SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The Logical Consistency of Categorical Set Theory

av

Filip Åkerman

2018 - No K28

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET, 106 91 STOCKHOLM

The Logical Consistency of Categorical Set Theory

Filip Åkerman

Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Erik Palmgren

2018

The Logical Consistency of Categorical Set Theory

Filip Åkerman

August 29, 2018

Abstract

The categorical set theory ETCS is an attempt to bypass some issues of classical set theory. In this paper it will be presented in its original version formulated by William Lawvere in 1963. The theory consists of axioms that could be thought of as demands we make on a certain category. We give an account for a model of this theory and use this model to show that the theory is consistent.

Acknowledges

I would like to thank my superviser Erik Palmgren who has introduced me to category theory, a tiny universe under the branch of mathematical logic where everything is possible.

Contents

Introduction

For a foundation of mathematics we wish to have enough axioms to define and derive everything we want to use in mathematics, including every theorem and every proof. Many such foundations have been suggested and applied in the setting of axiomatic set theory. Sometimes with great success.

However, sets are defined recursively by their elements. Elements in turn are defined as sets and thus they are defined by their elements, which are defined by their elements and so on. If we continue asking we will arrive at a position when we need to say what a set *really* is. There are plenty of different answers, which would depend on your beliefs about mathematics. One such answer is offered by *categorical set theory*. This is a set theory based on functions, instead than the membership-relation [14, p.39].

In category theory we study mathematical structures without specificing the content of the structure. ETCS, *Elementary Theory of the Category of Sets*, is a set theory where sets are objects in a category. As we will see, problems like the one described above will vanish by presenting sets in a categorical setting. If we want this foundation it is necessary that ETCS is consistent, and to confirm this is the main task of this paper.

In chapter 1 we will present another set theory: RZC, or *Restricted Zermelo Set Theory with Choice*. We will show that this weaker version of classical Zermelo-Fraenkel set theory is consistent.

To make sense of ETCS we need chapter 2, dedicated to introduce key concepts in category theory.

In chapter 3 we will use RZC as a metatheory as we will finally prove the consistency of ETCS.

In chapter 4 we will return to the philosophical issue mentioned above: how should we choose our foundations?

Chapter 1

The Formal System RZC

Kurt Gödel's *Second Incompleteness Theorem* (1931) states that any consistent theory that is not too weak cannot prove the consistency of itself. Saying that a theory is not too weak means that it includes elementary arithmetic.

This theorem is the main reason why our first chapter is dedicated to the set theory **Restricted Zermelo-Fraenkel with Choice** (RZC). To show the consistency of ETCS, we will need RZC as a metatheory. Since we want every reader to understand what this means, the next section is a very short introduction to model theory. We have used the definitions from Jech [6, p.155].

1.1 Consistency

Definition A **formal language** $\mathcal{L}$ is a set of relation symbols, function symbols and constant symbols.

The difference from what we usually call languages is that the symbols of a formal language are fixed and does not change as we use them or by any other reason, except when we add more symbols to the set.

Definition A **theory** $\mathcal{T}$ is a set of sentences in a formal language.

In this paper, the theories RZC and ETCS will be presented as a set of sentences in formal language of first-order predicate logic.

Definition A **model** $\mathcal{M}$ in a language $\mathcal{L}$ is a pair consisting of a universe $\mathcal{U}$ of the model, and an interpretation $\mathcal{I}$ that determines how every symbol of the language $\mathcal{L}$ will be interpreted in $\mathcal{U}$.

If $\mathcal{M}$ *models* a theory $\mathcal{T}$ we write $\mathcal{M} \models \mathcal{T}$ and this means that the sentences in $\mathcal{T}$ are true in $\mathcal{M}$. If a theory has very few sentences there will be many possible models, but if we add more to the theory naturally we will get higher requirements on the model. Perhaps we add contradictory sentences and in that case we will not have any models at all. Often we want to add not too few axioms to make a theory less trivial.

Definition A theory is **consistent** if it is not possible to derive falsity ($\bot$) by the rules of natural deduction. If this is the case we write:

$$\mathcal{T} \nvdash \bot$$

Now we will present the formal system (i.e. a theory consisting of axioms, inference rules and a language) RZC as a list of axioms. To prove the consistency of RZC we will use a model $\mathcal{M}$ and for every axiom A in RZC we will show that $\mathcal{M} \models A$. That is, A is true given our model and as we show this for every axiom we get $\mathcal{M} \models \text{RZC}$.

Why does this imply that RZC is consistent? By *The Soundness Theorem* we have that $\mathcal{T} \vdash \varphi$ implies $\mathcal{T} \models \varphi$. Using the contraposition of this we know that as RZC does not model $\bot$ we also must have RZC $\nvdash \bot$, i.e. RZC is consistent.

1.2 The axioms of RZC

Extensionality If two sets have the same elements, then the sets are the same.

$$\forall x\, \forall y\, (\forall z\, (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

Null Set There exist a set $\emptyset$ with no members.

$$\exists x\, \forall y\, \neg(y \in x)$$

Pair Set For all sets x and y there exists a set which consists of exactly x and y.

$$\forall x\, \forall y\, \exists z\, (u \in z \leftrightarrow (u = x \vee u = y))$$

Union For every set x there exists a set $\cup x$ that consists exactly of the members of the members of x.

$$\forall x\, \exists y\, (u \in y \leftrightarrow \exists v \in x\, (u \in v))$$

Power Set For every set x there exists a set $\mathcal{P}(x)$ that consists of every subset of x.

$$\forall x\, \exists y\, \forall z\, (z \subseteq x \leftrightarrow z \in y)$$

Regularity Every nonempty set x have some element y that does not intersect x.

$$\forall x\ (x \neq \emptyset \rightarrow \exists y \in x\ (y \cap x = \emptyset))$$

Restricted Comprehension Scheme For every x and unary restricted formula φ there exists a set y with the following property: $a \in y$ if and only if $a \in x$ and $\varphi(a)$. This axiom is often refered to as Restricted *Separation* Scheme.

$$\exists y\ \forall a\ (a \in y \leftrightarrow a \in x \wedge \varphi(a))$$

Axiom of Infinity There exists a set ω that contains $\emptyset$ and for each $y \in \omega$ also $y \cup \{y\} \in \omega$.

$$\exists \omega\ (0 \in \omega \wedge \forall y \in \omega\ (y \cup \{y\} \in \omega))$$

Axiom of Choice Let A and B be sets and $R \subseteq (A \times B)$ a binary relation. The axiom then states that if for every x in A there exists some y such that $(x, y) \in R$, then there exists a *choice funtion* $f : A \rightarrow B$ that satisfies $R(x, f(x))$ for every x in A.

$$\forall x \in A \exists y \in B(R(x, y)) \rightarrow \exists f : A \rightarrow B(\forall x \in A(R(x, f(x))))$$

The reason we have chosen RZC and not the natural option Zermelo-Fraenkel with Choice (ZFC) is because in ETCS there are no corresponding way to quantify over a complete category, in the sense that we are able to quantify over a complete set universe when we have the usual non-restricted Axiom of Comprehension. With the exception of this axiom RZC is very similar to ZFC.

More precisely the RZC version of Comprehension states that φ must be a restricted formula, meaning that every quantifier of the formula is bounded to some already well-defined set in the universe. For instance, in the universe of sets, $\forall a(a \in x)$ would not be a restricted formula but $\forall a \in y(a \in x)$ is.

We must however mention that by transfinite recursion it is possible to construct a bigger model than the one we soon will describe, such that the Axiom of Comprehension is no longer restricted [7, p.3].

1.3 von Neumann's Universe

A universe for a theory is a set whose members constitute a model for the theory. We will show that the members of **von Neumann's Universe** constitute a model for RZC. The universe is built up recursively in the following manner. First, let $\emptyset$ be a member. Then, let the power set of $\emptyset$ be another member. Continuing taking power sets we get infinitely many members of the universe. Altogether we get the following hierarchy of sets.

$$V_0 = \emptyset$$

$$V_{\alpha+1} = \mathcal{P}(V_\alpha)$$

for finite ordinals α

$$V_\lambda = \cup_{n<\lambda} V_n$$

for infinite ordinals λ

The members are indexed by *the ordinal numbers*, or ordinals. We define an ordinal number α as a transitive well-ordered set with respect to membership $\in$ [6, p. 19]. The finite ordinal numbers (which are the natural numbers) are defined as $0 = \emptyset$, $1 = \{0\}$, $2 = \{0, 1\}$, $3 = \{0, 1, 2\}$, etc.

The ordinal number $\mathbb{N}$ or ω is the number of elements in the set of all finite ordinal numbers, and it is the least infinite ordinal. We will work with this universe up to the level of $V_{\omega+\omega} = \cup_{n<\omega} V_{\omega+n}$ (there is no reason to think about higher infinite ordinals for our purposes).

Important for our purporses is however that every V_α could be shown to be *transitive*, meaning that,

$$x \in V_\alpha \Rightarrow x \subseteq V_\alpha$$

This property will allow us to find every set required by the RZC axioms.

A good example how transitivity helps us to achieve new sets is how we does not find the set 3 in the recursion, but we will still have this set in $\mathcal{V}$ because we have the set $V_3 = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

The union of all V_α is denoted by $\mathcal{V}$ is *the class of pure grounded sets* and further commentary on this fact can be find in Moschovakis [15, p.187-188].

1.4 Consistency of RZC

Assuming we got $\mathcal{V}$, up to the level of $V_{\omega+\omega}$, we want to deduce that the axioms of RZC holds. This implies that $\mathcal{V}$ is model of RZC and by the soundness theorem RZC is consistent.

In the following discussion every α and β will be considered ordinal numbers smaller than $\omega + \omega$. We will sometimes use the fact that $\alpha + n$ also will be smaller than $\omega + \omega$ when $n < \omega$. Some of the axioms demands existence for certain sets. In these cases it will be enough to find the required sets on some level below $V_{\omega+\omega}$ in the hierarchy since our model is the union of all those sets.

Extensionality This axiom will only fail if there were distinct sets containing exactly the same elements. This is not possible, because if two sets x and y contain the same elements we will get by transitivity that they also have the same subsets. Then, by construction, $x = y$.

Null Set $\emptyset$ is by construction an element of V_1 as well as other sets in $\mathcal{V}$, so this axiom holds.

Pair Set Let $x \in V_\alpha$ and $y \in V_\beta$. Since any member higher in the hierarchy (e.g. $V_{\omega+\omega}$) is well-ordered by the ordinal numbers we know that also $x \in V_\beta$ (if $\alpha < \beta$) or $y \in V_\alpha$ (if $\alpha > \beta$). The pair set $\{x, y\}$ will consequently be in $V_{\max(\alpha,\beta)+1}$. It follows that $\{x, y\} \in V_{\omega+\omega}$.

Union Let $x \in V_\alpha$. $\cup x$ is defined as the set of all sets that are members of some member of x. If $u \in x$ we know that $u \in V_\alpha$ and by transitivity also $u \subseteq V_\alpha$. Members of u will be in V_α as well. These members constitute $\cup x$, so $\cup x$ itself must be contained a level above V_α. Therefore every union of a set in V_α will be in $V_{\alpha+1}$, and the axiom holds.

Power Set Indeed, as we constructed the universe with power sets we clearly have some power sets in $\mathcal{V}$, but we have to argue for the general case. However, this is also clear because if $x \in V_\alpha$, then $\mathcal{P}(x) \in V_{\alpha+1}$ and $V_{\alpha+1}$ will still be on the level of $V_{\omega+\omega}$ if V_α was.

Regularity This axiom only require that every set has an element satisfying certain properties. If $y \in x \in V_\alpha$ by transitivity $y \in x \subseteq V_\alpha$ and therefore $y \in V_\alpha$.

Restricted Comprehension Scheme Let $x \in V_\alpha$ and let φ be an arbitrary formula. We want to show the existence of the set y in the universe, where y is defined as the set of all $a \in x$ such that $\varphi(a)$. By transitivity we have that $x \subseteq V_\alpha$. We also have $y \subseteq x$ by the definition of y. Taken together, we have that $y \subseteq V_\alpha$. Then $y \in V_{\alpha+1}$.

Axiom of Infinity If we associate every set in the hierarchy with with the indexed integer we can note that $V_{\omega+1}$ (the set of every subset of V_ω) contain the set of all natural numbers $\mathbb{N}$ as they were defined above. The axiom of infinity demands for a certain set and we will see that $\mathbb{N}$ will do.

First observe that $\emptyset \in \mathbb{N}$. As we defined ordinal numbers, the other requirement is that the successor of every finite ordinal number will be an finite ordinal number as well. By rules of ordinal computing we however have $\alpha + 1 < \omega$ whenever α is a finite ordinal less than ω and this fact follows.

Axiom of Choice Let $x \in V_\alpha$ and $y \in V_\beta$. We want to find the level of the choice function $f : x \to y$. Since f is a relation it is a subset of the set $x \times y$. The set $x \times y$ consists of ordered pairs (a, b) defined as $\{\{a\}, \{a, b\}\}$ and with $a \in x$ and $b \in y$ [15, p.34]. This means that f will be an ordered pair (a, b) for some sets $a \in x$ and $b \in y$.

As both $\{a\}$ and $\{a, b\}$ are elements in $V_{\max(\alpha,\beta)+1}$ we know that the union (a, b) will be in $V_{\max(\alpha,\beta)+2}$. Since $\max(\alpha, \beta) + 2 < \omega + \omega$ we know that $f \in V_{\omega+\omega}$ and this confirms that the axiom of choice holds in the model.

This finishes the proof that $V_{\omega+\omega}$ is a model of RZC. In fact, of all models of RZC that could be constructed from $\mathcal{V}$ it is the minimal, because we required an infinite set and for the verifications we also required some sets higher in the hierarchy.

Chapter 2

Categories

Category Theory was first presented in 1945 "General Theory of Natural Equivalences" by Samuel Eilenberg and Saunders Mac Lane [12, p.12]. The motivation was to describe transformation between different sections of mathematics. To make such a transformation we use *functors*. Since functors are everywhere in mathematics there is a broad spectrum of applications in algebraic topology, abstract algebra, logic, computer science and more.

2.1 Preliminaries

DEFINITION A **category** is a collection of **objects** (a,b,c,...) and **arrows** (f,g,h,...) with the following operations.

1. *Domain* assigns to each arrow f an object $a = \operatorname{dom} f$
2. *Codomain* assigns to each arrow f an object $b = \operatorname{cod} f$

   We write $f : a \to b$

3. *Composition* assigns to each pair of arrows $f : a \to b$, $g : b \to c$ with $\operatorname{cod} f = \operatorname{dom} g$ a new arrow $g \circ f : a \to c$
4. *Identity* assigns to each object a an arrow $1_a : a \to a$

   We also require:

5. *Unit* For all arrows $f : a \to b$, $g : b \to c$, the identity arrow 1_b works as a unit in regards to composition, i.e. $1_b \circ f = f$ and $g \circ 1_b = g$
6. *Associativity* For all arrows $f : a \to b$, $g : b \to c$ and $k : c \to d$ we have $(k \circ g) \circ f = k \circ (g \circ f)$

Note that in every theory based on categories we take the notions of mappings, domain, codomain and composition for granted similar to how in every theory based on sets we presuppose the elementhood relation.

The definition of a category is very flexible. Naturally, we can consider the objects to be sets and the arrows to be functions, and we will get the category **Set**. On the other hand, as long as the requirements are fulfilled objects and arrows could be anything. More examples of categories includes **Grp**, the category with groups as objects and homomorphisms as arrows, and **Top**, the category with topological spaces as objects and continuous functions as arrows.

We describe properties of categories with diagrams. These diagrams, unless nothing else stated, will be considered *commutative*. This means that for every pair of objects A and B in the diagram, and every pair of well-defined compositions of arrows,

$$f_1 \circ ... \circ f_n : A \to B$$

$$g_1 \circ ... \circ g_m : A \to B$$

We will have,

$$f_1 \circ ... \circ f_n = g_1 \circ ... \circ g_m$$

Definition A **functor** F is a mapping between two categories such that objects are mapped to objects and arrows are mapped to arrows in such a way that F preserves domain, codomain, composition and identity arrows in the following manner [2, p.8-9],

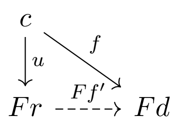
$$F(f : A \to B) = F(f) : F(A) \to F(B)$$

$$F(1_A) = 1_{F(A)}$$

$$F(g \circ f) = F(g) \circ F(f)$$

A functor could map from one category to another, as for instance $\pi : \mathbf{Top}_* \to \mathbf{Grp}$ that takes a topological space (with base point $*$) to its fundamental group. A functor could also be $\mathcal{P} : \mathbf{Set} \to \mathbf{Set}$ that take a set to its power set.

Definition Let $F : D \to C$ be a functor. Let c be an object in C. Then a **universal arrow** is a pair (r, u), where r is an object in D and $u : c \to Fr$ is an arrow in C, such that for every object d in D and arrow $f : c \to Fd$ there exist a unique arrow $f' : r \to d$ such that $Ff' \circ u = f$. We say that the construction have **the universal mapping property** [10, p.55].



This is our first commutative diagram. When we assign the existence of an unique arrow we draw it dashed, as Ff' in this case. Note that F could be a functor from C to itself and this will indeed be the case for all further universal constructions.

From a categorical point of view universal constructions are everywhere in mathematics. By this reason, a formal theory based on categories will require existence of such constructions. Some of these constructions are called **limits**. We would require some further terminology to distinguish limits from other constructions, but it will not be of any importance to do so. However, it will be important that every limit have a corresponding **colimit**. For a limit the colimit is the same construction except that for every arrow the domain and the codomain is interchanged.

Finally, we will need the abstractions of surjective and injective functions of category theory.

Definition An arrow α is an **epimorphism** if for every pair of arrows f, g we have that $f \circ \alpha = g \circ \alpha$ implies $f = g$.

Definition An arrow β is a **monomorphism** if for every pair of arrows f, g we have that $\beta \circ f = \beta \circ g$ implies $f = g$.

2.2 Limits and other universal constructions

In this section we will look upon three limits and their corresponding colimits. As the first axiom of ETCS will be the existence of the following limits we attach every definition with a formula in the language of first order logic. These are formulas that hold in our model when we say limits exists. This is also the case for two more constructions presented in this section, also presented in first order logic.

Definition An object 1 is a **terminal object** if for every object X there exists a unique arrow $h : X \to 1$.

$$\forall X \ \exists! h \ (h : X \to 1)$$

Definition An object 0 is an **initial object** if for every object X there exists a unique arrow $h : 0 \to X$

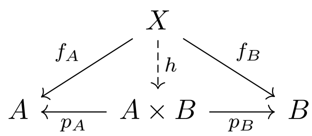
$$\forall X \ \exists! h \ (h : 0 \to X)$$

0 and 1 are unique up to isomorphism by the following argument [2, p.34]. Say we did have two initial objects 0 and $0'$ and let $f : 0 \to 0'$ and $g : 0' \to 0$ be two arrows. Since the arrow $1_0 : 0 \to 0$ must be unique by defintion we know that $g \circ f = 1_0$ and since also the arrow $1_{0'} : 0' \to 0'$ is unique we have

$f \circ g = 1_{0'}$. This implies that 0 is isomorphic to $0'$ (i.e. g is a two-sided inverse to f). An analogous argument shows that 1 is unique up to isomorphism.

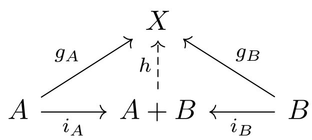
As we will later see (section 3.2), demanding these limits will be crucial steps as we construct set theory in a categorical theory, allowing us to talk about inclusion of elements and more.

Definition A **product** of two objects A and B is an object $A \times B$ together with two arrows p_A and p_B called **projections**. In addition, the projections have the following universal property: for every object X with arrows $f_A : X \to A$ and $f_B : X \to B$ there is a unique arrow h such that the following diagram commutes.



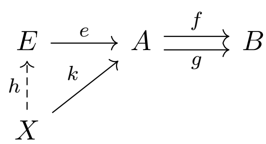
$$\forall X \; \forall f_A \; \forall f_B \; \exists! h \; (p_A \circ h = f_A \wedge p_B \circ h = f_B)$$

Definition A **coproduct** of two objects A and B is an object $A+B$ together with two arrows i_A and i_B called *injections*. In addition, the injections have the following universal property: for every object X with arrows $g_A : A \to X$ and $g_B : B \to X$ there is a unique arrow h such that the following diagram commutes.



$$\forall X \; \forall g_A \; \forall g_B \; \exists! h \; (h \circ i_A = g_A \wedge h \circ i_B = g_B)$$

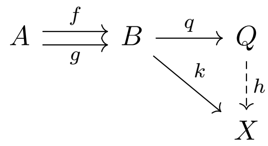
Definition For every pair of parallel arrows $f, g : A \rightrightarrows B$ a **equalizer** is a an object E together with an arrow $e : E \to A$ with the property $f \circ e = g \circ e$. An equalizer also has an universal arrow, such that for every $k : X \to A$ with $f \circ k = g \circ k$ there is a unique arrow $h : X \to E$ such that the following diagram commutes.



$$\forall X \; \forall k \; (f \circ k = g \circ k \to \exists! h \; [k = e \circ h])$$

Definition For every pair of parallel arrows $f, g : A \rightrightarrows B$ a **coequalizer** is an object Q together with an arrow $q : B \to Q$ with the property $q \circ f = q \circ g$.

Again we have an universal arrow, such that for every $k : B \to X$ with $k \circ f = k \circ g$ there is a unique arrow $h : Q \to X$ such that the following diagram commutes.

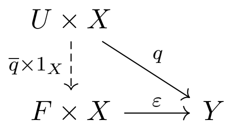


$$\forall X\ \forall k\ (k \circ f = k \circ g \to \exists! h\ [k = h \circ q])$$

There are other limits and colimits than those introduced above, but it will be possible to construct those in ETCS as long as only these limits above exists [8, p.11-12].

We will need two more universal constructions to present ETCS.

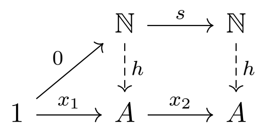
Definition An **exponent** (in a category with products) from an object X to an object Y is an object F (or Y^X) together with an arrow $\varepsilon : F \times X \to Y$. In addition, the exponent have the universal property that for every object U with an arrow $U \times X \to Y$ there is a unique arrow $\bar{q} : U \to F$ such that the following diagram commutes.



$$\forall X\ \forall Y\ \forall q\ \exists! \bar{q}\ (\varepsilon \circ (\bar{q} \times 1_X) = q)$$

An exponent is an abstraction of function sets. In ETCS, every exponent will be such a set, that for some other sets A and B, consists of exactly the mappings $f : A \to B$.

Definition A **natural number object** (NNO) is a triple ($\mathbb{N}$,0,s) with arrows $0 : 1 \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$. In addition, for every pair of arrows with $1 \xrightarrow{x_1} A \xrightarrow{x_2} A$ there is a unique arrow $h : \mathbb{N} \to A$ such that the following diagram commutes.



$$\forall A\ \forall x_1\ \forall x_2\ \exists! h\ (h \circ 0 = x_1 \land h \circ s = x_2 \circ h)$$

Chapter 3

The Formal System ETCS

In a classical set theory we assume we have a universe of sets, we say that certain sets exists and we demand some further requirement. Then we hopefully can prove as many true mathematical statements as possible. Contrast this with ETCS. Now we say we have a category, we say some objects exists and again demand some further requirements. Again we hope that most of mathematics will follow. Even if we have to think about things differently, the idea of an axiomatic system remains intact.

3.1 The axioms of ETCS

The axioms should be thought of as requirements for a fixed category. The model we will use to prove the consistency of ETCS will have sets as objects, but this is not presupposed in the axioms. That is, when we say object it will not necessarily be a set. Nevertheless the axioms uses set-theoretic symbols and notions. This is because we will have a special interpretation of these concepts. In section 3.2 we will explain how this works.

Existence of limits There exist a terminal object 1 and an initial object 0. For all objects A and B there exist a product $A \times B$ and a coproduct $A + B$. For all pair of arrows $A \rightrightarrows B$ there exist an equalizer $e : E \to A$ and a coequalizer $q : B \to Q$.

Exponent For all objects A and B there exists an exponent F with an evaluation arrow $\varepsilon : F \times A \to B$.

Natural Number Object There exists a Natural number object with arrows $0 : 1 \to \mathbb{N}$ and $s : \mathbb{N} \to \mathbb{N}$.

Extensionality If $f, g : A \rightrightarrows B$ and we have that $f \circ a = g \circ a$ holds for every $a \in A$, then we also have $f = g$.

$$\forall f, g : A \to B\ ((\forall a \in A \to f \circ a = g \circ a) \to f = g)$$

Axiom of Choice If $f : A \to B$ is an arrow with nonempty domain there exists an arrow $g : B \to A$, the *quasi-inverse* to f, such that $f \circ g \circ f = f$.

$$\forall f : A \to B\ (A \neq \emptyset \to \exists g : B \to A\ (f \circ g \circ f = f))$$

Unique Empty Object Every object not isomorphic to 0 has elements.

$$\forall A (A \ncong 0 \to \exists a\ (a \in A))$$

Disjoint Union Every element in a coproduct is a member of one but not both its injections. That is, a sum is a disjoint union of its terms.

$$\forall A \forall B\ (x \in A + B \to ((x \mathrel{\dot{\in}} i_A \wedge \neg x \mathrel{\dot{\in}} i_B) \vee (\neg x \mathrel{\dot{\in}} i_A \wedge x \mathrel{\dot{\in}} i_B)))$$

Existence of Bigger Objects There are objects with more than one element.

$$\exists A\ (\exists a\ \exists b\ (a \in A \wedge b \in A \wedge a \neq b))$$

3.2 Understanding the axioms

In this section we will give more motivation how the complete list of axioms will work as a substitute to classical set theory, or rather how to built up mathematics from these axioms.

Since the objects and the arrows will be sets, it is natural to think that the purpose of the first three axioms is to guarantee certain sets.

For a theory with all those constructions still some categories that are too weak will model that theory. This is why we want to add the remaining axioms. For instance in the category of abelian groups products and coproducts may intersect and there are no guarantee that disjoint unions exist [3, p.4-5]. In the category consisting of only objects 0 and 1, and with no other arrows than the identity arrows and an arrow $0 \to 1$ there will be a vast amount of mathematics not doable [8, p.19].

Furthermore, these axioms allow us to make new interpretations of basic concepts of set theory: elementhood, subsets and membership. Now we redefine these notions in our new categorical setting.

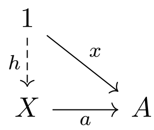
Definition If there exists a terminal object in a category, we define **elementhood** $a \in A$ to be the case if and only if $a : 1 \to A$ is an arrow in the category [8, p.9].

Hence $\in$ is a special case of arrow, unlike in set theory where it is presupposed.

Moreover we see that evaluation $f(x)$ will be a special case of composition of the arrows f and x. Given these definitions of elementhood and evaluation, it will always be possible to understand properties of objects without considering the content of the objects. We have bypassed the problem of defining fundamental relations recursively mentioned in the introduction.

Definition We define a **subset** $a \subseteq A$ as a monomorphism with $\operatorname{cod} a = A$.

Definition For a subset $a \subseteq A$ we say that $x \in A$ is **member** of a, and write $x \dot{\in} a$, if there exists an arrow h such that the following diagram commutes.

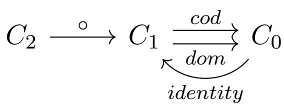


From now on, these well-known concepts will be interpreted as described in this section. The axioms of ETCS above will now be intelligible.

3.3 The Category Neu

Our goal is to show that ETCS is a consistent theory by presenting a model where ETCS holds. Similar to how a model for a classical set theory is a universe of sets, a model for ETCS will be a category with sets as objects and functions between sets as arrows. We will name this category **Neu** as it is really the von Neumann universe thought of as a category instead of as a set universe.

First, inspired by Awodey, note that every category could be visualised as below [2, p.22].



Here C_0 is the objects in the category, C_1 is the arrows, and C_2 is the collection of pairs of arrows (f, g) such that $\operatorname{cod} f = \operatorname{dom} g$. When we have decided what C_0, C_1 and C_2 consists of we have the model we will call **Neu** and we are ready to test the axioms.

We choose C_0 to be the sets in the class $V_{\omega+\omega}$. In the previous chapter we confirmed that pair, cartesian product and union of any sets in $V_{\omega+\omega}$ will also be sets in the universe. We choose C_1 to be the sets of arrows $\langle\langle x, y\rangle, f\rangle$ with $f : x \to y$ as a *function* between sets (we will use pointy brackets when talking about functions, but as sets this is really the same as ordered pairs).

f being a function means that f is a *functional* and *total* relation $f \subseteq x \times y$. For every objects a, b and c we have that f is functional if $\langle a, b\rangle \in f$ and $\langle a, c\rangle \in f$ implies that $b = c$. We say f is total if for every element $a \in x$ there exists a $b \in y$ such that $\langle a, b\rangle \in f$.

We also note that since x, y and f all are sets we have that C_1 is a subset to the collection $(C_0 \times C_0) \times C_0$, which in turn is a subset of $V_{\omega+\omega}$ by the previous argument.

Composition of functions, which is associative, is defined as $\langle\langle y, z\rangle, g\rangle \circ \langle\langle x, y\rangle, f\rangle = \langle\langle x, z\rangle, g \circ f\rangle$. As a set we write the composition of the arrows $f \subseteq x \times y$ and $g \subseteq y \times z$ like below.

$$g \circ f = \{\langle a, c\rangle \in x \times z \mid \exists b \in y \ (\langle a, b\rangle \in f \wedge \langle b, c\rangle \in g)\}$$

Such functions are functional and total. Totality follows from construction. To show functionality, let $\langle a, b\rangle, \langle a, c\rangle \in g \circ f$. By definition of composition, $\langle a, b\rangle \in g \circ f$ implies that there exist a d such that $\langle a, d\rangle \in f$ and $\langle d, b\rangle \in g$. Also, $\langle a, c\rangle \in g \circ f$ implies that there exist a d' such that $\langle a, d'\rangle \in f$ and $\langle d', c\rangle \in g$. As $\langle a, d\rangle \in f$ and $\langle a, d'\rangle \in f$, functionality of f implies that $d = d'$. This means that $\langle d, b\rangle \in g$ and $\langle d, c\rangle \in g$, so by functionality of g we have $b = c$. That is, $g \circ f$ is functional.

Furthermore, we confirm that $C_2 \subseteq C_1 \times C_1$, and this means that this class of arrows also is contained in the von Neumann universe.

We also confirm that **Neu** is indeed a category. The domain of an arrow $\langle\langle x, y\rangle, f\rangle$ is x and the codomain is y and therefore these two required operations are well-defined. We also have identity arrows and unit. All other requirements are already checked.

3.4 Consistency of ETCS

Initial- and terminal object We choose the initial object in **Neu** to be $\emptyset$ and we choose the terminal object to be $\{\emptyset\}$. We will show that these choices satisfies the demanded universal properties.

We first show that for every object X there is a unique arrow $h : 0 \to X$. In **Neu** this means $h \subseteq \emptyset \times X$ but we will often stick to the arrow notation as it is usually how we write functions. However, in this particular case $\emptyset \times X$ is not a pair (there is no function with empty domain) and the only subset to h is $\emptyset$. Hence $h = \emptyset$ and h is unique.

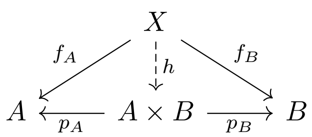
Then, we must show that for every X there also must be a unique arrow $h : X \to 1$, or $h \subseteq X \times 1$. Since h is total we know that for every $x \in X$ there is a $y \in 1$ such that $\langle x, y\rangle \in h$. Since there is only one element $\emptyset$ in 1 we get that by the extensionality axiom in our metatheory RZC that every two functions contained in $X \times 1$ will be the same sets, as they consists of the same elements.

Before we cover the other universal constructions, we want to check if the categorical extensionality axiom holds in **Neu**, since this property will become useful later.

Extensionality Assume that $f, g : A \to B$ and $f(x) = g(x)$ holds for every $x \in A$. Under these conditions, extensionality holds if $f = g$. We want to show that $\langle a, b\rangle \in f$ if and only if $\langle a, b\rangle \in g$ for all $\langle a, b\rangle$. This will imply that $f = g$, by the usual extensionality axiom that belongs to our metatheory.

Let $\langle a, b\rangle \in f$. Since $a \in A$ we know that $f(a) = g(a)$. $f(a)$ is the set of pairs $(\emptyset, b)$ in $1 \times B$ such that $\langle a, b\rangle \in f$ and if $g(a)$ is the same set this implies that also $\langle a, b\rangle \in g$. We get the other direction by the same argument with g instead of f.

Product We will choose the product $A \times B$ to be the set of ordered pairs (a, b) where $a \in A$ and $b \in B$. Let the projections be the functions $p_A(a, b) = a$ and $p_B(a, b) = b$. These sets and functions are well-defined in **Neu**. Given arrows $f_A : X \to A$ and $f_B : X \to B$ we must show that we have an universal arrow $h \subseteq X \times (A \times B)$ such that the following diagram commutes.

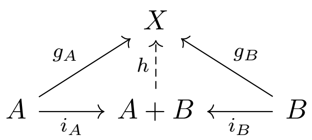


For every $x \in X$ we must have $h(x) = (a, b)$ for some $(a, b) \in A \times B$, but this holds if and only if $a = p_A(a, b) = f_A(x)$ and $b = p_B(a, b) = f_B(x)$, so we can write,

$$h = \{\langle x, (a, b)\rangle \mid x \in X, \langle x, a\rangle \in f_A, \langle x, b\rangle \in \operatorname{cod} f_B\}$$

But this is really just the function $h(x) = (f_A(x), f_B(x))$. Such an arrow must exists and it is unique by construction.

Coproduct We will choose the coproduct $A + B$ to be the set of sets on the form $(a, 0)$ or $(b, 1)$ where $a \in A$ and $b \in B$. We will have $i_A(a) = (a, 0)$ and $i_B(b) = (b, 1)$ as the injections, because then the coproduct will be a disjoint union of A and B (for any two elements $x \in A$ and $y \in B$ we have $i_A(x) \neq i_B(y)$). We want to show that for this coproduct there is a unique arrow h such that the following diagram commutes.



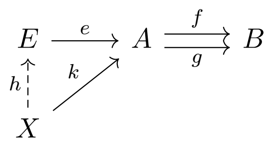
Consider the following function, which clearly is well-defined in **Neu**.

$$h(u, v) = \begin{cases} g_A(u) & v = 0 \\ g_B(u) & v = 1 \end{cases}$$

Working by cases, we note that both when $v = 0$ and $v = 1$ we get a similar situation as in the product verification, since we defined the universal arrow from the arrows $g_A, g_B : A, B \rightrightarrows X$. We have $h(u, 0) = x$ if and only if $g_A(u) = x$ and $h(u, 1) = x$ if and only if $g_B(u) = x$, but this implies directly that h above is the only possible option.

Equalizer Let $f, g : A \rightrightarrows B$ be two parallel arrows between two objects in **Neu**. We choose the equalizer E to be the set of elements a of A such that $f(a) = g(a)$. It is possible from our metatheory to construct such a set by the Axiom of Restricted Comprehension. From this definition of E it is clear that the equalizer arrow $e : E \to A$ is simply an inclusion function.

Assuming that $k : X \to A$ is an arrow such that $f \circ k = g \circ k$ we want to show that there exist a unique function $h \subseteq X \times E$ such that the following diagram commutes.



This function will be k with codomain restricted to E, that is,

$$h = \{\langle x, y\rangle \in X \times E \mid k(x) = e(y)\}$$

This arrow is indeed a function. To show this, we have to show that h is functional and total. We will use that every inclusion function is injective. If we have $\langle x, y_1\rangle \in h$ and $\langle x, y_2\rangle \in h$ we have both $k(x) = e(y_1)$ and $k(x) = e(y_2)$ so since e is injective we get $y_1 = y_2$ and h is functional.

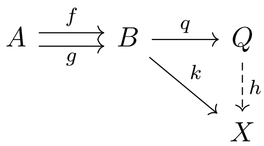
Now pick one $x \in X$ and name $k(x) = y \in E$. e is an inclusion arrow so $e(y) = y$. Since $k(x) = e(y)$ we know it exists $y \in E$ such that $\langle x, y\rangle \in h$, so h is total.

To show that h is unique we will again use that e is injective. If we have two different h_1 and h_2 satisfying $e(h_1(x)) = e(h_2(x))$ for some $x \in X$ we get from this property that $h_1(x) = h_2(x)$. This is true for every x so by extensionality we get $h_1 = h_2$. Note that we also proved that e was a monomorphism.

Coequalizer To show the existence of coequalizer $q : B \to Q$, start by defining an equivalence relation by $\sim := \cap \{\tilde{\sim} \subseteq B \times B \mid \tilde{\sim}$ is an equivalence relation such that $f(a) \,\tilde{\sim}\, g(a),\ \forall a \in A\}$. We use that the intersection of several equivalence relations is an equivalence relation as well. Also note that $\sim$ is not empty, because the equivalence relation that identifies every element in B to every other element will be in the intersection.

The definition of $\sim$ guarantees that it is the smallest equivalence relation such that $f(a) \sim g(a)$. That is, small in the sense that most elements neither in the image of f nor g will be identified with other elements.

We further define $Q = \{[b]_\sim \mid b \in B\}$ as the set of equivalence classes of this relation. Then q will map every element of B to its corresponding equivalence class. For this construction we want to show that there is an universal arrow.



Recall that k by definition has the property that $k \circ f = k \circ g$. From this we know that $k(f(a)) = k(g(a))$ for every $a \in A$. Since $f(a) \sim g(a)$ we know that two elements in the same equivalence class will never be mapped to different values in X. This means that h could be k with domain Q.

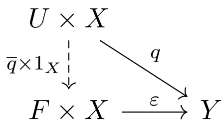
$$h = \{\langle y, x\rangle \in Q \times X \mid k(q^{-1}(y)) = x\}$$

h is functional because every x is uniquely determined by $k(q^{-1}(y))$ and it is total simply because $k(q^{-1}(y))$ takes a value for every y.

It is clear by the definition that q is a surjective and we will use that to show that h is unique. First assume that $h_1 \circ q = h_2 \circ q$. Since q is surjective we know that for every y there is a x such that $y = q(x)$. We now get that $h_1(y) = h_1(q(x)) = h_2(q(x)) = h_2(y)$ and since consequently $h_1(y) = h_2(y)$ holds for every y we get by extensionality that $h_1 = h_2$. In addition we showed that q is an epimorphism.

We must say something about the existence of equivalence relations in **Neu**. This will get a special meaning when working in a category, see Lawvere [8, p.25-26]. B under the equivalence relation $\sim$ will be a subset of $B \times B$, but we will also need characteristic functions that partition sets into subsets that do or do not satisfies properties. We can meet this need in a category with a *subobject classifier*. Such a construction will be derivable from the other axioms of ETCS, see Goldblatt [5, p.79-82].

Exponent For every two objects X and Y the exponent in **Neu** will be a set F consisting of functions $f \subseteq X \times Y$. These functions are subsets of $\mathcal{P}(X \times Y)$ and since this is a set in $V_{\omega+\omega}$ we know that F is an object in **Neu**. We let the evaluation map ε be a function that takes pair sets (f, x) to the composition $f(x)$. Now we want to show that we have an universal arrow $\bar{q}$ such that the following diagram commutes.



We want $\bar{q}$ to be like q except that its domain is restricted to U. We define,

$$\bar{q}(u) = q(u, -)$$

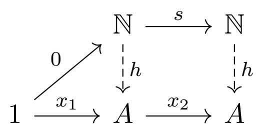
Take an arbitrary $u \in U$ and $x \in X$. This choice of $\bar{q}$ will imply that,

$$(\bar{q} \times 1_X)(u, x) = (\bar{q}(u), x) = (q(u, -), x)$$

By the definition of ε we moreover have $\varepsilon(q(u, -), x) = q(u, x)$ and this means $\varepsilon \circ (\bar{q} \times 1_X) = q$ as required.

This is also the only arrow of this kind. Assume $\widetilde{q} : U \to F$ is another arrow such that $\varepsilon \circ (\widetilde{q} \times 1_X) = q$. Then we would have $\varepsilon(\widetilde{q}(u, -), x) = q(u, x)$ and moreover $\widetilde{q}(u, x) = q(u, x)$, which implies that $\bar{q} = \widetilde{q}$.

Natural Number Object Recall that a NNO is a triple $(\mathbb{N}, 0, s)$. Let $\mathbb{N}$ be the finite ordinal numbers. Define $s(n) = n \cup \{n\}$, which will be well-defined in **Neu**. Assuming there are arrows $x_1 : 1 \to A$ and $x_2 : A \to A$, we want to show that there is a unique arrow h such that the following diagram commutes.



First we will show that for every $n \in \mathbb{N}$ there exists an arrow h_n with $\operatorname{dom} h_n = \{k \mid k \leq n\}$ and $\operatorname{cod} h_n = A$ such that the diagram above commutes for that particular element of $\mathbb{N}$. We will show by induction that the following recursive function qualifies.

$$h_n(m) = \begin{cases} x_1 & m = 0 \\ x_2(h_{n-1}(m)) & \text{otherwise} \end{cases}$$

Let $x_1, x_2 \in A$. We will first do the base case, confirming that $h_0(0) = x_1$. All we need to check is that $\langle 0, x_1 \rangle$ is in h_0, and this is clear by the definition of h_0.

Assume now as induction hypothesis that there is some natural number n such that $h_n(0) = x_1$ and $h_n(s) = x_2(h_n)$. We will use this to show that $h_{n+1}(0) = x_1$ and $h_{n+1}(s) = x_2(h_{n+1})$. We first observe that $\langle 0, x_1 \rangle \in h_{n+1}$, so we know that $h_{n+1}(0) = x_1$.

Let $m \in \mathbb{N}$ and $a, a' \in A$. If we assume $\langle m, a \rangle \in h_{n+1} \circ s$ we get that there exists a u such that $\langle m, u \rangle \in s$ and $\langle u, a \rangle \in h_{n+1}$. Using the definition of h and the induction hypthesis we have,

$$a = h_{n+1}(u) = h_{n+1}(s(m)) = x_2(h_n(s(m))) = x_2(x_2(h_n(m)))$$

If we assume $\langle m, a' \rangle \in x_2 \circ h_{n+1}$ we get that there exists a v such that $\langle m, v \rangle \in h_{n+1}$ and $\langle v, a' \rangle \in x_2$. Using the definition of h we have,

$$a' = x_2(v) = x_2(h_{n+1}(m)) = x_2(x_2(h_n(m)))$$

We see that $a = a'$, which implies that $h_{n+1}(s) = x_2(h_{n+1})$. By induction, h_n will make the diagram above commute for every n.

Next thing is to show that h_n also is unique for every n. Assume that h'_n is another function satisfying the requirements. We will show that $h_n = h'_n$ for every n, again by using induction. The base case follows directly, if $h_0(0) = x_1$ and $h'_0(0) = x_1$ then by extensionality $h_0 = h'_0$.

As induction hypothesis we assume there are some $n \in \mathbb{N}$ such that $h_n = h'_n$ and we want to show that $h_{n+1} = h'_{n+1}$. Assume that $h_{n+1}(m) = a_m$ and $h_n(m+1) = a_{m+1}$. This holds if and only if $\langle m, a_{m+1} \rangle \in h_n \circ s$. By the commutativity property this means $\langle m, a_{m+1} \rangle \in x_2 \circ h_n$. Hence, there exists a $y \in A$ such that $h_n(m) = y$ and $x_2(y) = a_{m+1}$. It follows that $y = a_m$. It did not matter if we did this for h or h' and therefore,

$$h_{n+1}(m) = a_m \Leftrightarrow h_n(m) = a_m$$

$$h'_{n+1}(m) = a_m \Leftrightarrow h'_n(m) = a_m$$

Using the induction hypothesis we then get for every m,

$$h_{n+1}(m) = a_m \Leftrightarrow h'_{n+1}(m) = a_m$$

So $h_{n+1} = h'_{n+1}$ by extensionality and therefore, by induction, $h_n = h'_n$ for every n.

Finally, now we can use these functions to define the universal arrow we wanted,

$$h = \cup_{n \in \mathbb{N}} h_n : \mathbb{N} \to A$$

By construction, this function is unique and makes the diagram above commute.

Axiom of Choice Let $f : X \to Y$ be a function with nonempty domain. We want to show that there is a $g : Y \to X$ such that $f \circ g \circ f = f$. We will use a alternative version of the set theoretic version of axiom of choice than the one presented in chapter 1. Our choice of metatheory allows us to use this version of the axiom. It states that for every $y \in Y$ there exists a choice function $g : Y \to \cup_{y \in Y} H(y)$ with $g(y) \in H(y)$, as long as every $H(y) \neq \emptyset$. Here H is a function that we define, for some $x_0 \in X$, as below.

$$H(y) = \begin{cases} \{x_0\} & f^{-1}(y) = \emptyset \\ f^{-1}(y) & f^{-1}(y) \neq \emptyset \end{cases}$$

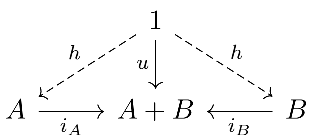
Hence $H(y) \neq \emptyset$ and the function g exist by the axiom. To show that g satisfies the requirements for the categorical axiom of choice, note that $g(y) \in f^{-1}(y)$ for every y, and that this implies that $f(g(y)) = y$. Now choose $x \in f^{-1}(y)$ (this also means $f(x) = y$) and compute:

$$(f \circ g \circ f)(x) = f(g(f(x))) = f(g(y)) = y = f(x)$$

In other words, the requirements are fulfilled.

Unique Empty Object As a set is empty if and only if it is the empty set, indeed 0 will be the only empty object in **Neu**.

Disjoint Union If we let $u \in A + B$ this axiom requires that either $u \dot{\in} i_A$ or $u \dot{\in} i_B$. Recall that this means that the following diagram commutes for some h, with the injections i_A and i_B being monomorphisms.



To prove this we take a detour to some results by Lawvere. He shows that i_A and i_B has no members in common, so if we show that $u \dot{\in} i_A$ it will follow that $u \dot{\notin} i_B$, and the object of a coproduct is really a disjoint union [8, p.20].

It is clear by construction of coproducts in **Neu** that every element of $A + B$ is a element of A or B, so we can suppose that the arrow h exist. We only have to verify that the injections are monomorphisms.

To show this, we will use the exponent axiom. Lawvere has shown that in a category with exponents the distributivity relation below holds [9, p.126-129].

$$A \times X_1 + A \times X_2 \cong A \times (X_1 + X_2)$$

The other part of the proof is that in a distributive category injections are monomorphisms [4, p.153]. We skip to repeat these cited verifications but conclude that the Disjoint Union Axiom hold.

Existence of Bigger Objects As all sets in von Neumann's universe up until $V_{\omega+\omega}$ are objects in **Neu**, certainly there are objects with more than one element.

Chapter 4

Further comments

4.1 The relation between the two theories

By the arguments in the previous chapter von Neumann's universe constitute a model of ETCS, and this in particular shows that this theory is consistent. We have used RZC barely as means to reach this goal, but in fact more could be said about the relation between RZC and ETCS. These two theories are **equiconsistent**, or inter-interpretable, meaning that one of the two is consistent if and only if the other one will is consistent.

As we have already proved that consistency of ETCS provided that RZC is consistent, it would be natural to do the opposite to prove the statement above. However, Mac Lane and Moerdijk have a proof that covers both directions at once [11, p.331-343]. They make a *mutual interpretation* of RZC and ETCS. The idea is to think a set x as a tree with itself as its root, and then consider the children of x as elements of the set x. We will not explain further but in this way the authors interpret both theories at the same time. This implies that the theories are equiconsistent.

In fact, they do not work with categories fulfilling the ETCS axioms as we have presented it in this text, but with a *well-pointed topos*. This is a category with limits, exponents, NNO, subobject classifier (mentioned on page 21), endowed with extensionality and choice. The reason why the argument by Mac Lane and Moerdijk applies for ETCS is that it can be showed that a category fulfills the conditions for being a well-pointed topos if and only if it satisfies the axioms of ETCS.

As we now see that RZC and ETCS are equally strong theories it is not obvious which one is the better. It all boils down to the philosophical issue on the ontology of mathematics.

4.2 Philosophically speaking

In this paper a set theory based on categories have been presented and proven to be consistent. To some readers it still may not seem obvious why such a set theory would be relevant, especially when it turned out that this theory have the same strength as RZC.

One reason to prefer categorical set theory is that you are a *structuralist*. In that case you believe every mathematical theory describes structures between mathematical objects that in themselves lacks internal properties. In a category we have such a structure that has no such internal properties of objects. This is a difference from classical set theory.

As we have seen the sets of categorical set theory are not like sets in other set theories. Set theories based on recursive definitions, as described in the introduction, regards sets merely as codings of mathematical objects. For instance, in chapter 1 we defined 0 as $\emptyset$. However, $\emptyset$ is still $\emptyset$ and not zero. In ETCS, on the other hand, the arrow $0 : 1 \to \mathbb{N}$ should be thought of as zero. If something is isomorphic to zero, it is isomorphic to this particular arrow [13, p.495-496].

Non-structuralists could argue that nothing in a category really exists in the common use of the word [1, p.10]. If not the idea of categories is rooted somewhere, but are independent from everything else, have we not neglected an important ontological question? Indeed, we must actually deal with such questions when thinking about foundations.

Those who thinks this is an issue must however admit that, ontologically, neither are sets any straight-forward concept. But structuralists would even claim that the lack of demands for existence is a benefit of ETCS, because in this way the theory is applicable to many structures instead of only one [14, p.42].

Moreover, even if you not regard the essence of mathematics to be structures, you still may consider a set theory like ETCS to be legit. Even if you think something else lies behind structures, you still may think that a theory based on categories better describes how mathematics is built up.

Another argument to work with ETCS rather than RZC or ZFC is that the latter theories includes all the structural properties that ETCS includes, but they also include other properties. There are no model of ZFC that only includes the axioms that all possible ZFC model shares [13, p.493-497]. A model for a categorical set theory however, will always be in this way, since sets only have structural properties.

In sum, categorical set theory may or may not be closer to your philosophical convictions. Some may argue that categories are more like how we think about mathematics, but some may argue that it says little or nothing about the contents of mathematics. However, ETCS will not make us poorer as it will give us all the mathematics that we need.

Bibliography

[1] S. Awodey. An answer to G. Hellman's question: "Does category theory provide a framework for mathematical structuralism?" [Philos. Math. (3) **11** (2003), no. 2, 129–157]. *Philos. Math. (3)*, 12(1):54–64, 2004. URL `https://doi.org/10.1093/philmat/12.1.54`.

[2] S. Awodey. *Category theory*, volume 52 of *Oxford Logic Guides*. Oxford University Press, Oxford, second edition, 2010.

[3] F. Borceux. *Handbook of Categorical Algebra / Volume 2. Categories and Structures*. Cambridge University Press, Cambridge, 1995.

[4] A. Carboni, S. Lack, and R. Walters. Introduction to extensive and distributive categories. *Journal of Pure and Applied Algebra*, 84:145–158, 1993. URL `https://core.ac.uk/download/pdf/81979461.pdf`.

[5] R. Goldblatt. *Topoi, the categorial analysis of logic*. North-Holland, Amsterdam, rev. ed edition, 1984.

[6] T. Jech. *Set theory*. Acad. P., New York, 1978.

[7] A. Joyal and I. Moerdijk. *Algebraic Set Theory*. Cambridge University Press, Cambridge, 1995.

[8] F. W. Lawvere. An elementary theory of the category of sets (long version) with commentary. *Repr. Theory Appl. Categ.*, (11):1–35, 2005. Reprinted and expanded from Proc. Nat. Acad. Sci. U.S.A. **52** (1964), With comments by the author and Colin McLarty.

[9] F. W. Lawvere and R. Rosebrugh. *Sets for mathematics*. Cambridge Univ. Press, Cambridge, 2003.

[10] S. Mac Lane. *Categories for the working mathematician*, volume 5 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1998.

[11] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic*. Universitext. Springer-Verlag, New York, 1994. A first introduction to topos theory, Corrected reprint of the 1992 edition.

[12] J.-P. Marquis and G. E. Reyes. The history of categorical logic: 1963–1977. In *Sets and extensions in the twentieth century*, volume 6 of *Handb. Hist. Log.*, pages 689–800. Elsevier/North-Holland, Amsterdam, 2012. URL `https://doi.org/10.1016/B978-0-444-51621-3.50010-4`.

[13] C. McLarty. Numbers can be just what they have to. *Noûs*, 27(4): 487–498, 1993. URL `https://doi.org/10.2307/2215789`.

[14] C. McLarty. Exploring categorical structuralism. *Philos. Math. (3)*, 12 (1):37–53, 2004. URL `https://doi.org/10.1093/philmat/12.1.37`.

[15] Y. Moschovakis. *Notes on set theory*. Undergraduate Texts in Mathematics. Springer, New York, second edition, 2006.