \in I$, $S_i = \{1, 2, ..., m_i\}$ is the player's finite set of strategies, for positive integers $m_1, ..., m_n$ and $S = \times_i S_i$, is the finite set S of strategy profiles. This thesis will focus on finite two-player games. We shall sometimes write a strategy profile $(s_1, ..., s_n)$ as (s_i, s_{-i}), where $-i$ denotes player role $j \neq i$.

2.2 The equilibrium concept

A fundamental concept in the theory of games is the equilibrium concept. Equilibrium is a condition in which all acting influences are balanced, resulting in a stable or unchanging system. The most widely used method of predicting the outcome of a strategic interaction is Nash equilibrium. A pure-strategy Nash equilibrium is an action profile with the property that no single player can obtain a higher payoff by deviating unilaterally from this profile. We shall further denote Nash equilibrium as NE. [3]

Sometimes, games may possess many equilibria and the problem which one of these should be chosen as solution arises, why predictions obtained may be incomplete. In order to obtain sharper predictions, the Nash equilibrium concept has been refined. One of the most well-known refinements, introduced in Selten (1975), is "trembling hand perfection" (THP). The idea behind THP is that your opponents might deviate from their intended strategy due to error (a "tremble") and you should prepare for that in choosing your strategy. THP takes into account this probability and protects the player if the opponent makes a mistake. [1]

Definition 2 *A strategy profile* $(s_i^*, s_{-i}^*) \in S$ *is a pure strategy Nash equilibrium if, and only if, for all* $i \in I$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ *for any* $s_i \in S_i$. [4]

Example 1 Consider a game involving two players. Player 1 has the strategy set $S_1 = \{s_1, s_2\}$ and player 2 has the strategy set $S_2 = \{t_1, t_2\}$. The payoff matrix for the different action profiles can be seen below:

$$\begin{array}{c|c|c|} & t_1 & t_2 \\ \hline s_1 & 1,2 & 0,0 \\ \hline s_2 & 0,0 & 1,1 \\ \hline \end{array} \tag{2.3}$$

To find the Nash equilibria, we examine each action profile in turn. By playing (s_1, t_1), neither player can increase his payoff by choosing a different action, and thus this action profile is a Nash equilibrium. If playing (s_1, t_2), player 1 could obtain a payoff of 1 rather than 0, by choosing s_2 rather than s_1, given player 2's action. Also, player 2 can increase his payoff by choosing t_1 rather than t_2. Thus this action profile is not a Nash equilibrium. If playing (s_2, t_1), player 1 could obtain a payoff of 1 rather than 0, by choosing s_1 rather than s_2, given player 2's action. Also, player 2 can increase his payoff by choosing t_2 rather than t_1. Thus this action profile is not a Nash equilibrium. If playing (s_2, t_2), neither player can increase his payoff by choosing a different action, and thus this action profile is a Nash equilibrium. We conclude that this game has two pure Nash equilibria, (s_1, t_1) and (s_2, t_2).

Example 2 Consider the following two-player game:

$$\begin{array}{c|c|c|} & t_1 & t_2 \\ \hline s_1 & 2,1 & 1,2 \\ \hline s_2 & 1,2 & 2,1 \\ \hline \end{array} \tag{2.4}$$

Since one player can always increase his payoff by choosing a different action, we conclude that this game has no pure Nash equilibrium.

2.3 Mixed strategies

Sometimes, instead of simply choosing an action, players may be able to choose probability distributions over the set of available pure strategies. Such randomizations over the set of actions are called mixed strategies.

A mixed strategy x_i for player i is a vector x_i in $\mathbb{R}^{m_i}$ composed of the probabilities associated with available actions, its h:th coordinate $x_{ih} \in [0, 1]$ being the probability assigned by x_i to the player's h:th pure strategy. A mixed-strategy profile is a vector $x = (x_1, ..., x_n) \in \square(S) = \times_{i \in I} \Delta(S_i)$ of mixed strategies, one for each player. For each player i, the mixed-strategy set in a finite game is

$$X_i = \Delta(S_i) = \left\{ x_i \in \mathbb{R}_+^{m_i} : \sum_{h \in S_i} x_{ih} = 1 \right\}, \tag{2.5}$$

the unit simplex in $\mathbb{R}^{m_i}$. The mixed-strategy set X_i of player i is a non-empty, compact and convex Euclidean subspace $\Delta(S_i) \subset \mathbb{R}^{mi}$ with dimension $m_i - 1$. The vertices of $\Delta(S_i)$ are the unit vectors $1_i^1 = (1, 0, 0..., 0)$, $1_i^2 = (0, 1, 0..., 0)$, ..., $1_i^m = (0, 0, 0..., 1)$. Each such vertex 1_i^h represents the mixed strategy for player i that assigns probability one to his h:th pure strategy, so the pure strategies, in turn, are special cases of mixed strategies, namely the unit vectors in $\mathbb{R}^{m_i}$. [1]

Every mixed strategy $x_i \in \Delta(S_i)$ is a linear combination of the pure strategies, 1_i^h:

$$x_i = \sum_{h=1}^{m_i} x_{ih} \cdot 1_i^h = (x_{i1}, x_{i2}, ..., x_{im_i}). \tag{2.6}$$

This linear combination is a convex combination since the coefficients sum to one and are non-negative. Hence the mixed-strategy simplex $\Delta(S_i)$ is the convex hull of its vertices.

Example 3 Let's assume that player i has three pure strategies: $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. The mixed strategy x_1 is then a linear combination of the pure strategies; $\mathrm{x}_1 = \mathrm{x}_{11} \cdot (1, 0, 0) + x_{12} \cdot (0, 1, 0) + x_{13} \cdot (0, 0, 1)$, where the linear coefficients are the probabilities. The mixed strategy set of player i can be projected to the (x_{i1}, x_{i2})-plane. It is a triangle, given as the set of convex combinations of the unit vectors, which are the vertices of the triangle:

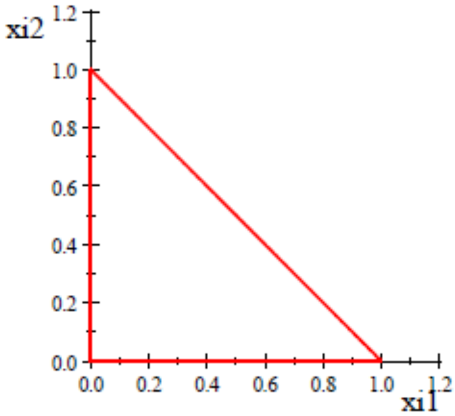

Figure 1: The mixed strategy simplex as the set of convex combinations of the unit vectors. [1]

Different players' randomizations are assumed to be statistically independent in a mixed-strategy set. The probability that a particular pure strategy profile $s = (s_1, ..., s_n) \in S$ will be used, when a mixed-strategy profile $x \in \square(S)$ is played is

$$x(s) = \Pi_{i=1}^{n} x_i(s_i), \tag{2.7}$$

the product of the probabilities $x_i(s_i)$ assigned by each player i's mixed strategy $x_i \in \Delta(S_i)$ to his pure strategy $s_i \in S_i$.

2.4 Expected payoffs

The expected value of the payoff to player i under a mixed strategy profile $x \in \square(S)$ is

$$\tilde{u}_i(x) = \sum_{s \in S} x(s) \cdot u_i(s). \tag{2.8}$$

It is the weighted sum of the payoffs that player i obtains under each pure strategy, weighted by the probability of that combination. This payoff is a linear function of each player's mixed strategy, x_j, because for any $x \in \square(S)$ and any two players i and j

$$\tilde{u}_i(x) = \sum_{k=1}^{m_j} \tilde{u}_i(1_j^k, x_{-j}) \cdot x_{jk}. \tag{2.9}$$

To see this, we note that playing a pure strategy $s_j = k \in S_j$ is probabilistically equivalent to playing the mixed strategy $1_j^k \in \Delta(S_j)$, so we may write $\tilde{u}_i(1_j^k, x_{-j})$ for the payoff that player i obtains when player j uses his k:th pure strategy.

In a finite two-player game, we may define the payoff to player 1 as $u_1(h,k) = a_{hk}$ and the payoff to player 2 as $u_2(h,k) = b_{hk}$. For any pair of mixed strategies $x_1 \in \Delta(S_1)$ and $x_2 \in \Delta(S_2)$, the expected values are then defined by the relations

$$\tilde{u}_1(x) = \sum_{h=1}^{m_1} \sum_{k=1}^{m_2} x_{1h} a_{hk} x_{2k} \tag{2.10}$$

and

$$\tilde{u}_2(x) = \sum_{h=1}^{m_1} \sum_{k=1}^{m_2} x_{1h} b_{hk} x_{2k}. \tag{2.11}$$

Because the players choose their pure strategies h and k independently, the probability that they choose the pure strategy pair (h,k) is the product $x_{1h}x_{2k}$ of these probabilities, which is the coefficients of the payoffs a_{hk} and b_{hk} in (2.10) and (2.11). [1]

Example 4 Consider the following two-player game:

	t₁	t₂
s₁	1,2	0,0
s₂	0,0	2,1

(2.12)

Suppose that player 1 plays strategy s_1 with probability $\frac{1}{3}$ and strategy s_2 with probability $\frac{2}{3}$ and that player 2 plays strategy t_1 with probability $\frac{2}{3}$ and strategy t_2 with probability $\frac{1}{3}$. The expected payoff to player 1 is

$$\begin{aligned}&\tilde{u}_1(x) = x_{11}a_{11}x_{21} + x_{11}a_{12}x_{22} + x_{12}a_{21}x_{21} + x_{12}a_{22}x_{22} = \\ &\frac{1}{3}\cdot 1\cdot\frac{2}{3} + \frac{1}{3}\cdot 0\cdot\frac{1}{3} + \frac{2}{3}\cdot 0\cdot\frac{2}{3} + \frac{2}{3}\cdot 2\cdot\frac{1}{3} = \frac{2}{9} + 2\cdot\frac{2}{9} = \frac{2}{3}\end{aligned} \tag{2.13}$$

and the expected payoff to player 2 is:

$$\tilde{u}_2(x) = x_{11}b_{11}x_{21} + x_{11}b_{12}x_{22} + x_{12}b_{21}x_{21} + x_{12}b_{22}x_{22} = \frac{1}{3}\cdot 2 \cdot \frac{2}{3} + \frac{1}{3}\cdot 0 \cdot \frac{1}{3} + \frac{2}{3}\cdot 0 \cdot \frac{2}{3} + \frac{2}{3}\cdot 1 \cdot \frac{1}{3} = 2\cdot\frac{2}{9} + \frac{2}{9} = \frac{2}{3}. \tag{2.14}$$

Players may attempt to maximize the expected value of their own payoffs, as defined in (2.8). If a player was given the opportunity to know about the other players' moves, he would want to switch his strategy to a best reply. Suppose, that everybody's strategy is a best reply to everybody else's. Then no one will have any incentive to change the situation. We will be in a stable situation, in a Nash equilibrium. In order to extend the definition of Nash equilibrium where mixed strategies can be taken into account, we need to require that the mixed strategy of every player is the best reply to the mixed strategies of the other players.

2.5 Best reply correspondence

A pure best reply for player i to a mixed-strategy profile $x \in \square(S)$ is a strategy $s_i \in S$ such that no other available pure strategy gives him a higher payoff against x. The i:th player's pure-strategy best-reply correspondence $\beta_i : \square(S) \rightrightarrows S_i$ is defined by

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(1_i^h, x_{-i}) \geq u_i(1_i^k, x_{-i}) \quad \forall k \in S_i\}. \tag{2.15}$$

For each player $i \in I$ and pure strategy $h \in S_i$, let

$$B_{ih} = \{x \in \square(S) : h \in \beta_i(x)\}, \tag{2.16}$$

which defines the set of mixed-strategy profiles to which h is a best reply for player i. [1]

Definition 3 *A mixed best reply for player i to a strategy $x \in \square$ is a strategy $x_i^* \in \Delta(S_i)$ such that no other mixed strategy gives a higher payoff to i against x.*

Every mixed strategy $x_i \in \Delta(S_i)$ is a convex combination of pure strategies. Since $\tilde{u}_i(x_i', x_{-i})$ is linear in x_i', no mixed strategy can give a higher payoff to player i against $x \in \square(S)$ than any one of his best pure replies to x. Formally, for any $x \in \square(S)$, $x_i' \in \Delta(S_i)$ and $h \in \beta(x)$, we have

$$\tilde{u}_i(x'_i, x_{-i}) = \sum_{k=1}^{m_i} \tilde{u}_i(1_i^k, x_{-i}) \cdot x'_{ik} \leq \sum_{k=1}^{m_i} \tilde{u}_i(1_i^h, x_{-i}) \cdot x'_{ik} = \tilde{u}_i(1_i^h, x_{-i}). \quad (2.17)$$

Hence, the set of pure best replies is identical with the set of pure replies that give the maximal payoff among the player's mixed strategies:

$$\beta_i(x) = \{h \in S_i : \tilde{u}_i(1_i^h, x_{-i}) \geq \tilde{u}_i(x'_i, x_{-i}) \quad \forall x'_i \in \Delta(S_i)\} \quad (2.18)$$

Every pure best reply, viewed as a mixed strategy, is also a mixed best reply. By linearity of $\tilde{u}_i(x'_i, x_{-i})$ in x'_i, any convex combination of pure best replies is a mixed best reply.

Accordingly, the i:th player's mixed-strategy best-reply correspondence $\tilde{\beta}_i : \Box(S) \rightrightarrows \Delta(S_i)$ is defined by

$$\tilde{\beta}_i(x) = \{x_i^* \in \Delta(S_i) : \tilde{u}_i(x_i^*, x_{-i}) \geq u_i(x'_i, x_{-i}) \quad \forall x'_i \in \Delta(S_i)\}. \quad (2.19)$$

Example 5 Consider a game given by the following bimatrix:

	t_1	t_2
s_1	4,2	4,1
s_2	2,2	5,0

(2.20)

The best reply of player 1 to the strategy t_1 of player 2 is the strategy s_1, i.e. $\beta_1(t_1) = s_1$. Similarly, the best reply of player 1 to the strategy t_2 is the strategy s_2, i.e. $\beta_1(t_2) = s_2$. Similarly for the best replies of player 2, we have $\beta_2(s_1) = t_1$, $\beta_2(s_2) = t_1$. In this case, it is easy to find the pair of strategies that are mutually best replies; it is the pair (s_1, t_1) which is an equilibrium point of the game.

Example 6 Consider the following game:

	t_1	t_2
s_1	-1,1	1,-1
s_2	1,-1	-1,1

(2.21)

Since $\beta_1(t_1) = s_2$, $\beta_1(t_2) = s_1$, $\beta_2(s_1) = t_1$ and $\beta_2(s_2) = t_2$, no pair of pure strategies consists from mutually best replies. It is necessary to consider mixed strategies. Let player 1 play s_1 with probability p and s_2 with probability $1-p$. Let player 2 play t_1 with probability q and t_2 with probability $1-q$.

Expected payoffs for particular players are the following:

$$\tilde{u}_1(p,q) = -pq+p(1-q)+(1-p)q-(1-p)(1-q) = (2q-1)+p(2-4q) \quad (2.22)$$

$$\tilde{u}_2(p,q) = pq-(1-p)q-p(1-q)+(1-p)(1-q) = (1-2p)+q(4p-2) \quad (2.23)$$

Now we will search for best replies of player 1 to various choices of probability q of player 2. We consider the expected payoff $(2q-1)+p(2-4q)$ for fixed q and variable p with $0 \leq p \leq 1$. This is a linear function of p and the maximum will depend on the slope of the function $2-4q$, whether it is positive, negative or 0.

If $0 \leq q < \frac{1}{2}$, then $\tilde{u}_1(p,q)$ is a linear function with positive slope, which is therefore increasing. Maximum occurs for the greatest possible value of p, i.e. for $p=1$.

If $q = \frac{1}{2}$, then $\tilde{u}_1(p,\frac{1}{2}) = 0$ is a constant function for which each value is maximal and minimal. Hence player 1 is indifferent between both strategies, $\beta_1(\frac{1}{2}) = \langle 0,1 \rangle$.

If $\frac{1}{2} < q \leq 1$, then $\tilde{u}_1(p,q)$ is a linear function with negative slope, which is therefore decreasing. Maximum occurs for the least possible value of p, i.e. for $p=0$.

For player 1 we have:

$$\beta_1(x_2) = \begin{cases} 1 & \text{for } 0 \leq q < \frac{1}{2} \\ \langle 0,1 \rangle & \text{for } q = \frac{1}{2} \\ 0 & \text{for } \frac{1}{2} < q \leq 1 \end{cases}$$

and similarly for player 2:

$$\beta_2(x_1) = \begin{cases} 0 & \text{for } 0 \leq p < \frac{1}{2} \\ \langle 0,1 \rangle & \text{for } p = \frac{1}{2} \\ 1 & \text{for } \frac{1}{2} < x_1 \leq 1 \end{cases}$$

To find the Nash equilibria we argue as follows: If $q > \frac{1}{2}$ then the best reply is $p=0$ but the best reply to $p=0$ is $q=0$. This contradicts $q > \frac{1}{2}$ and thus does not lead to a Nash equilibrium. If $q < \frac{1}{2}$ then the best reply is $p=1$

but the best reply to $p = 1$ is $q = 1$ and again, this does not lead to a Nash equilibrium. If $q = \frac{1}{2}$ then the best reply is any p and so if we choose $p = \frac{1}{2}$ then the best reply to $p = \frac{1}{2}$ is any q, in particular $q = \frac{1}{2}$. The equilibrium point is therefore $\left(\frac{1}{2}, \frac{1}{2}\right)$.

The best response functions for player 1 and player 2 are represented in the plane below.

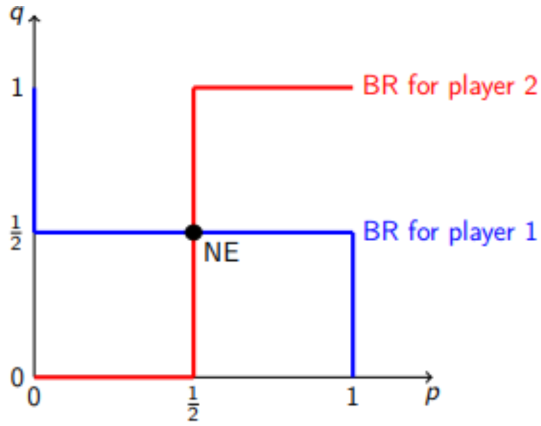

Figure 2: Best response functions in Example 6. [4]

2.6 Assumptions

In the theory of games, there are certain conditions that need to be properly satisfied, in order to obtain an appropriate solution of a problem. These conditions are often termed as the assumptions of the game theory. [1]

- **Rationality:** Each player i chooses a strategy $s_i \in S_i$ to maximize his expected payoff consistent with his beliefs about the other players' strategy choices. In forming his probabilistic belief about the others, each player assumes that the other player's strategy choices are statistically independent of each other.
- **Complete information:** Each player is fully aware of the rules of the game. Hence, each player knows what strategies are available to himself and what strategy sets and payoff functions are available to all of the other players'.

- **Common knowledge:** The fact that each player is rational and knows the game is common knowledge among players of the game. That is, it is known by all players that all players know the game $G = \langle I, S, u \rangle$ and are rational. It is also known by all players that it is known by all players that all players know the game and are rational and so on.

A consequence of rationality is that a player will use only strategies that are best replies to some beliefs he might have about the strategies of the other players. These beliefs are consistent with the other players' rationality, i.e., if player i believes that player j will play strategy t_j, then t_j maximizes j's payoff with respect to a belief of j about other players' strategies. These beliefs are also consistent with the other players' rationality. This means that we can iteratively delete strategies that are never best replies. For a player, the set of strategies that survives this iterated deletion of never best replies, is called his set of rationalizable strategies. Every Nash equilibrium is a rationalizable equilibrium, which will be shown in section 2.7.

2.7 Dominance relations

A strategy is called a strictly dominating strategy if it always provides a higher payoff, irrespective of what other players do. Hence, a strictly dominant strategy strictly dominates all other strategies. A strategy is called strictly dominated if, for every choice of strategies of the other players, this strategy always earns a player a smaller payoff than some other strategy would do. [1]

Definition 4 *$x_i^* \in \Delta(S_i)$ strictly dominates $x_i^{'} \in \Delta(S_i)$ if $\tilde{u}_i(x_i^*, x_{-i}) > \tilde{u}_i(x_i^{'}, x_{-i})$ for all $x \in \Box(S)$.*

Example 7 Consider the following two-player game:

$$\begin{array}{c|c|c|} & \mathrm{t}_1 & \mathrm{t}_2 \\ \hline \mathrm{s}_1 & 3{,}2 & 5{,}1 \\ \hline \mathrm{s}_2 & 0{,}0 & 2{,}2 \\ \hline \end{array} \tag{2.24}$$

Strategy s_1 is a strictly dominant strategy for player 1 and hence, strategy s_2 is strictly dominated by strategy s_1. Player 2 has no strictly dominated strategy.

A pure strategy may also be strictly dominated without being strictly dominated by any pure strategy.

Example 8 Consider the following game:

$$\begin{array}{c|c|c|} & t_1 & t_2 \\ \hline s_1 & 4,a & 0,b \\ \hline s_2 & 0,c & 4,d \\ \hline s_3 & 1,e & 1,f \\ \hline \end{array} \tag{2.25}$$

The payoffs to player 2 are omitted for simplicity. Player 1's pure strategy s_3, is not strictly dominated by any of his other pure strategies, but it is still strictly dominated by a strategy that would mix his pure strategies s_1 and s_2. Denote this mixed strategy $x_1^* = (\frac{1}{2}, \frac{1}{2}, 0)$. Irrespective of what player 2's strategy is, the expected payoff to player 1 from using x_1^* is 2. Because for any $x_2 \in \Delta(S_2)$,

$$\tilde{u}_1(x_1^*, x_2) = \frac{1}{2} \cdot 4 \cdot x_{21} + \frac{1}{2} \cdot 4 \cdot x_{22} = 2. \tag{2.26}$$

A strategy is said to weakly dominate another strategy if the first strategy never earns a lower payoff than the second, and sometimes earn a higher payoff.

Definition 5 *$x_i^* \in \Delta(S_i)$ weakly dominates $x_i^{'} \in \Delta(S_i)$ if $\tilde{u}_i(x_i^*, x_{-i}) \geq \tilde{u}_i(x_i^{'}, x_{-i})$ for all $x \in \Box(S)$, with strict inequality for some $x \in \Box(S)$.*

Example 9 Consider the following game:

$$\begin{array}{c|c|c|} & t_1 & t_2 \\ \hline s_1 & 1,1 & 0,0 \\ \hline s_2 & 0,0 & 1,0 \\ \hline \end{array} \tag{2.27}$$

Strategy s_2 is weakly dominated by strategy s_1 for player 1 and strategy t_2 is weakly dominated by strategy t_1 for player 2.

Proposition 1 *If (x_i^*, x_{-i}^*) is a dominant strategy solution, then (x_i^*, x_{-i}^*) is a Nash equilibrium.* [5]

Proof The strategy x_i^* dominates every other strategy in $\Delta(S_i)$. Thus $u_i(x_i^*, x_{-i}^*) \geq u_i(x_i, x_{-i}^*)$ for all $x_i \in \Delta(S_i)$ and x_i^* is the unique best response to x^*_{-i}. Similarly, x^*_{-i} is the unique best response to x^*_i. Thus (x^*_i, x^*_{-i}) is a Nash equilibrium.

A method for finding an equilibrium condition in a normal form game is called "Iterated Elimination of Strictly Dominated Strategies" (IESDS). This method implies systematic elimination of strictly dominated strategies for each player, because rational players will never play strictly dominated strategies. [4]

Example 10 Consider the following game:

	t_1	t_2	t_3
s_1	20,10	10,20	100,0
s_2	30,0	25,10	50,0
s_3	0,100	0,200	0,500

(2.28)

The strategy s_3 of player 1 is strictly dominated by the strategy s_2. Hence, we eliminate s_3 for player 1.

	t_1	t_2	t_3
s_1	20,10	10,20	100,0
s_2	30,0	25,10	50,0

(2.29)

Now both strategy t_1 and t_3 of player 2 are strictly dominated by the strategy t_2. By eliminating these two strategies we get:

	t_2
s_1	10,20
s_2	25,10

(2.30)

Strategy s_1 of player 1 is now strictly dominated by s_2. Hence, we eliminate s_1.

	t_2
s_2	25,10

(2.31)

An equilibrium point of the game is (s_2, t_2).

For two-player games, the set of all rationalizable strategies can be found by IESDS. For this method to hold, one also needs to consider strict domination by mixed strategies. We may continue eliminating strictly dominated strategies from the reduced form, even if they were not strictly dominated in the original matrix. In games with more than two players, there may be strategies that are not strictly dominated, but which can never be a best response. By iterated elimination of all such strategies one can find the rationalizable strategies for a multiplayer game.

Proposition 2 *If (x_i^*, x_{-i}^*) is a Nash equilibrium, then (x_i^*, x_{-i}^*) is not eliminated by IESDS.* [5]

Proof Suppose (x_i^*, x_{-i}^*) is eliminated during IESDS. Then one of the strategies is removed at some stage. Suppose that x_i^* is removed before x_{-i}^*. Then x_i^* and x_{-i}^* are possible strategies at this stage of the construction and, because it is about to be eliminated, there is a strategy $x_i' \in \Delta(S_i)$ such that x_i' strictly dominates x_i^*. But then $u_i(x_i', x_{-i}^*) > u_1(x_i^*, x_{-i}^*)$ and x_i^* is not a best response for player i to x_{-i}^*. This is a contradiction to the assumption that (x_i^*, x_{-i}^*) is a Nash equilibrium.

Strictly dominated strategies cannot be a part of a Nash equilibrium, since it is irrational for any player to play them. Weakly dominated strategies may be part of Nash equilibrium. For instance, consider the payoff matrix in example 9. Since player 2 does better by playing t_1 instead of t_2 and never does worse, t_1 weakly dominates t_2. Despite this, (s_2, t_2) is a Nash equilibrium. No strategy is strictly dominated and hence all strategies are rationalizable. But since s_1 of player 1 is a best response to t_1 of player 2 and s_2 i a best response to t_2, the only NE are (s_1, t_1) and (s_2, t_2). [4]

Rationalizability requires a player to play optimally with respect to some reasonable belief about the other players' actions. On the other hand, Nash equilibrium requires that a player play optimally with respect to what his opponents are actually playing. That is, the belief he holds about the other players' actions has to be correct. This point makes clear that each player's strategy in a Nash equilibrium profile is rationalizable, but lots of rationalizable profiles are not Nash equilibria.

2.8 Nash equilibrium

The Nash equilibrium concept requires a strategy profile $x \in \square(S)$ so that each component strategy x_i is optimal under some belief of the i:th player about the others' strategies and also it should be optimal under the belief $x_{-i} \in \times_{-i \neq i} \Delta(S_{-i})$ that all other play accordingly to x. In other words, no player has any incentive to deviate from his strategy profile if he knows that players' rationality and their beliefs about what the others play are mutually known. We can therefore say that a strategy profile is called a Nash equilibrium if it is a best reply to itself. [1]

Definition 6 *$x \in \square(S)$ is a Nash equilibrium if $x \in \tilde{\beta}(x)$.*

Let $X^{NE} \subseteq \square(S)$ denote the set of Nash equilibria. It follows from the definition that the three following statements are equivalent:

- $x \in X^{NE}$
- $x_{ih} > 0 \Rightarrow h \in \beta_i(x)$
- $h \notin \beta_i(x) \Rightarrow x_{ih} = 0$

Central to the proof of the existence of Nash equilibria in finite games will be Brouwer's Fixed-Point Theorem about continuous functions.

Theorem 1 (Brouwer's Fixed-Point Theorem) *Suppose that $X \subset \mathbb{R}^n$ is non-empty, compact and convex. If $f : X \to X$ is continuous, then there exists at least one fixed point, that is, there exists a $x^* \in X$ such that $x^* = f(x^*)$.*

See [6] for a proof.

The existence of Nash equilibrium in mixed strategies for finite games follows directly from Theorem 1.

Theorem 2 (Nash, 1950) *The mixed-strategy extension of any finite game has at least one Nash equilibrium.*

Proof Given a strategy profile $x \in \square(S)$, for each player i, let $u^+_{ih}(x) = \max\{0, \tilde{u}_i(1^h_i, x_{-i}) - \tilde{u}_i(x)\}$, being the excess payoff of pure strategy $h \in S_i$, that is, the extra payoff that i would earn if he were to deviate from his strategy x_i to pure strategy $h \in S_i$. We then define a function $f : \square(S) \to \square(S)$, onto itself, where

$$f_{ih}(x) = \frac{x_{ih} + u^+_{ih}(x)}{1 + \sum_{k \in S_i} u^+_{ik}(x)} \qquad \forall i \in I, h \in S_i. \tag{2.32}$$

This function maps a strategy profile s to a new strategy profile s' in which each agent's actions that are better responses to s receive increased probability mass. This function will modify the mixed strategy of player i by shifting some of the weight of the distribution to give more weight to the set of strategies $h \in S_i$. The function f is continuous, since each u^+_{ih} is continuous. Further, since S is convex and compact and $f : \Box(S) \to \Box(S)$, f must have at least one fixed point, by Theorem 1. Suppose that $x^* = f(x^*)$. Then

$$x^*_{ih} = \frac{x_{ih} + u^+_{ih}(x)}{1 + \sum_{k \in S_i} u^+_{ik}(x)} \tag{2.33}$$

$$\leftrightarrow x^*_{ih} \sum_{k \in S_i} u^+_{ik}(x^*) = u^+_{ih}(x^*) \tag{2.34}$$

for all players $i \in I$ and pure strategies $h \in S_i$, by Theorem 1. This means that $u^+_{ih}(x^*) = 0$ if and only if $x^*_{ih} = 0$. Therefore $u^+_{ih}(x^*) > 0$ for all pure strategies h with $x^*_{ih} > 0$. This is impossible, since all pure strategies in use cannot earn above average. Hence $\sum_k u^+_{ik}(x^*) = 0$. Thus $\tilde{u}_i(1^k_i, x^*_{-i}) \leq \tilde{u}_i(x^*)$ for all players i and pure strategies $k \in S_i$, implying a Nash equilibrium, $x^* \in X^{NE}$.

Example 11 Consider the following game:

	t$_1$	t$_2$
s$_1$	1,-1	-1,1
s$_2$	-1,1	1,-1

(2.35)

This game has no pure strategy Nash equilibrium, but it has one Nash equilibrium in mixed strategies. Suppose player 1 plays $\frac{3}{4}$ of strategy s_1 and $\frac{1}{4}$ of strategy s_2, then player 2 by playing pure strategy t_2 can get an expected payoff of $\frac{3}{4} \cdot 1 + \frac{1}{4} \cdot (-1) = \frac{1}{2}$. This cannot happen at equilibrium since player 1 then wants to deviate to the pure strategy s_2, deviating from the original mixed strategy. Hence, the unique Nash equilibrium is where both players play both pure strategies with probability one half, $x^*_1 = x^*_2 = (\frac{1}{2}, \frac{1}{2})$. Since $u^+_{ih}(x^*) = 0$, it can be seen that

$$x_{ih}^{*} \sum_{k \in S_i} u_{ik}^{+}(x^{*}) = u_{ih}^{+}(x^{*}) = 0 \tag{2.36}$$

and thus there is no room for improvement which is by definition a Nash equilibrium. The expected payoff to both players is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0$ and neither can do better by deviating to another strategy. This profile x^* is a fixed point under $\tilde{\beta}$ and indeed $\tilde{\beta}_i(x^*) = \Delta(S_i)$ for $i = 1, 2$, with $\Delta(S_i)$ denoting the unit simplex in $\mathbb{R}^2$. [1]

3 Prosocial preferences

Everyone of us has preferences in our approaches that affect how we perceive and act. In most applications of economic models, it is assumed that human behavior is driven by self-interest, that is maximization of own payoff, without caring for social goals. In recent years, a large body of research has shown that, when individuals who interact know each other's preferences, natural selection leads to preferences that deviate from maximization of own objective payoff. The predictive power of pure selfishness in certain interactions has been questioned and alternative preferences which include moral values as part of human motivation have been suggested. An interesting question is what preferences and moral values humans should be expected to have. Alger and Weibull (2013) [7] show that, when each individual's preferences is his private information, natural selection leads to a certain one-dimensional family of moral preferences that springs out from the mathematics. This family consists of all convex combinations of selfishness and morality. Individuals with such preferences are called Homo moralis and the weight attached to the moral goal is called the degree of morality.

3.1 Utility theory

As we just have noted, a player is an entity with preferences. Players make choices on the basis of preferences, that take into account all factors that can influence their behavior. To represent a player's preferences, we use utilities. As defined in (2.1), the utility function, $u_i : S \to \mathbb{R}^n$, represent how an individual subjectively values the material payoff π according to its preferences.

Example 12 Consider a n-player interaction. If π_i is the payoff to player i, let the utility or goal function to player i be represented in the form

$$u_i = (1 - \alpha_i)\pi_i + \alpha_i \sum_{j=1}^{n} \pi_j \tag{3.1}$$

for some $0 \leq \alpha_i \leq 1$. Here, player i may care to some extent, α_i, about the sum of monetary rewards to all players. If player i is purely selfish, then $\alpha_i = 0$. If player i is only concerned about social welfare, then $\alpha_i = 1$.

Example 13: Consider the following two-player game:

$$\begin{array}{c|c|c|}
 & t_1 & t_2 \\ \hline
s_1 & 3,3 & 0,4 \\ \hline
s_2 & 4,0 & 2,2 \\ \hline
\end{array} \tag{3.2}$$

If the payoffs reflect the player's preferences, then rationality implies that the strategy combination (s_2, t_2) will be chosen. Suppose instead that the numbers in the bimatrix are monetary gains. If both players only care about their own monetary gains, then (s_2, t_2) is still the unique outcome if both players are rational. Now, suppose that the players do care about each others' monetary gains and that their payoffs are their own monetary gain plus the other's monetary gain, weighted by some coefficient,

$$u_i = \pi_i + \alpha_i \cdot \pi_{-i} \tag{3.3}$$

where π_{-i} denotes the monetary gain for player role $-i \neq i$. The payoff-matrix will then be as shown below,

$$\begin{array}{c|c|c|}
 & t_1 & t_2 \\ \hline
s_1 & 3+3\alpha_1,\ 3+3\alpha_2 & 4\alpha_1,4 \\ \hline
s_2 & 4,\ 4\alpha_2 & 2+2\alpha_1,\ 2+2\alpha_2 \\ \hline
\end{array} \tag{3.4}$$

where $\alpha_1, \alpha_2 \in \mathbb{R}$. If α_i is positive, it means that player i is altruistic towards the other player and $\alpha_i = 0$ means that i is selfish. A negative coefficient α_i means that i is spiteful towards the other player. Now pure strategy s_2 (t_2) strictly dominates pure strategy s_1 (t_1) for player 1 (2) if and only if $a_i \leq \frac{1}{3}$, that is, if and only if player i is spiteful or selfish or not too altruistic. If player 1 (2) is sufficiently altruistic, $a_i > \frac{1}{3}$, then strategy s_1 (t_1) is a better reply than strategy s_2 (t_2), if player 1 expects player 2 to play t_1. If both players are sufficiently altruistic, $\alpha_1, \alpha_2 > \frac{1}{3}$, then (s_1, t_1) is even a Nash equilibrium.

In a sequential prisoners' dilemma experiment, Miettinen, Kosfeld, Fehr and Weibull (2016) [8], analyzed different goal functions and examined different models' explanatory power. The data they obtained in the experiment were used to compare the explanatory power of a few models of other-regarding and moral preferences. These are described below and are some of the most common models of prosocial preferences. The monetary payoff earned by a subject in player role $i = 1, 2$ when using pure strategy $x_i \in X_i$ against an opponent who uses strategy $x_{-i} \in X_{-i}$ (where $-i$ denotes player role $j \neq i$) was $\pi_i(x_i, x_{-i})$.

Homo economicus
If an individual is purely self-interested and only cares about a maximization of own material payoff, we say that he is a Homo economicus. His utility is then

$$u_i(x_i, x_{-i}) = \pi_i(x_i, x_{-i}), \tag{3.5}$$

equivalent with his expected payoff.

Altruism
If an individual cares about his own material payoff and also attaches a positive weight to the material payoffs to others, we say that he is an altruist. His utility is the sum of his own payoff and the payoff to the other player, where the latter term is weighted by a factor $\alpha \in [0, 1]$. We call α the degree of altruism. The utility is given by

$$u_i(x_i, x_{-i}) = (1 - \alpha)\pi_i(x_i, x_{-i}) + \alpha \cdot \pi_{-i}(x_{-i}, x_i). \tag{3.6}$$

Altruism takes place when an individual acts with an unselfish regard for others, even at a risk or cost to himself.

Homo moralis
If an individual cares about his own material payoff and also attaches a weight to what his material payoff would be if others use the same strategy as him, we say that he is a Homo moralis. The utility to a Homo moralis with degree morality $\kappa \in [0, 1]$ is

$$u_i(x_i, x_{-i}) = (1 - \kappa)\pi_i(x_i, x_{-i}) + \kappa\pi_i(x_i, x_i), \tag{3.7}$$

where $\pi_i(x_i, x_{-i})$ is maximizing own fitness and $\pi_i(x_i, x_i)$ defines doing what would be right for both, in terms of payoffs, if the other player did the same. Clearly, a Homo economicus is a Homo moralis goal function, namely, the one with morality profile when $\kappa = 0$, i.e. attaches zero weight to morality. At the opposite extreme of the spectrum of Homo moralis we find Homo kantientis, that is pure ethical reasoning in line with Kant's categorical imperative[3]. [7]

[3]Kant's (1785) categorical imperative can be formulated as follows: "Act only in accordance with that maxim through which you can at the same time will that it become a universal law." [2]

This variety of Homo moralis occurs when $\kappa = 1$, i.e. attaches unit weight to morality. Individuals of this "pure Kantian" type always choose a strategy that would maximize all players' payoffs, if it was adopted by every player. The behavior of all other varieties of Homo moralis lies between these two extremes.

Inequity aversion

In the Fehr-Schmidt model, focus is being put on own payoff and inequity aversion. A negative weight is given to payoff differences between players (with a bigger weight when the difference is to the player's own disadvantage). The utility is

$$\begin{aligned} u_i^{FS}(x_i, x_{-i}) = \pi_i(x_i, x_{-i}) - \alpha(\pi_{-i}(x_{-i}, x_i) - \pi_i(x_i, x_{-i}))_+ \\ -\beta(\pi_i(x_i, x_{-i}) - \pi_{-i}(x_{-i}, x_i))_+, \end{aligned} \tag{3.8}$$

for $i = 1, 2$. Both α and β are non-negative, $\alpha \geq \beta$ and $(x)_+$ denotes $\max\{0, x\}$.

4 Classes of games

It is possible to construct several classifications of games, based on the properties and criteria one wants to look at. A classification of a 2-person, 2-strategy (2×2) game was made by Pangallo, Sanders, Galla, and Farmer (2017). [9] By looking at relevant combinations of parameters, the classification forms a game taxonomy where games are defined in terms of the orderings in the payoff matrix. They were only concerned with the number of Nash equilibria and with their type, i.e. whether they were pure or mixed strategy Nash equilibria. They found three classes of 2×2 games, where all such games belong to one of these classes.

4.1 Game taxonomy

Consider a general 2×2 payoff bimatrix:

$$\begin{array}{c|c|c|} & \text{A} & \text{B} \\ \hline \text{A} & \text{a,e} & \text{b,g} \\ \hline \text{B} & \text{c,f} & \text{d,h} \\ \hline \end{array} \tag{4.1}$$

The number and type of the Nash equilibria depend on the pairwise ordering of the payoffs for each player, namely (a, c) and (b, d) for the row player and (e, g) and (f, h) for the column player. The classes of 2×2 games found can be seen below.

Coordination games
In coordination games, all pure strategy Nash equilibria exist when players choose the same or corresponding strategies. A 2×2 coordination game has two pure strategy Nash equilibria, namely the strategy profiles {A, A} and {B, B} and one mixed strategy Nash equilibria. The following inequalities $a > c$, $d > b$, $e > g$, $h > f$ should hold in the payoff matrix. This setup can be extended for a game with more than two pure strategies, as well as more than two players.

Stag hunt
The Stag hunt game describes a conflict between safety and social cooperation.

This game has two pure strategy Nash equilibria, one where the players choose a safe action and one where the players choose a risky action. The safe action is risk dominant, while the risky action is payoff dominant, implying that when both choose the risky actions, they end up at a high-payoff equilibrium and when both choose the safe action, they end up at a low-payoff equilibrium. Both players may want to obtain high payoff but uncertainty about the opponent's action may prevent them to take such strategic risk.

Formally, a generic symmetric stag hunt is given by the inequalities $a > c \geq d > b$, for the payoffs in the payoff matrix illustrated in (4.2).

$$\begin{array}{c|c|c|} & \text{S} & \text{H} \\ \hline \text{S} & \text{a,a} & \text{b,c} \\ \hline \text{H} & \text{c,b} & \text{d,d} \\ \hline \end{array} \tag{4.2}$$

In addition to the pure strategy Nash equilibria, there is one mixed strategy Nash equilibrium, which depends on the payoffs in the payoff matrix.

Anticoordination games

Games where a player's best response is to choose an action unlike that of the other player, are called anti-coordination games. These games are defined by the orderings $a < c$, $d < b$, $e < g$, $h < f$ in the payoff matrix. From a mathematical point of view, anticoordination 2×2 games are largely similar to coordination games, in that they also have two pure strategy and one mixed strategy Nash equilibria. The difference is that in the pure strategy Nash equilibria, the players choose strategies with different labels, i.e. (A, B) and (B, A), instead of same labels.

Hawk-Dove

A well-known example of a 2-player anti-coordination game is the Hawk-Dove game. Two animals compete for a piece of food. One of the animals plays the strategy Hawk (H) and behaves aggressively, while the other animal plays the strategy Dove (D) and behaves passively. Given that the resource is given the value v, the damage from losing a fight is given cost c. If both animals are aggressive, as in strategy pair (H, H), they risk destroying the food and injuring each other. Then each animal receives payoff $\frac{v}{2} - \frac{c}{2}$, because it wins the resource half of the time but always pays a cost c. If a Hawk meets a Dove he gets the full resource v to himself, while the Dove will back off and get nothing. If both animals are passive, i.e. Dove meets a Dove, both share the resource and get $\frac{v}{2}$. [10] This leads to a payoff matrix as shown below:

$$
\begin{array}{c|c|c|}
 & \text{H} & \text{D} \\
\hline
\text{H} & \frac{v-c}{2}, \frac{v-c}{2} & \text{v, 0} \\
\hline
\text{D} & \text{0, v} & \frac{v}{2}, \frac{v}{2} \\
\hline
\end{array} \tag{4.3}
$$

Depending on the value of c relative to v, there are different types of equilibria. If $v \geq c$, then there exists a unique Nash equilibrium, (H, H). If $v < c$, then there exists three Nash equilibria: (H, D), (D, H) and a mixed strategy equilibrium.

Discoordination games
The opposite of a coordination game is a discoordination game. One player's incentive is to coordinate while the other player tries to avoid this. These games are defined by the orderings $a > c$, $d > b$, $e < g$, $h < f$, or $a < c$, $d < b$, $e > g$, $h > f$ in the payoff matrix. They have a unique mixed strategy NE and no pure strategy NE, since the players have incentives to coordinate on different strategy profiles.

Matching Pennies
A prototypical discoordination game is Matching pennies. It is played between two players simultaneously placing a penny on the table, with the payoff depending on whether the pennies match. If the pennies match, i.e. both are heads or both are tails, then one player keeps both pennies. If the pennies do not match, i.e. one shows heads and one shows tails, then the other player wins and keeps both pennies. The game can be written in a payoff matrix shown below:

$$
\begin{array}{c|c|c|}
 & \text{H} & \text{T} \\
\hline
\text{H} & 1,-1 & -1,1 \\
\hline
\text{T} & -1,1 & 1,-1 \\
\hline
\end{array} \tag{4.4}
$$

This game has no pure strategy NE, since if the players' strategies match, i.e. both playing heads (H, H) or both playing tails (T, T), then player 1 prefers to switch strategies. If the strategies do not match, then player 2 prefers to switch. Instead, the unique NE of this game is in mixed strategies. Each player chooses heads or tails with equal probability and makes the other indifferent between choosing heads or tails, so neither player has an incentive to deviate to another strategy. [11]

Dominance solvable games

In any case, if by IESDS, there is only one strategy left for each player, the game is called a dominance-solvable game. Dominance-solvable games are defined by all 12 remaining possible orderings, for example $a > c$, $b > d$, $e > g$, $f > h$. They have a unique pure strategy Nash equilibrium, obtainable from the elimination of strictly dominated strategies.

Prisoner's dilemma

The prisoner's dilemma is a 2×2 dominance-solvable game and is probably the most widely used game in game theory. Two prisoners are suspected of committing a crime. They are being interrogated in separate rooms and face the same scenario. They are offered the same deal and know the consequences of each action and are completely aware that the other prisoner has been offered the same deal. Both of them want to minimize their prison sentences. Each prisoner can either betray the other by testifying that the other committed the crime ("defecting"), or to cooperate with the other by remaining silent. If both prisoners betray the other, each serves 3 years in prison. If only one prisoner betrays the other, then that prisoner goes free while the other prisoner gets 5 years in prison. If both prisoners cooperate with the other by remaining silent, each will only serve 1 year in prison. The decision matrix is shown below:

	Cooperate	Defect
Cooperate	-1,-1	-5,0
Defect	0,-5	-3,-3

(4.5)

Eliminating all dominated strategies, can solve this game. Since "Cooperate" is strictly dominated by "Defect" for player 1, the rational thing to do for player 1 is do defect. Similarly for player 2, no matter what prisoner 1 does, prisoner 2 is better off defecting. Therefore, “to defect” is the dominant strategy. The dilemma then is that mutual cooperation (remaining silent) yields a better outcome than mutual defecting but it is not the rational outcome because at the individual level, the choice to cooperate, is irrational. As a result, both prisoners defect and receive multiple years in prison. If they had cooperated, they could have served only one year. [2]

The prisoner's dilemma game can be expressed in the more general form below

	C	D	
C	a,a	b,c	(4.6)
D	c,b	d,d	

and to be a prisoner's dilemma game, the following inequalities $c > a > d > b$ must hold for the payoffs, where $a > d$ implies that mutual cooperation is superior to mutual defection and $c > a$ and $d > b$ imply that defection is the dominant strategy for both players.

4.2 Analysis

Consider the two-player prisoner's dilemma game form shown in (4.6). Let us suppose that this game represent a one-time interaction. Then, this game seems to leave no hope for cooperation if the players act rationally, but could altruism or morality motivate a person to cooperate? Consider now a transformation of the game form in (4.6) with an altruistic interpretation. The factor α will be the altruism parameter of both players, described in (3.6). The payoff received by one player is added to the other player's payoff multiplied by α, and similarly for the other player. We obtain the following modified payoff matrix:

	C	D	
C	$(1-\alpha)a+\alpha a,\ (1-\alpha)a+\alpha a$	$(1-\alpha)b+\alpha c,\ (1-\alpha)c+\alpha b$	(4.7)
D	$(1-\alpha)c+\alpha b,\ (1-\alpha)b+\alpha c$	$(1-\alpha)d+\alpha d,\ (1-\alpha)d+\alpha d$	

This game is payoff-symmetric and, hence, we can simplify it by only looking at one player's actions. In the game form below,

C	a	$(1-\alpha)b+\alpha c$	(4.8)
D	$(1-\alpha)c+\alpha b$	d	

the entries of the payoff matrix refer to the row player, player 1. It can be seen that strategy choice C dominates when $a > (1-\alpha)c+\alpha b$ and $(1-\alpha)b+\alpha c > d$.

We can identify two important conditions;

$$\alpha > \alpha_1 = \frac{c-a}{c-b}$$

and

$$\alpha > \alpha_2 = \frac{d-b}{c-b}.$$

When both inequalities are satisfied, strategy choice C dominates. When neither inequality is satisfied, strategy choice D dominates. When only one inequality is satisfied, we will have a coordination game with two pure NE and a mixed NE.

The actual outcome of the game depends on the relationship between α_1, α_2 and α. Let us consider that the average degree of altruism between interacting individuals is a number between 0 and 1. If $a + d > b + c$, then $\alpha_2 > \alpha_1$. Now, if $\alpha_2 > \alpha_1 > \alpha$, then strategy D will dominate. If $\alpha_2 > \alpha > \alpha_1$, both cooperation and defection will be possible actions. If $\alpha > \alpha_2 > \alpha_1$, then strategy C will dominate. Similarly, if $a + d < b + c$, then $\alpha_1 > \alpha_2$. If $\alpha_1 > \alpha_2 > \alpha$, then strategy D will dominate. If $\alpha_1 > \alpha > \alpha_2$, then both cooperators and defectors might exist. If $\alpha > \alpha_1 > \alpha_2$, then strategy C will dominate. Therefore, it follows that the prisoner's dilemma game leads to mutual cooperation if $\alpha > \max\{\frac{c-a}{c-b}, \frac{d-b}{c-b}\}$.

Suppose, instead, that the players have preferences of Homo moralis, described in (3.7). For the prisoner's dilemma game in (4.6), we will now have the following modified payoff matrix,

$$\begin{array}{c|c|c|} \hline C & (1-\kappa)a + \kappa a & (1-\kappa)b + \kappa a \\ \hline D & (1-\kappa)c + \kappa d & (1-\kappa)d + \kappa d \\ \hline \end{array} \tag{4.9}$$

where strategy choice C dominates when $(1-\kappa)a + \kappa a > (1-\kappa)c + \kappa d$ and $(1-\kappa)b + \kappa a > (1-\kappa)d + \kappa d$. We can can identify the following conditions;

$$\kappa > \kappa_1 = \frac{c-a}{c-d}$$

and

$$\kappa > \kappa_2 = \frac{d-b}{a-b}.$$

As previous, when both inequalities are satisfied, strategy choice C dominates and when neither inequality is satisfied, strategy choice D dominates. The outcome of the game will depend on the relationship between κ_C, κ_D and κ. If $a+d > b+c$, then $\kappa_2 > \kappa_1$. If $\kappa_2 > \kappa_1 > \kappa$, then strategy D will dominate and if $\kappa > \kappa_2 > \kappa_1$, then strategy C will dominate. Similarly, if $a+d < b+c$, then $\kappa_1 > \kappa_2$. If $\kappa_1 > \kappa_2 > \kappa$, then strategy D will dominate and if $\kappa > \kappa_1 > \kappa_2$, then strategy C will dominate. It follows that the prisoner's dilemma leads to mutual cooperation if $\kappa > \max\{\frac{c-a}{c-d}, \frac{d-b}{a-b}\}$.

Finally, let us suppose that the players have preferences of Inequity aversion, described in (3.8). The modified payoff matrix will be as shown below:

C	a	$b-\alpha(c-b)-\beta(b-c)$
D	$c-\alpha(b-c)-\beta(c-b)$	d

(4.10)

where strategy choice C dominates when $a > c-\alpha(b-c)-\beta(c-b)$ and $b-\alpha(c-b)-\beta(b-c) > d$. We can can identify the following conditions;

$$\beta - \alpha > \frac{c-a}{c-b}$$

and

$$\beta - \alpha > \frac{d-b}{c-b}.$$

Since $\alpha \geq \beta$ through definition, $\beta - \alpha < 0$ and, hence, the above conditions are impossible. Thus, in theory, this game will not lead to cooperation. Even though players with preferences of Inequity aversion does not like payoff differences, they dislike payoff differences to their own disadvantage more than payoff differences to their opponents' disadvantages, which explains why the above conditions does not lead to cooperation.

For altruists and moralists, however, we can apply the above results on the payoffs in the prisoner's dilemma game in (4.5). It can be seen that $\alpha_2 > \alpha_1$ and $\kappa_2 > \kappa_1$, since $a+d = -4 > b+c = -5$. Also, notice that $\kappa_2 > \alpha_2$. The different values of α_1, α_2, κ_1 and κ_2 are shown in the table below:

α	κ
$\alpha_1 = \frac{0-(-1)}{0-(-5)} = \frac{1}{5}$	$\kappa_1 = \frac{0-(-1)}{0-(-3)=} \frac{1}{3}$
$\alpha_2 = \frac{-3-(-5)}{0-(-5)} = \frac{2}{5}$	$\kappa_2 = \frac{-3-(-5)}{-1-(-5)} = \frac{1}{2}$

(4.11)

Another example of a prisoner's dilemma game, where instead, $a + d < b + c$, is shown in the following bimatrix:

	C	D
C	-2,-2	-4,0
D	0,-4	-3,-3

(4.12)

The payoffs will, as previously stated, fulfill the conditions $\alpha_1 > \alpha_2$ and $\kappa_1 > \kappa_2$. Also, observe that $\kappa_1 > \alpha_1$. The different values of α_1, α_2, κ_1 and κ_2 are shown below:

α	κ
$\alpha_1 = \frac{-0-(-2)}{0-(-4)} = \frac{1}{2}$	$\kappa_1 = \frac{0-(-2)}{0-(-3)=} \frac{2}{3}$
$\alpha_2 = \frac{-3-(-4)}{0-(-4)} = \frac{1}{4}$	$\kappa_2 = \frac{-3-(-4)}{-2-(-4)} = \frac{1}{2}$

(4.13)

Apparently, if we have a one-shot prisoner's dilemma game, with $c > a > d > b$, then, if the condition $a + d > c + d$ holds, $\kappa_2 = \frac{d-b}{a-b} > \frac{d-b}{c-b} = \alpha_2$ and if $a + d < c + d$ holds, then $\kappa_1 = \frac{c-a}{c-d} > \frac{c-a}{c-b} = \alpha_1$. Hence, in both cases, since $\max\{\kappa_1, \kappa_2\} > \max\{\alpha_1, \alpha_2\}$, it can be seen that it takes less altruism than morality for mutual cooperation.

4.3 Summary

In social dilemmas, such as the prisoner's dilemma, people choose between personal gains and the common good. Regardless of what others do, rational people are better off following their self-interest than acting in the collective interest. However, many people cooperate, thus fostering the good of the collective, while they set aside their own self-interests. Experimental evidence suggests that people are not entirely selfish and evolution does not favour selfish people. [7] In the one-shot prisoner's dilemma analysis above, it can be seen that altruistic individuals are more cooperative than moral individuals. If we compare cooperativeness between altruists and moralists for the general prisoner's dilemma game in (4.6), we will see that the degree α needed for mutual cooperation is less for altruists than the degree κ for moralists and thus, it takes less altruism than morality to turn cooperation into a

Nash equilibrium. Altruists care about the actual consequences of defecting unilaterally and that it generates a payoff loss for his opponent. This payoff loss might exceed the own payoff gain from defecting together to defecting unilaterally. On the other hand, moralists care about what would be his payoff if others were to act like himself and thus does not care about this payoff loss. Hence, in the unrepeated prisoner's dilemma, altruistic preferences favour cooperation and allow natural selection to favour cooperation over defection, more than moral individuals do.

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