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Braid Groups and Configuration Spaces

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Abstract

In this thesis we show that the Artin braid group is isomorphic to the fundamental group of the configuration space of the Euclidean plane. We give enough group theory to define the braid groups as well as some of its subgroups. We then define the homotopy groups and fiber bundles, and show that fiber bundles induce a long exact sequence of homotopy groups. After defining the configuration space of a topological space, we show that a certain map between configuration spaces is a fiber bundle, and we then use the long exact sequence of homotopy groups along with the results about the braid groups to prove the main theorem.

We end with a brief discussion about another result we conclude using this fiber bundle, namely that the configuration space of the Euclidean plane is a classifying space of the Artin braid group.

Contents

1 Introduction

The goal of this text is to show that the *Artin braid group* on n strings, B_n, is isomorphic to the fundamental group of the n:th configuration space of the plane $\mathbb{R}^2$; concepts which we will introduce in Section 2 and Section 3 respectively.

The notion of a braid was first introduced by Emil Artin in the 1920s to formalize intertwining of strings, hence the name. He pointed out that braids with a fixed number of strings form a group.

To get the most out of this text, the reader would benefit from having taken a course in topology where concepts such as the fundamental group is being covered, as well as being familiar with group theory. The content of Section 2 and 3 are mostly from [2] and [3], with the details filled out by us. The part about the Seifert-Van Kampen theorem in Section 3 comes mainly from [5]. Section 4 is mostly from [1], with some modifications.

In Section 2 we will try to get as much as possible out of the group theory part of the paper. We start by defining the free group and group presentations to get a nice way of describing the *Artin braid group* B_n. We then define two subgroups of the Artin braid group - one of them being the *pure braid group* P_n, and the other one the kernel of a certain homomorphism $f_n : P_n \to P_{n-1}$ denoted U_n. We then go on to show how the generators of U_n will tell us more about P_n, with proofs left for Section 4. We end by proving the *five lemma* which will be used in later sections.

Moving to Section 3 we will start by giving some of the properties of covering spaces, and the define the higher homotopy groups $\pi_n(X, x_0)$ for a topological space X with base point $x_0 \in X$, which are higher dimensional analogues of the fundamental group of a topological space. Then we compute the homotopy groups of the wedge sum of n circles, using mainly the Seifert-Van Kampen theorem. We then define a *fiber bundle* and show that that construction gives us a long exact sequence of homotopy groups, which we will make use of in Section 4 after we define the *configuration space* $\mathscr{F}_n(X)$ of a topological space X and show that a certain map between to such spaces is a fiber bundle.

Section 4 will be dedicated to the main proof of the text. We piece together the two previous sections to show that the Artin braid group B_n is isomorphic to the fundamental group of the unordered configuration space of $\mathbb{R}^2$, $\pi_1(\mathscr{C}_n(\mathbb{R}^2))$. We will start by showing some of the properties of configuration spaces resulting from the theory presented in Section 3, and in particular the long exact sequence of homotopy groups.

The very brief Section 5 will be dedicated to show that the homotopy groups of the n:th configuration space of $\mathbb{R}^2$, $\pi_k(\mathscr{C}_n(\mathbb{R}^2))$, all vanish for $k \geq 2$.

2 Artin Braid Group

In this section we define the Artin braid group, and mention some of its properties as well. We will also prove the five lemma which will be used in Section 4.

Definition 2.1. Let G,H be groups. The *free product* of G and H, denoted $G*H$, is the set of all finite sequences of the form

$$a_1 * a_2 * a_3 * \dots * a_n$$

where a_i is an element of either G or H for all i, subject to the following relations:

$$\dots * a_i * 1 * a_j * \dots = \dots * a_i * a_j * \dots$$
$$\dots * a_i * g_1 * g_2 * a_j * \dots = \dots * a_i * (g_1 g_2) * a_j * \dots,$$

and similarly for elements of H. This set forms a group with $*$ as the operation, in the sense that

$$(a_1 * \dots * a_n) * (b_1 * \dots * b_m) = a_1 * \dots * a_n * b_1 * \dots * b_m.$$

We will now define the free group. To do this we begin by defining a free group on one generator. Let S be a set and $\sigma \in S$. The *free group generated by* σ is the set $\{\sigma\} \times \mathbb{Z}$ with the operation defined by $(\sigma,n)(\sigma,m) = (\sigma,n+m)$. We abbreviate (σ,n) as σ^n. The *free group generated by S* is the group

$$F(S) = \mathop{\Huge *}_{\sigma \in S} F(\sigma).$$

Proposition 2.1 (Universal property of free groups)**.** *Let G be a group and S a set. Let $\varphi : S \to G$ be a function from S to the underlying set of G. Then there is a unique group homomorphism $\Phi : F(S) \to G$ such that the following diagram*

$$\begin{tikzcd} S \arrow[r] \arrow[dr, "\varphi"] & F(S) \arrow[d, dashed, "\Phi"] \\ & G \end{tikzcd}$$

commutes. The horizontal arrow is just the inclusion of S.

Definition 2.2. Let S be a set and R be a set of elements of $F(S)$. We define a *group presentation* as

$$\langle S | R \rangle = F(S)/\bar{R},$$

where $\bar{R}$ is the smallest normal subgroup of $F(S)$ containing R, in the sense that if we have a normal subgroup T of $F(S)$ such that $R \subset T \subset \bar{R}$, then we must have that $T = \bar{R}$. We usually call S the *generators*, and R the *relations*.

For example, the presentation $\langle \sigma \mid \varnothing \rangle = F(\sigma) \cong \mathbb{Z}$ and $\langle \sigma_1, \sigma_2 \mid [\sigma_1, \sigma_2] \rangle \cong \mathbb{Z}^2$, where the bracket denotes the commutator $[x,y] = xyx^{-1}y^{-1}$.

To define a homomorphism φ from a presentation $\langle S|R\rangle$ to a group G is the same as defining a homomorphism $\tilde{\varphi}$ from $F(S)$ to G such that $\tilde{\varphi}(r) = 1$ for all relations $r \in R$. We can summarize this property in the following diagram

$$\begin{tikzcd} F(S) \arrow[r, "\tilde{\varphi}"] \arrow[d] & G \\ F(S)/R \arrow[ur, dashed, "\varphi"] & \end{tikzcd}$$

where we factor $\tilde{\varphi}$ through the kernel.

Definition 2.3. The *Artin braid group on n strings*, B_n, is the group generated by the $n-1$ generators $\sigma_1,...,\sigma_{n-1}$ and the relations

$$\sigma_i\sigma_j = \sigma_j\sigma_i$$

for all $i, j = 1,2,...,n-1$ with $|i-j| \geq 2$, and

$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

for all $i, j = 1,2,...,n-2$.

These relations are usually referred to as the *"braid relations"*. We will usually just call the group the *braid group* on n strings, and the elements *braids*.

The group B_1 is the trivial group, and for B_2 we get an infinite cyclic group isomorphic to $\mathbb{Z}$.

If we add the relation $\sigma_i^2 = \sigma_i$ to the braid relations, we get a presentation for the symmetric group S_n, where the σ_i's correspond to the transpositions of the form $(i, i+1)$. It is well known that every permutation can be written as a product of such transpositions, and one can check that they indeed satisfy the relations.

A nice way to visualize these braids is to see them as intertwined strings like in Figure 1. We can think of the generators σ_i as twisting the i:th and $(i+1)$:th strings, like in Figure 2. The composition of two braids can be seen as placing one braid above the other and tying together the strings, like in Figure 5 further bellow. With this in mind, we can think of the permutations in S_n as braids, but where twisting two string clockwise gives the same braid as twisting anti-clockwise.

We will now define two important subgroups of B_n, the first one being the *pure braid group*, and the second a subgroup of that. These subgroups will tell us more about B_n and the exact reason will become apparent later in Section 4. We start with the following lemma.

Lemma 2.2. *If $s_1,...,s_{n-1}$ are elements of a group G that satisfy the braid relations, then there is a unique homomorphism $f : B_n \to G$ such that $f(\sigma_i) = s_i$ for all $i = 1,...,n-1$.*

Proof. Let $F(S)$ be the free group generated by $S = \{\sigma_1,...,\sigma_{n-1}\}$. By the universal property of free groups there exists a unique homomorphism $\tilde{f} : F(S) \to G$ such that

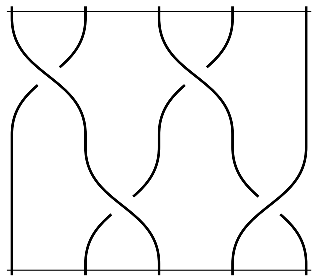

Figure 1: Braid in B_5

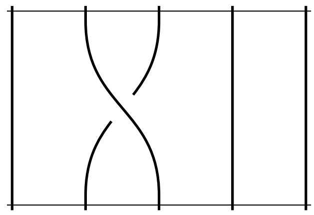

Figure 2: The element σ_2 in B_5

$\tilde{f}(\sigma_i) = s_i$ for all $i = 1,...,n-1$. Since G is assumed to satisfy the braid relations we get

$$\tilde{f}(\sigma_i\sigma_j) = \tilde{f}(\sigma_i)\tilde{f}(\sigma_j) = s_is_j = s_js_i = \tilde{f}(\sigma_j\sigma_i)$$

for $|i-j| \geq 2$, and

$$\begin{aligned}\tilde{f}(\sigma_i\sigma_{i+1}\sigma_i) &= \tilde{f}(\sigma_i)\tilde{f}(\sigma_{i+1})\tilde{f}(\sigma_i) = s_is_{i+1}s_i = s_{i+1}s_is_{i+1} = \tilde{f}(\sigma_{i+1})\tilde{f}(\sigma_i)\tilde{f}(\sigma_{i+1})\\ &= \tilde{f}(\sigma_{i+1}\sigma_i\sigma_{i+1}),\end{aligned}$$

so $\tilde{f}$ induces a homomorphism $f : B_n \to G$, provided that the braid relations get mapped to the identity.

□

In particular, if we pick $G = S_n$ where S_n is the symmetric group, since the transpositions $(i,i+1) \in S_n$ satisfy the braid relations, we have a unique homomorphism $\pi : B_n \to S_n$ such that $\pi(\sigma_i) = (i,i+1)$ for all $i = 1,...,n-1$, and since the transpositions generate the symmetric group, this homomorphism is surjective. We call the kernel of this homomorphism, $\ker(\pi : B_n \to S_n)$, the *pure braid group* on n strings, denoted P_n.

Proposition 2.3. *Define for* $1 \leq i < j \leq n$ *and for generators* $\sigma_1,...\sigma_{n-1} \in B_n$

$$A_{i,j} = \sigma_{j-1}\sigma_{j-2}...\sigma_{i+1}\sigma_i^2\sigma_{i+1}^{-1}...\sigma_{j-2}^{-1}\sigma_{j-1}^{-1}.$$

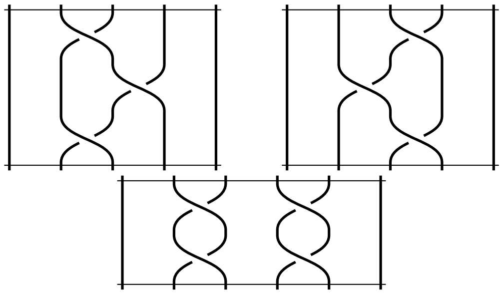

Figure 3: Braid relations

The elements $A_{i,j}$ generate P_n, with relations

$$A_{r,s}^{-1}A_{i,j}A_{r,s} = \begin{cases} A_{i,j} & \text{if } s < i \text{ or } i < r < s < j \\ A_{r,j}A_{i,j}A_{r,j}^{-1} & \text{if } s = i \\ A_{r,j}A_{s,j}A_{i,j}A_{s,j}^{-1}A_{r,j}^{-1} & \text{if } i = r < s < j \\ [A_{r,j},A_{s,j}]A_{i,j}[A_{s,j},A_{r,j}] & \text{if } r < i < s < j. \end{cases}$$

Proof. See [1]. □

For a picture of the generators of P_n, see Figure 4. The pure braids will the braids where the start and endpoint of each string lies on a vertical line.

The next subgroup will be defined as the kernel of the forgetting homomorphism $f_n : P_n \to P_{n-1}$ which "forgets" about the n:th string of a braid in P_n. We define f_n as follows:

$$f_n(A_{i,j}) = \begin{cases} A_{i,j} & j < n \\ 1 & j = n. \end{cases}$$

It is fairly easy to see that this is well defined. What we do is examine the relations for P_n above and let one of the letters i, j, r, s be equal to n, and then check that they indeed get mapped to the same element. For example, for the first relation

$$A_{r,s}^{-1}A_{i,j}A_{r,s} = A_{i,j} \quad \text{if } s < i \text{ or } i < r < s < j,$$

the only case we have to check is when $j = n$. So we get

$$f_n(A_{r,s}^{-1}A_{i,n}A_{r,s}) = A_{r,s}^{-1}1A_{r,s} = 1,$$

and

$$f_n(A_{i,j}) = 1,$$

and similarly for the other relations.

See Figure 4 for a visualization of f_n.

We denote the subgroup by $U_n = \ker(f_n : P_n \to P_{n-1})$.

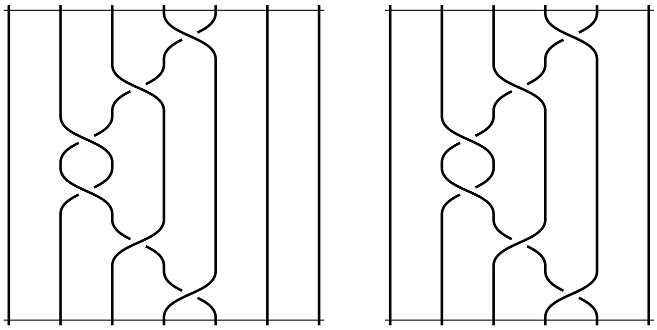

Figure 4: $f_7 : P_7 \ni A_{2,5} \mapsto A_{2,5} \in P_6$

Definition 2.4. Let $\{G_i\}_{i\in\mathbb{Z}}$ be a family of groups and $\{d_i : G_i \to G_{i+1}\}_{i\in\mathbb{Z}}$ a family of group homomorphisms. A sequence

$$\dots \xrightarrow{d_{i-1}} G_i \xrightarrow{d_i} G_{i+1} \xrightarrow{d_{i+1}} G_{i+2} \xrightarrow{d_{i-2}} \dots$$

is said to be *exact* if for all i, $im(d_i) = ker(d_{i+1})$.

In particular, for a short exact sequence

$$1 \longrightarrow G \xrightarrow{f} H \xrightarrow{g} K \longrightarrow 1,$$

where 1 denotes the trivial group, we get that f has to be injective and g surjective.

Definition 2.5. Let G be a group. If H is a subgroup and N a normal subgroup such that $H \cap N = \{1\}$ and $G = NH$. We then say that G is the *semidirect product* of N and H, written $G = N \rtimes H$.

Proposition 2.4. *Let*

$$1 \longrightarrow K \xrightarrow{f} G \xrightarrow{g} H \longrightarrow 1$$

be a short exact sequence of groups. If g has a section, i.e. there exists a homomorphism $s : H \to G$ such that $g \circ s = id_H$, then $G = im(f) \rtimes im(s)$.

Proof. Firstly, since the sequence is exact $im(f) = ker(g)$, so $im(f)$ is a normal subgroup.

We now want to show that if $x \in G$ then there exists $y \in im(s)$ and $z \in im(f)$ such that $x = yz$. Since g is surjective, $im(s) = im(s \circ g)$. Let $y = s(g(x))$ and let $z = xy^{-1}$. We will now show that $z \in im(f)$.

$$\begin{aligned} g(z) = g(xy^{-1}) = g(x)g(y^{-1}) = g(x)g(y)^{-1} = g(x)(g(s(g(x))))^{-1} = \\ = g(x)id_H(g(x))^{-1} = g(x)g(x)^{-1} = 1, \end{aligned}$$

so $z \in ker(g) = im(f)$, $y \in im(s)$ and since $x = yz$, we have that $G = im(f)im(s)$.

Now we need to show that $im(f) \cap im(s) = \{1\}$. Let $x \in im(f) \cap im(s)$. Then $x = f(k) = s(h)$ for some $k \in K$ and $h \in H$. Since $im(f) = ker(g)$ we get

$$1 = g(f(k)) = g(s(h)) = id_H(h) = h,$$

so

$$x = f(k) = s(h) = s(1) = 1$$

which shows that $im(f) \cap im(s) = \{1\}$. □

We have the inclusion $\iota : B_{n-1} \to B_n$ defined by $\iota(\sigma_i) = \sigma_i$. From the braid relations, it is clear that this defines a homomorphism. In particular, if we restrict this map to the pure braid group P_{n-1} we have a homomorphism $P_{n-1} \xrightarrow{\iota} P_n$. This particular restriction turns out to be rather important, and we will make use of it in Section 4 for the main proof of the text.

Lemma 2.5. *Since the sequence $1 \longrightarrow U_n \longrightarrow P_n \xrightarrow{f_n} P_{n-1} \longrightarrow 1$ is exact, and f_n has a section $\iota : P_{n-1} \to P_n$ which is just the inclusion map, we can write $P_n = U_n \rtimes P_{n-1}$. Every $\beta \in P_n$ can be expanded uniquely as*

$$\beta = \iota(\beta')\beta_n,$$

where $\beta' \in P_{n-1}$ and $\beta_n \in U_n$. Here $\beta' = f_n(\beta)$ and $\beta_n = \iota(\beta')^{-1}\beta$. We can see that $\beta' \in im(\iota)$ since f_n is surjective, and $\beta_n \in U_n$ since $f_n(\iota(\beta')^{-1}\beta) = f_n(\iota(\beta')^{-1})f_n(\beta) = \beta'^{-1}\beta' = 1$. Applying this expansion inductively, we can conclude that every pure braid $\beta \in P_n$ can be written uniquely as

$$\beta = \beta_2\beta_3...\beta_n,$$

for $\beta_j \in U_j \subset P_j \subset P_n$, $j = 2, 3, ..., n$.

Theorem 2.6. *The group U_n is free on the generators $\{A_{i,n}\}_{i=1,2,...,n-1}$.*

A proof of this theorem will be given in Section 4.

We will now prove the *five lemma* which will be used later in the main proof of the text.

Lemma 2.7 (Five lemma)**.** *Consider the commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_1 & \xrightarrow{g_1} & G_2 & \xrightarrow{g_2} & G_3 & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle\varphi_1} & & \downarrow{\scriptstyle\varphi_2} & & \downarrow{\scriptstyle\varphi_3} & & \\
1 & \longrightarrow & H_1 & \xrightarrow{h_1} & H_2 & \xrightarrow{h_2} & H_3 & \longrightarrow & 1
\end{array}$$

of groups, with rows exact. If φ_1 and φ_3 are isomorphisms, then so is φ_2.

Proof. We start by showing φ_2 is surjective. Let x be and element in H_2. Since φ_3 is an isomorphism there exists a $y \in G_3$ such that $\varphi_3(y) = h_2(x)$, and since the rows are exact there exists a $z \in G_2$ such that $g_2(z) = y$. Now by commutativity of the right square we get

$$h_2(x) = \varphi_3(g_2(z)) = h_2(\varphi_2(z))$$

which implies

$$h_2(\varphi_2(z)x^{-1}) = 1$$

By exactness of the bottom row we get that $\varphi_2(z)x^{-1} \in im(h_1)$ so there exists an $a \in H_1$ such that $h_1(a) = \varphi_2(z)x^{-1}$, and since φ_1 is an isomorphism there exists $b \in G_1$ such that $\varphi_1(b) = a$. Now we consider the element $g_1(b)$ and apply φ_2. By commutativity of the left square we get

$$\varphi_2(g_1(b)) = h_1(\varphi_1(b)) = h_1(a) = \varphi_2(z)x^{-1}$$

which implies

$$x = \varphi_2(g_1(b^{-1})z)$$

so φ_2 is surjective.

For injectivity, we start by taking an arbitrary element $x \in G_2$ such that $\varphi_2(x) = 1$. Now we want to show that $x = 1$. By commutativity of the right square we get

$$1 = h_2(1) = h_2(\varphi_2(x)) = \varphi_3(g_2(x))$$

which means $g_2(x) \in ker(\varphi_3)$ and since φ_3 is an isomorphism $g_2(x) = 1$ so, by exactness of the upper row $x \in im(g_1)$ so there exists a $y \in G_1$ such that $x = g_1(y)$. By commutativity of the left square we get

$$h_1(\varphi_1(y)) = \varphi_2(g_1(y)) = \varphi_2(x) = 1$$

and since h_1 is injective by exactness of the bottom row, $\varphi_1(y) = 1$ and since φ_1 is an isomorphism $y = 1$. So we have that

$$x = g_1(y) = g_1(1) = 1$$

Since the kernel is trivial, φ_2 is injective, and hence an isomorphism. $\square$

3 Homotopy Theory

In this section we will lay out some of the basic concepts and definitions of homotopy theory, and show that we get a long exact sequence of homotopy groups which will lead us to the generators of the group U_n from the previous section, which in turn will lead us to the main proof of the next section. We start by going through the definition and some properties of covering spaces.

Definition 3.1. A continuous map $E \xrightarrow{p} B$ between topological spaces E and B is a *covering map* if it is surjective, and if for every $b \in B$ there exists an open neighborhood U of b such that $p^{-1}(U)$ is a union of disjoint open sets, each of which maps homeomorphically onto U by p. We will call such U *evenly covered*, and for $b \in B$, we denote the set $p^{-1}(b)$ by F_b, the *fiber over b*. The space E is called a *covering space*.

One classic example of a covering space is $p : \mathbb{R} \to S^1$, where S^1 is viewed as the unit vectors in $\mathbb{C}$, and where the covering map is $p(x) = e^{2\pi i x}$.

Proposition 3.1. *Let $E \xrightarrow{p} B$ be a covering map. Every path $f : I \to B$ with $f(0) = b$ lifts uniquely to a path $\tilde{f} : I \to E$ with $\tilde{f}(0) = e \in F_b$. I.e. for such f there exists an $\tilde{f}$ such that the following diagram*

$$\begin{tikzcd} & E \arrow[d, "p"] \\ I \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \end{tikzcd}$$

commutes.

Definition 3.2. Let X and Y be topological spaces, and $f, g : X \to Y$ be continuous maps. We say that f is *homotopic* to g if there exists a continuous map $H : X \times I \to Y$, where I denotes the close unit interval $[0,1] \subset \mathbb{R}$, called a *homotopy*, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all $x \in X$. If f is homotopic to g, write $f \simeq g$. We can think of homotopies as families of continuous maps $\{h_t : X \to Y\}_{t \in I}$.

Definition 3.3. We say that to topological spaces X and Y are *homotopy equivalent* if there exist continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$. We say that f is a *homotopy equivalence* with *homotopy inverse* g. A space X that is homotopy equivalent to a one-point space is called *contractible*.

In fact, $\simeq$ defines an equivalence relation.

Definition 3.4. Let $\mathrm{Maps}(X,Y)$ be the set of continuous maps from X to Y. We call the set of equivalence classes $\mathrm{Maps}(X,Y)/\simeq$, the set of *homotopy classes* of maps $f : X \to Y$.

We will mostly be interested in the cases where the topological spaces are pointed, i.e. when spaces have a given base point $x_0 \in X$. We sometimes write such a space as (X, x_0) or, if there is no confusion about what we mean, we just write X as in the non-pointed case.

With this in mind, we would like to have a case where maps $f : X \to Y$ takes the base point of X to the base point of Y. So if x_0 is the base point in X and y_0 the one in Y, we write $f : (X,x_0) \to (Y,y_0)$ for a map such that $f(x_0) = y_0$. More generally, if $A \subset X$ and $B \subset Y$ we write $f : (X,A) \to (Y,B)$ for a map that carries A to B, in the sense that $f(A) \subset B$.

A homotopy of base point preserving maps is a homotopy $H : X \times I \to Y$ from f to g such that for all $t \in I$, $H(x_0,t) = y_0$. In the pointed case, the set $\mathrm{Maps}(X,Y)/\simeq$ as above but where the maps are base point preserving, is called the set of *based homotopy classes*. We denote this set by $[X,Y]_*$.

For a topological space X with base point $x_0 \in X$, we define $\pi_n(X,x_0)$ to be the set of homotopy classes of maps $f : (I^n, \partial I^n) \to (X,x_0)$, where homotopies f_t are required to satisfy $f_t(\partial I^n) = x_0$ for all t. This set forms a group by the operation defined as

$$(f+g)(x_1,x_2,...,x_n) = \begin{cases} f(2x_1,x_2,...,x_n) & x_1 \in [0,1/2] \\ g(2x_1-1,x_2,...,x_n) & x_1 \in [1/2,1] \end{cases}.$$

Inverses here are given by $-f(x_1,x_2,...,x_n) = f(1-x_1,x_2,...,x_n)$. Note that for $n = 1$ we recover the fundamental group $\pi_1(X)$ of X, which will be the main focus of this text, but the reason for the additive notation here is that for $n \geq 2$, we get that $\pi_n(X)$ is actually abelian, although this is not at all obvious. For more details, see for example [3]. For $n = 0$ we can extend this definition by letting I^0 be a one point space and $\partial I^0 = \varnothing$, so $\pi_0(X)$ becomes the set of path-components of X. However, this is not a group.

The homotopy invariance of the fundamental group turns out to hold for all homotopy groups. Namely, if $f : (X,x_0) \to (Y,y_0)$ is a homotopy equivalence in the base point preserving sense, then the induced map $f_* : \pi_n(X,x_0) \to \pi_n(Y,y_0)$ is an isomorphism for all n. Also, as for the fundamental group, if the space X is path-connected, different choices of base point x_0 yield isomorphic homotopy groups $\pi_n(X,x_0)$ for all n. Therefore, if the space in question is path-connected we sometimes omit the base point and simply write $\pi_n(X)$. See for example [3]. In some cases, the higher homotopy groups behave much nicer than the fundamental group.

Proposition 3.2. *Let $p : (E,e_0) \to (B,b_0)$ be a covering map. Then p induces an isomorphism of homotopy groups $p_* : \pi_n(E,e_0) \to \pi_n(B,b_0)$ for $n \geq 2$.*

We will use this proposition to compute the higher homotopy groups for $n \geq 2$ of a certain space which will be useful for us later in the text, namely the wedge sum of a fixed number of circles. Since we will also need to make use of the fundamental group of the same space, we will compute that first.

Definition 3.5. Let $q : X \to Y$ be a continuous map between topological spaces. We say that q is a *quotient map* if it is surjective, and if Y has the quotient topology induced by q, that is if $U \subset Y$ is open if and only if $q^{-1}(U)$ is open in X.

If $\sim$ is an equivalence relation on a topological space X, then the natural projection $q : X \to X/\sim$ mapping every $x \in X$ to its equivalence class, is a quotient map.

Definition 3.6. Let $q : X \to Y$ be a continuous map. A subset $U \subset X$ is called *saturated with respect to q* if $U = q^{-1}(V)$ for some subset $V \subset Y$.

Proposition 3.3. *A continuous, surjective map $q : X \to Y$ is a quotient map if and only if it takes saturated open subsets to open subsets.*

Proof. Assume q is a quotient map. If $U \subset X$ is saturated and open, then $U = q^{-1}(V)$ for some $V \subset Y$, and by the definition of quotient map, V is open in Y.

Conversely, assume q is a continuous, surjective map that takes saturated open subsets to open subsets. We want to show that $V \subset Y$ is open if and only if $q^{-1}(V)$ is open. If $V \subset Y$ open, then $q^{-1}(V)$ is open since q is assumed to be continuous. Since $q^{-1}(V)$ is saturated by definition, if it in addition is open, then $q(q^{-1}(V)) = V$ is open. $\square$

Definition 3.7. Let $X_1,...,X_n$ be topological spaces, with base points $x_i \in X_i$. The *wedge sum*, denoted $X_1 \vee ... \vee X_n$ is the space obtained by taking $\coprod_{i=1}^{n} X_i / \sim$, where $\sim$ identifies the base points, and no other identifications are being made. The canonical choice of base point of this space is the equivalence class of the base points $x_1,...,x_n$.

Proposition 3.4. *The fundamental group of the wedge sum of n circles, $S^1 \vee ... \vee S^1$, is free on n generators.*

To prove this proposition, we can use the Seifert-Van Kampen theorem. We will not prove the theorem in this text, but see for example [5] for a proof.

Theorem 3.5 (Seifert-Van Kampen)**.** *Let X be a topological space. Suppose that $U,V \subset X$ are open subsets such that $U \cup V = X$, and with U,V and $U \cap V$ path-connected. Let $x_0 \in U \cap V$, and define a subset $C \subset \pi_1(U,x_0) * \pi_1(V,x_0)$ by*

$$C = \{(i_*\gamma)(j_*\gamma)^{-1} \mid \gamma \in \pi_1(U \cap V, x_0)\},$$

where i_*, j_* are maps induced by the inclusions $i : U \cap V \to U$ and $j : U \cap V \to V$. Then

$$\pi_1(X,x_0) \cong \big(\pi_1(U,x_0) * \pi_1(V,x_0)\big)/\bar{C}.$$

In particular, $\pi_1(X,x_0)$ is generated by the images of $\pi_1(U,x_0)$ and $\pi_1(V,x_0)$ under the homomorphisms induced by inclusions.

Corollary 3.5.1. *Assume that the hypotheses of the Seifert-Van Kampen theorem. Suppose also that $U \cap V$ is simply connected. Then*

$$\pi_1(X,x_0) \cong \pi_1(U,x_0) * \pi_1(V,x_0).$$

Ideally we would like to apply this corollary to a wedge of two spaces $X_1 \vee X_2$ with $U = X_1$ and $V = X_2$ considered as subspaces of the wedge sum ($X_1 = S^1 = X_2$ in our case), but the problem is that these spaces will not be open in $X_1 \vee X_2$. Luckily for us, this can be resolved.

Definition 3.8. Let X be a topological space and $A \subset X$ a subspace of X. A continuous map $r : X \to A$ is called a *retraction* if the restriction of r to A is the identity map of A. If there exists a retraction from X to A, we call A a *retract* of X. Let $\iota_A : A \to X$ be the inclusion of A. If $\iota_A \circ r$ is homotopic to the identity map of X, we say that r is a *deformation retraction*, and we say that r is a *strong deformation retraction* if it in addition to being a deformation retraction, there exists a homotopy H from id_X to $\iota_A \circ r$ that is stationary on A, meaning that

$$H(x,t) = id_X(x) \quad \text{for all } x \in A,\ t \in I.$$

If $r : X \to A$ is a strong deformation retraction, we say that A is a *strong deformation retract* of X.

Definition 3.9. A point x_0 of a topological space X is called a *nondegenerate base point* of X if it has a neighborhood that admits a strong deformation retraction onto x_0.

Lemma 3.6. *Suppose $x_i \in X_i$ is a nondegenerate base point for $i = 1,...,n$. Then the base point x_0 of $X_1 \vee ... \vee X_n$ is nondegenerate.*

See [5] for a proof.

Let $q : \coprod_{i=1}^n X_i \to X_1 \vee ... \vee X_n$ denote the quotient map. The inclusion of X_i into $\coprod_{i=1}^n X_i$ composed with q induces continuous injective maps $\iota_j : X_j \to X_1 \vee ... \vee X_n$.

Theorem 3.7 ([5] (p.256))**.** *Let $X_1,...,X_n$ be topological spaces with nondegenerate base points $x_i \in X_i$. The map*

$$\Phi : \pi_1(X_1,x_1) * ... * \pi_1(X_n,x_n) \to \pi_1(X_1 \vee ... \vee X_n, x_0)$$

induced by $\iota_{j_*} : \pi_1(X_j,x_j) \to \pi_1(X_1 \vee ... \vee X_n, x_0)$ is an isomorphism.

Proof. We start with the wedge sum of two spaces $X_1 \vee X_2$. Choose neighborhoods W_i in which x_i is a strong deformation retract, and let $U = q(X_1 \coprod W_2)$, $V = q(W_1 \coprod X_2)$ where q is the quotient map $X_1 \coprod X_2 \xrightarrow{q} X_1 \vee X_2$. Since both $X_1 \coprod W_2$ and $W_1 \coprod X_2$ are saturated open sets in $X_1 \coprod X_2$, the restriction of q to each of them is a quotient map onto its image, so U and V are both open in $X_1 \vee X_2$.

We will now show that the following inclusions

$$\{x_0\} \hookrightarrow U \cap V$$
$$X_1 \hookrightarrow U$$
$$X_2 \hookrightarrow V$$

are all homotopy equivalences, because each space on the left hand side is a strong deformation retract of the corresponding right hand side. For the first space, this just follows Lemma 3.6. For $X_1 \hookrightarrow U$, let $H : W_2 \times I \to W_2$ be a homotopy which gives a strong deformation retraction of W_2 to x_2. Define $G : (X_1 \coprod W_2) \times I \to X_1 \coprod W_2$ to be the identity on $X_1 \times I$ and H on $W_2 \times I$. The map G descends to the quotient and yields a strong deformation retraction of U onto X_1. A similar argument shows the case $X_2 \hookrightarrow V$.

Now, since $U \cap V$ is contractible, using 3.5.1 gives us that the inclusions $U \hookrightarrow X_1 \vee X_2$ and $V \hookrightarrow X_1 \vee X_2$ induce an isomorphism

$$\pi_1(U,x_0) * \pi_1(V,x_0) \xrightarrow{\cong} \pi_1(X_1 \vee X_2, x_0).$$

Moreover, the maps $X_1 \xrightarrow{\iota_1} U$ and $X_2 \xrightarrow{\iota_2} V$ induce isomorphisms

$$\pi_1(X_1,x_1) \xrightarrow{\cong} \pi_1(U,x_0)$$
$$\pi_1(X_2,x_2) \xrightarrow{\cong} \pi_1(V,x_0).$$

Composing these isomorphisms proves the case $n = 2$, and the case $n > 2$ follows by induction, since Lemma 3.6 guarantees that the hypotheses of the theorem are satisfied by the spaces X_1 and $X_2 \vee ... \vee X_n$. □

Applying the previous result to a wedge of n circles show that $\pi_1(S^1 \vee ... \vee S^1) \cong \mathbb{Z} * ... * \mathbb{Z} \cong F_n$ where F_n denotes the free group on n generators.

The Cayley graph of F_n - i.e. a graph where every vertex corresponds to an element of F_n, and where we have an edge between to vertices if and only if they differ by multiplication by a generator - is a covering space of the wedge of n circles, and is constructed for $n = 2$ in [3], but generalizes for higher dimensions as well. In particular, this covering space is a tree and hence contractible. See [4]. This gives us the following result.

Proposition 3.8. *For $n \geq 2$, the homotopy groups $\pi_n(\mathbb{R}^2 \setminus \{x_1,...,x_k\}) = 0$, where $\{x_1,...,x_k\} \subset \mathbb{R}^2$ is a set of distinct points.*

Proof. Since $\mathbb{R}^2 \setminus \{x_1,...,x_k\}$ is homotopy equivalent to a wedge of n circles, and since the covering space projection induces an isomorphism of homotopy groups π_n for $n \geq 2$, we have our result. □

The reason for this computation will become apparent in Section 4.

A concept we will make use of is that of *relative homotopy groups* for a pair (X,A), where $x_0 \in A \subset X$ for base point x_0. We start by regarding I^{n-1} as the face of I^n with last coordinate $s_n = 0$. Let J^{n-1} be the closure of $\partial I^n \setminus I^{n-1}$. For $n \geq 1$, define $\pi_n(X,A,x_0)$ to be the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X,A,x_0)$, i.e maps from I^n to X where the boundary ∂I^n gets carried to A and J^{n-1} to the base point x_0, where homotopies are required to be on the same form for all t. Note that for $A = x_0$, we get that $\pi_n(X,x_0,x_0) = \pi_n(X,x_0)$, so the relative homotopy groups are generalizations of the homotopy groups from earlier.

The sum is defined in the same way as for $\pi_n(X)$, except that we no longer can use the last coordinate s_n. Thus the set $\pi_n(X,A,x_0)$ forms a group for $n \geq 2$.

Theorem 3.9. *For $x_0 \in A \subset X$ we have a long exact sequence*

$$... \to \pi_n(A,x_0) \xrightarrow{i_*} \pi_n(X,x_0) \xrightarrow{j_*} \pi_n(X,A,x_0) \xrightarrow{\partial} \pi_{n-1}(A,x_0) \to ... \to \pi_0(X,x_0).$$

The maps i_* and j_* are induced by the inclusions $(A,x_0) \xrightarrow{i} (X,x_0)$ and $(X,x_0,x_0) \xrightarrow{j} (X,A,x_0)$ respectively. The boundary map ∂ comes from restricting maps $(I^n,\partial I^n,J^{n-1}) \to (X,A,x_0)$ to I^{n-1}. For a proof, see [4] or [3].

Definition 3.10. A *fiber bundle* structure on a space E with fiber F consists of a projection map $p : E \to B$ such that for all $b \in B$ there exists a neighborhood $b \in U \subset B$ with a homeomorphism $h : p^{-1}(U) \to U \times F$ making the following diagram

$$\begin{tikzcd}
p^{-1}(U) \arrow[rr, "h"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\
& U &
\end{tikzcd}$$

commute.

The map h above is what is called a *local trivialization* of the bundle. Since the fiber bundle structure is determined by the projection map, we usually say that $p : E \to B$ is a fiber bundle or, if we want to indicate what the fibers are, we write $F \to E \xrightarrow{p} B$. We usually call E the *total space* and B the *base space*.

Definition 3.11. A map $p : E \to B$ is said to have the *homotopy lifting property* with respect to the space X if, given a homotopy $g_t : X \to B$ and a map $\tilde{g}_0 : X \to E$ lifting g_0, so $p\tilde{g}_0 = g_0$, then there exists a homotopy $\tilde{g}_t : X \to E$ lifting g_t.

So, given g_t and $\tilde{g}_0$ as above, there exists a homotopy $\tilde{g}_t$ such that the two triangles in the diagram

$$\begin{tikzcd}
X \arrow[r, "\tilde{g}_0"] \arrow[d] & E \arrow[d, "p"] \\
X \times I \arrow[r, "g"] \arrow[ur, dashed, "\tilde{g}"] & B
\end{tikzcd}$$

commute.

We say that the map $p : E \to B$ has the *homotopy lifting property for a pair* with respect to the pair (X,A) if every homotopy $f_t : X \to B$ lifts to a homotopy $\tilde{g}_t : X \to E$ starting with a given lift $\tilde{g}_0$ and extending a given lift $\tilde{g}_t : A \to E$.

The homotopy lifting property generalizes the path lifting property defined earlier in this section. This can be seen by taking X in the diagram above to be a one-point space.

Proposition 3.10. *A fiber bundle $E \xrightarrow{p} B$ has the homotopy lifting property with respect to n-cubes, I^n.*

Theorem 3.11. *Suppose $E \xrightarrow{p} B$ has the homotopy lifting property with respect to I^n. Choose base points $b_0 \in B$ and $x_0 \in F = p^{-1}(b_0)$. Then the induced map $p_* : \pi_n(E,F,x_0) \to \pi_n(B,b_0)$ is an isomorphism for all $n \geq 1$. Hence, if B is path connected, there exists a long exact sequence*

$$\ldots \to \pi_n(F,x_0) \to \pi_n(E,x_0) \xrightarrow{p_*} \pi_n(B,b_0) \to \pi_{n-1}(F,x_0) \to \ldots \to \pi_0(E,x_0) \to 0$$

We will prove the last statement of the theorem here. To see that p_* is surjective respectively injective, what we basically do is apply the homotopy lifting property repeatedly. See [3].

Proof. Consider the long exact sequence in 3.9 for the pair (E,F)

$$... \to \pi_n(F,x_0) \xrightarrow{i_*} \pi_n(E,x_0) \xrightarrow{j_*} \pi_n(E,F,x_0) \xrightarrow{\partial} \pi_{n-1}(F,x_0) \to ...,$$

but let j_* be the map $p_* \circ j_* : \pi_n(E,x_0) \to \pi_n(B,b_0)$. The sequence then becomes

$$... \to \pi_n(F,x_0) \xrightarrow{i_*} \pi_n(E,x_0) \xrightarrow{p_* \circ j_*} \pi_n(B,b_0) \xrightarrow{\partial} \pi_{n-1}(F,x_0) \to ...,$$

which is the long exact sequence we are after. For the map $\pi_0(F) \to \pi_0(E)$ at the end, surjectivity comes from the hypothesis that B is path connected, since a path in E from an arbitrary point $x \in E$ can be obtained by lifting a path from $p(x)$ to b_0 in B. $\square$

4 Configuration Spaces

In this section we will show that the Artin braid group B_n is isomorphic to the fundamental group of the configuration space of the plane, $\pi_1(\mathscr{C}_n(\mathbb{R}^2))$. We start with the definition of a configuration space, and then we move on to a few properties that follow from the theory of Section 3.

Definition 4.1. Let X be a topological space. We call the space $\mathscr{F}_n(X) = \{(x_1,...,x_n) \in X^n \mid i \neq j \Rightarrow x_i \neq x_j\} \subset X^n$, with the product topology, the *n:th (ordered) configuration space* of X.

We have an action of the symmetric group S_n on this space, where S_n acts by permuting the coordinates of $\mathscr{F}_n(X)$. We call the orbit space $\mathscr{C}_n(X) = \mathscr{F}_n(X)/S_n$ the *n:th (unordered) configuration space* of X.

Sometimes, if Q_m is a set of m distinguished points $\{x_1,...,x_m\} \subset X$, we use the notation $\mathscr{F}_{m,n}(X)$ for the space $\mathscr{F}_n(X \setminus Q_m)$. In this paper, X will usually be a *manifold*, and recall that a manifold of dimension n, sometimes called n-manifold, is a topological Hausdorff space M such that each point $x \in M$ has an open neighborhood homeomorphic to $\mathbb{R}^n$. The following proposition will show that the choice of points Q_m will not matter.

Proposition 4.1. *Let M be a connected topological manifold, and let $\{p_1,...,p_k\}$ and $\{q_1,...,q_k\}$ be two k-tuples of distinct points of M. Then there exists a homeomorphism $\varphi : M \to M$ such that $\varphi(p_i) = q_i$ for $i = 1,2,...,k$.*

Proposition 4.2. *The projection $p : \mathscr{F}_n(M) \to \mathscr{C}_n(M)$ is a covering space projection.*

Proof. Let $x = (x_1,...,x_n) \in \mathscr{C}_n(M)$ since all x_i are distinct, we can find a neighborhood $U = U_1 \times ... \times U_n$ of x such that $i \neq j$ implies $U_i \cap U_j = \varnothing$ for all $i, j \in \{1,...,n\}$. Fix a $\sigma \in S_n$ and define $U_\sigma := U_{\sigma(1)} \times ... \times U_{\sigma(n)}$. Then we have $p^{-1}(U) = \{U_\sigma \mid \sigma \in S_n\} = \bigcup_{\sigma \in S_n} U_\sigma$ and since the U_σ are disjoint we have our covering space projection. ☐

Recall that this implies every path $\gamma : I \to \mathscr{C}_n(M)$ with $\gamma(0) = p(x_0)$ for some $x_0 \in \mathscr{F}_n(M)$ lifts uniquely to a path $\tilde{\gamma} : I \to \mathscr{F}_n(M)$ with $\tilde{\gamma}(0) = x_0$.

Theorem 4.3. *([2], p.26) Let M be a connected manifold of dimension ≥ 2. For $1 \leq r < n$, define $p : \mathscr{F}_n(M) \to \mathscr{F}_r(M)$, by $p(u_1,...,u_n) = (u_1,...,u_r)$. Then*

$$\mathscr{F}_{r,n-r}(M) \longrightarrow \mathscr{F}_n(M) \xrightarrow{p} \mathscr{F}_r(M)$$

is a fiber bundle.

Proof. Pick a point $u^0 = (u_1^0,...,u_r^0) \in \mathscr{F}_r(M)$. The pre-image $p^{-1}(u^0)$ consists of the elements

$(u_1^0,...,u_r^0,v_1,...,v_{n-r}) \in M^n$ with $u_1^0,...,u_r^0,v_1,...,v_{n-r}$ all distinct. Setting $Q_r = \{u_1^0,...,u_r^0\}$ we get

$$\mathscr{F}_{n-r}(M \setminus Q_r) = \mathscr{F}_{r,n-r}(M) = \{(v_1,...,v_{n-r}) \in (M \setminus Q_r)^{n-r} \mid i \neq j \Rightarrow v_i \neq v_j\}$$

The map $\{u^0\} \times \mathscr{F}_{n-r}(M) \longrightarrow \mathscr{F}_{n-r}(M)$ defined by $(u_1^0, ..., u_n^0, v_1, ..., v_{n-r}) \mapsto (v_1, ..., v_{n-r})$ is a clearly a homeomorphism, so $p^{-1}(u^0) \cong \mathscr{F}_{n-r}(M)$.

Now for local triviality. For each $i = 1, ..., r$ let $U_i \subset M$ be an open neighborhood of u_i^0 such that its closure $\bar{U}_i$ is a closed ball with interior U_i. Since $u_1^0, ..., u_r^0$ are all distinct, we can assume $U_i \cap U_j = \varnothing$ for all $i, j = 1, ..., r$ whenever $i \neq j$, so $U = U_1 \times ... \times U_r$ will be an open neighborhood of $u^0 \in \mathscr{F}_r(M)$.

We shall see that $p|_U$ is a local trivialization, i.e. that there is a homeomorphism $p^{-1}(U) \longrightarrow U \times \mathscr{F}_{r,n-r}(M)$ commuting with the projections to U.

For each $i = 1, ..., r$ define a continuous map $\theta_i : U_i \times \bar{U}_i \to \bar{U}_i$ with the following properties [1]. For every $u \in U_i$, let $\theta_i^u : \bar{U}_i \to \bar{U}_i$ be the map $v \mapsto \theta_i(u, v)$. We require:

1. $\theta_i^u : \bar{U}_i \to \bar{U}_i$ is a homeomorphism fixing the boundary $\partial \bar{U}_i$ pointwise.
2. $\theta_i^u(u_i^0) = u$.

The first property allows us to extend this homeomorphism to the entire manifold M in the following way. For $u = (u_1, ..., u_r) \in U$, define a map $\theta^u : M \to M$ by

$$\theta^u(v) = \begin{cases} \theta_i(u_i, v) & \text{if } v \in U_i \text{ for some } i = 1, ..., r \\ v & \text{if } v \in M \setminus \bigcup_i U_i. \end{cases}$$

It is clear that $\theta^u : M \to M$ is a homeomorphism continuously depending on u, sending $u_1^0, ..., u_r^0$ to $u_1, ..., u_r$ respectively. The formula

$$(u, v_1, ..., v_{n-r}) \mapsto (u, \theta^u(v_1), ..., \theta^u(v_{n-r}))$$

defines a homeomorphism $\phi : U \times \mathscr{F}_{r,n-r}(M) \to p^{-1}(U)$ with inverse

$$\phi^{-1} : (u, v_1, ..., v_{n-r}) \mapsto (u, (\theta^u)^{-1}(v_1), ..., (\theta^u)^{-1}(v_{n-r}))$$

The diagram

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, shift left, "\phi^{-1}"] \arrow[dr, "p|_U"'] & & U \times \mathscr{F}_{r,n-r}(M) \arrow[ll, shift left, "\phi"] \arrow[dl] \\ & U & \end{tikzcd}$$

clearly commutes, and thus we have our fiber bundle

$$\mathscr{F}_{r,n-r}(M) \longrightarrow \mathscr{F}_n(M) \xrightarrow{p} \mathscr{F}_r(M)$$

□

Corollary 4.3.1. *Let M and p be as above. For any $m \geq 0$, the map*

$$p : \mathscr{F}_{m,n}(M) \longrightarrow \mathscr{F}_{m,r}(M)$$

is a fiber bundle with fiber $\mathscr{F}_{m+r,n-r}(M)$.

[1] The construction of θ_i is carried out in [2] in detail, but requires some knowledge about smooth manifolds, as opposed to topological ones.

Proof. This follows by applying the previous theorem to $M = M \setminus Q_m$. $\square$

Proposition 4.4. *If $\pi_2(M \setminus Q_m, *) = \pi_3(M \setminus Q_m, *) = 0$ for each $m \geq 0$, then $\pi_2(\mathscr{F}_n(M), *) = 0$.*

Proof. The exact sequence of homotopy groups of the fiber bundle $p : \mathscr{F}_{m,n}(M) \to \mathscr{F}_{m,1} = M \setminus Q_m$ from Theorem 4.3 is

$$... \to \pi_3(M \setminus Q_m, *) \to \pi_2(\mathscr{F}_{m+1,n-1}(M), *) \to \pi_2(\mathscr{F}_{m,n}(M), *) \to \pi_2(M \setminus Q_m, *) \to ...,$$

so since $\pi_2(M \setminus Q_m, *) = \pi_3(M \setminus Q_m, *) = 0$ for each $m \geq 0$ we get that

$$\pi_2(\mathscr{F}_{m+1,n-1}(M), *) \cong \pi_2(\mathscr{F}_{m,n}(M), *),$$

and applying this inductively, we get that

$$\begin{aligned}\pi_2(\mathscr{F}_{m,n}(M), *) &\cong \pi_2(\mathscr{F}_{m+1,n-1}(M), *) \cong ... \cong \pi_2(\mathscr{F}_{m+n-1,1}(M), *) = \\ &= \pi_2(M \setminus Q_{m+n-1}, *) = 0.\end{aligned}$$

$\square$

Corollary 4.4.1. *The group $\pi_2(\mathscr{F}_n(\mathbb{R}^2)) = 0$.*

Proof. This follows from Proposition 4.4, since $\pi_2(\mathbb{R}^2 \setminus Q_m) = \pi_3(\mathbb{R}^2 \setminus Q_m) = 0$. $\square$

Let $(x_1^0, ..., x_n^0)$ be the base point of $\pi_1(\mathscr{F}_n(M))$, and let $\mathscr{F}_{n-1,1}(M) = M \setminus \{x_1^0, ..., x_{n-1}^0\}$. Define $i : \mathscr{F}_{n-1}(M) \to \mathscr{F}_n(M)$ by $i(x) = (x_1^0, ..., x_{n-1}^0, x)$.

Theorem 4.5. *If $\pi_0(M \setminus Q_m, *) = \pi_2(M \setminus Q_m, *) = \pi_3(M \setminus Q_m, *) = 1$ for all $m \geq 0$, then the following sequence is exact*

$$\begin{aligned}1 \to \pi_1(\mathscr{F}_{n-1,1}(M), x^0) \xrightarrow{i_*} \pi_1(\mathscr{F}_n(M), (x_1^0, ..., x_n^0)) \xrightarrow{p_*} \\ \xrightarrow{p_*} \pi_1(\mathscr{F}_{n-1}(M), (x_1^0, ..., x_{n-1}^0)) \to 1,\end{aligned}$$

where p_* is the map induced by the fiber bundle from 4.3.

Proof. The sequence is part of the homotopy sequence induced from Theorem 4.3, where the 1's come from the facts that $\pi_2(\mathscr{F}_{n-1}(M), *) = 1$ established in the previous proposition, and that $\pi_0(\mathscr{F}_{n-1,1}(M)) = \pi_0(M \setminus Q_{n-1}) = 1$. $\square$

Definition 4.2. Let $f : I \to \mathscr{F}_n(\mathbb{R}^2)$, $f(t) = (f_1(t), ..., f_n(t))$ be a path in $\mathscr{F}_n(\mathbb{R}^2)$. Each coordinate function f_i defines an arc $\beta_i = (f_i(t), t)$ in $\mathbb{R}^2 \times I$. We call their union $\beta = \beta_1 \cup ... \cup \beta_n$ a *geometric braid*. We say that two geometric braids β and β' are equivalent if $\beta \simeq \beta'$.

We will now describe the elements that generate $\pi_1(\mathscr{C}_n(\mathbb{R}^2), x_0)$. Recall the covering space projection $p : \mathscr{F}_n(\mathbb{R}^2) \to \mathscr{C}_n(\mathbb{R}^2)$. For $y_0 = ((1,0), ..., (n,0)) \in \mathscr{F}_n(\mathbb{R}^2)$ pick the point $p(y_0)$ as base point x_0 for $\pi_1(\mathscr{C}_n(\mathbb{R}^2), x_0)$. We can lift loops based at x_0

in $\mathscr{C}_n(\mathbb{R}^2)$ to paths starting at $y_0 = ((1,0),...,(n,0))$ in $\mathscr{F}_n(\mathbb{R}^2)$. The generator $\tilde{\sigma}_i$ of $\pi_1(\mathscr{C}_n(\mathbb{R}^2), x_0)$ is then represented by the path

$$f(t) = ((1,0),...,(i-1,0), f_i(t), f_{i+1}(t), (i+2,0),...,(n,0))$$

in $\mathscr{F}_n(\mathbb{R}^2)$, where $f_i(t) = (i+t, -\sqrt{t-t^2})$ and $f_{i+1}(t) = (i+1-t, \sqrt{t-t^2})$. That is to say, $f(t)$ is constant on all strings except the i:th and $(i+1)$:th, and those two strings get interchanged in a nice way. Notice the similarity with the braid in Figure 2.

We refer to Figure 5 to see how the composition of two geometric braids - one above and one under the middle line - would look like.

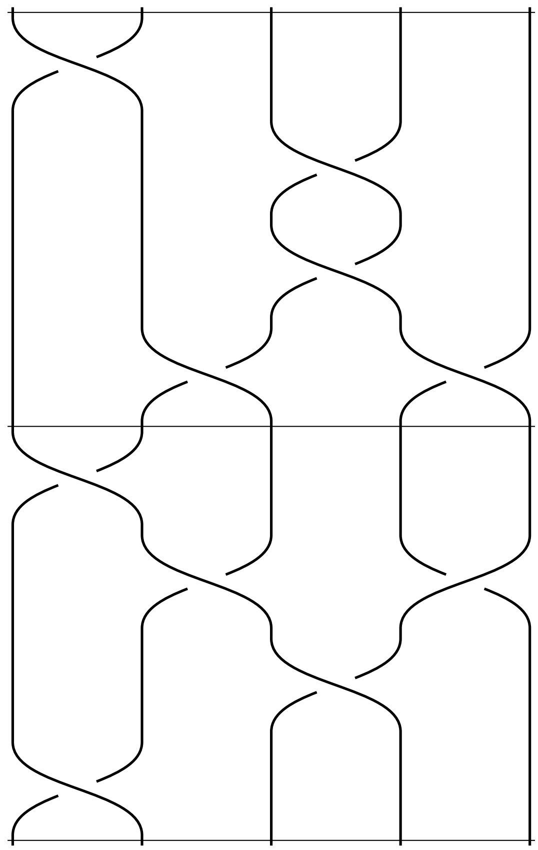

Figure 5: Example of composition of braids

Theorem 4.6 (Artin, 1925)**.** *The group $\pi_1(\mathscr{C}_n(\mathbb{R}^2), x_0)$ admits a presentation with generators $\tilde{\sigma}_1, ..., \tilde{\sigma}_{n-1}$ and defining relations*

$$\begin{aligned} \tilde{\sigma}_i\tilde{\sigma}_j &= \tilde{\sigma}_j\tilde{\sigma}_i \quad \textit{if} \quad |i-j| \geq 2,\ 1 \leq i,j \leq n-1 \\ \tilde{\sigma}_i\tilde{\sigma}_{i+1}\tilde{\sigma}_i &= \tilde{\sigma}_{i+1}\tilde{\sigma}_i\tilde{\sigma}_{i+1} \quad \textit{for all} \quad 1 \leq i \leq n-2. \end{aligned}$$

The proof of Theorem 4.6 will follow after the next lemma, which we will now set out to prove.

Let $b \in \pi_1(\mathscr{C}_n(\mathbb{R}^2), x_0)$ be represented by a loop $f : (I, \{0,1\}) \to (\mathscr{C}_n(\mathbb{R}^2), x_0)$ and let $\tilde{f} = (\tilde{f}_1, ..., \tilde{f}_n) : (I, \{0\}) \to (\mathscr{F}_n(\mathbb{R}^2), y_0)$ be the unique lift of f. We can see that any such lift induces a permutation of the set $\{1, ..., n\}$, which we call the *underlying permutation of* b. We define $u : \pi_1(\mathscr{C}_n(\mathbb{R}^2), x_0) \to S_n$ to be the map which sends each b to its underlying permutation τ_b which we write as

$$u(b) = \tau_b = \begin{pmatrix} \tilde{f}(0)_1, ..., \tilde{f}(0)_n \\ \tilde{f}(1)_1, ..., \tilde{f}(1)_n \end{pmatrix} \in S_n.$$

As an example: for b as in Figure 5 we would have that τ_b corresponds to the permutation $(243) \in S_5$.

Recall also the map $\pi : B_n \to S_n$ in Section 1 defined by $\pi(\sigma_i) = (i, i+1)$.

Lemma 4.7. *The homomorphism $\iota : B_n \to \pi_1(\mathscr{C}_n(\mathbb{R}^2))$ defined as $\iota(\sigma_i) = \tilde{\sigma}_i$ is an isomorphism if $\iota|_{P_n} : P_n \to \pi_1(\mathscr{F}_n(\mathbb{R}^2))$ is an isomorphism.*

To see that ι is well defined it is enough to note that the elements $\tilde{\sigma}_i$ satisfy the braid relations by for example examining Figure 3.

Proof. We get a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & P_n & \longrightarrow & B_n & \xrightarrow{\pi} & S_n & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \iota|P_n} & & \downarrow{\scriptstyle \iota} & & \downarrow & & \\ 1 & \longrightarrow & \pi_1(\mathscr{F}_n(\mathbb{R}^2)) & \longrightarrow & \pi_1(\mathscr{C}_n(\mathbb{R}^2)) & \xrightarrow{u} & S_n & \longrightarrow & 1 \end{array}$$

with rows exact so, applying the Five Lemma, we get that $\iota : B_n \to \pi_1(\mathscr{C}_n(\mathbb{R}^2))$ is an isomorphism. □

Now we just need to show that $i_n := \iota|P_n$ is an isomorphism. Corresponding to the forgetting homomorphism $f_n : P_n \to P_{n-1}$ we have the homomorphism $\pi_1(\mathscr{F}_n(\mathbb{R}^2)) \xrightarrow{p_*} \pi_1(\mathscr{F}_{n-1}(\mathbb{R}^2))$ from Theorem 4.5 with $ker(p_*) = \pi_1(\mathscr{F}_{n-1,1}(\mathbb{R}^2)) = \pi_1(\mathbb{R}^2 \setminus Q_{n-1})$, which is free on $n-1$ generators. Now consider the following diagram:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_n & \longrightarrow & P_n & \xrightarrow{f_n} & P_{n-1} & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle \iota|_{U_n}} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle i_{n-1}} & & \\ 1 & \longrightarrow & \pi_1(\mathscr{F}_{n-1,1}(\mathbb{R}^2)) & \longrightarrow & \pi_1(\mathscr{F}_n(\mathbb{R}^2)) & \xrightarrow{p_*} & \pi_1(\mathscr{F}_{n-1}(\mathbb{R}^2)) & \longrightarrow & 1. \end{array}$$

For $i = 1, 2, ..., n-1$, we can think of the image $\iota(A_{i,n})$ of the elements $A_{i,n}$ that generate U_n as a loop that starts at a point $x_0 \in \mathbb{R}^2$ and encircles the point x_i once, and separates it from the rest of the points in $Q_{n-1} = \{x_1, ..., x_{n-1}\}$. Then the set $\{\iota(A_{i,n}) \mid 1 \leq i \leq n-1\}$ is a generating set of $\pi_1(\mathscr{F}_{n-1,1}(\mathbb{R}^2)) = \pi_1(\mathbb{R}^2 \setminus Q_{n-1})$ which is free, and since $\iota|_{U_n}$ is surjective, U_n is free as well. In particular, $\iota|_{U_n}$ is an isomorphism for all n.

The proof of 4.6 will now follow by induction on n.

Proof of 4.6. For $n = 1$, both P_1 and $\pi_1(\mathscr{F}_1(\mathbb{R}^2))$ are trivial, so i_1 is an isomorphism. For the induction step, suppose that i_{n-1} is an isomorphism. Then since $\iota|U_n$ is an isomorphism for all n, if we apply the five lemma to the above diagram, we get that i_n is an isomorphism. Hence, by the previous lemma, $B_n \cong \pi_1(\mathscr{C}_n(\mathbb{R}^2))$. $\square$

5 Classifying Spaces

In this rather short section we will show that the groups $\pi_k(\mathscr{C}_n(\mathbb{R}^2))$ and $\pi_k(\mathscr{F}_n(\mathbb{R}^2))$ all vanish for $k > 1$.

Definition 5.1. Let G be a group. We say that G is a *topological group* if it comes equipped with a topology on the underlying set of G, such that the multiplication and inversion maps

$$\mu : G \times G \to G, \quad \mu(g_1, g_2) = g_1 g_2$$

and

$$i : G \to G, \quad i(g) = g^{-1},$$

are both continuous.

Definition 5.2. A *classifying space BG* of a topological group G is the quotient of a space EG, which has the property that all homotopy groups are trivial, by a free action of G, meaning that if there exists a point $x \in EG$ such that $gx = x$ for some $g \in G$, then g is the identity element.

If G is a topological group equipped with the discrete topology, then the classifying space of G is a path-connected space X such that

$$\pi_k(X) \cong \begin{cases} G & k = 1 \\ 0 & k \neq 1. \end{cases}$$

A space with the property that for some $n = 1, 2, ...$, $\pi_n(X) \cong G$ for some group G, and $\pi_k(X) = 0$ for $k \neq n$ is known as a *Eilenberg-MacLane space* $K(G, n)$. For $n = 1$, such spaces exist for arbitrary groups, and can explicitly be constructed. They exist for $n > 1$ as well, with the additional condition that G is abelian. See [4].

Proposition 5.1. *The groups $\pi_k(\mathscr{C}_n(\mathbb{R}^2))$ and $\pi_k(\mathscr{F}_n(\mathbb{R}^2))$ vanish for all $k > 1$.*

Proof. The proof will be by induction on n. Firstly, we look at $\mathscr{F}_n(\mathbb{R}^2)$. For $n = 1$, $\mathscr{F}_1(\mathbb{R}^2) = \mathbb{R}^2$, and since $\mathbb{R}^2$ is contractible, i.e. homotopy equivalent to a one point space, all homotopy groups vanish and, in particular they vanish for $k > 1$. Now suppose $\pi_k(\mathscr{F}_{n-1}(\mathbb{R}^2)) = 0$ for $k > 1$. The long exact homotopy sequence of the fiber bundle $\mathscr{F}_{n-1,1}(\mathbb{R}^2) \to \mathscr{F}_n(\mathbb{R}^2) \xrightarrow{p} \mathscr{F}_{n-1}(\mathbb{R}^2)$ is

$$... \to \pi_{k+1}(\mathscr{F}_{n-1}(\mathbb{R}^2)) \to \pi_k(\mathscr{F}_{n-1,1}(\mathbb{R}^2)) \to \pi_k(\mathscr{F}_n(\mathbb{R}^2)) \to \pi_k(\mathscr{F}_{n-1}(\mathbb{R}^2)) \to ...$$

and since $\pi_k(\mathscr{F}_{n-1}(\mathbb{R}^2)) = 0$ we get

$$... \to 0 \to \pi_k(\mathscr{F}_{n-1,1}(\mathbb{R}^2)) \to \pi_k(\mathscr{F}_n(\mathbb{R}^2)) \to 0 \to ...,$$

which gives us that $\pi_k(\mathscr{F}_{n-1,1}(\mathbb{R}^2)) \cong \pi_k(\mathscr{F}_n(\mathbb{R}^2))$, but as established earlier, $\mathscr{F}_{n-1,1}(\mathbb{R}^2) = \mathbb{R}^2 \setminus Q_{n-1}$ is homotopy equivalent to the wedge sum of n circles which has a contractible covering space, so the homotopy groups vanish for $k > 1$, and hence $\pi_k(\mathscr{F}_n(\mathbb{R}^2)) =$

0. Since we have a covering space projection $\mathscr{F}_n(\mathbb{R}^2) \to \mathscr{C}_n(\mathbb{R}^2)$, the homotopy groups of the two respective spaces are isomorphic for $k > 1$, so the groups $\pi_k(\mathscr{C}_n(\mathbb{R}^2))$ vanish as well. $\square$

This now shows that $\mathscr{C}_n(\mathbb{R}^2) = BB_n$ and that $\mathscr{F}_n(\mathbb{R}^2) = BP_n$.

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