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Weyls Law for Quantum Graphs

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Abstract

This thesis concerns the asymptotic distribution of eigenvalues on a quantum graph with certain vertex conditions. The operator of consideration is known as the Hamiltonian which acts as the negative second-order differential operator on the functions defined on the edges of a compact metric graph along with some appropriate vertex conditions. We will derive the asymptotic formula for the eigenvalue counting function of the Hamiltonian acting on the graph in two separate cases. Moreover, the thesis include a close study of the sesquilinear form corresponding to the Hamiltonian as well as an introduction to a few selected topics from the theory of Hilbert spaces.

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Introduction

In 1911, the German mathematician Hermann Weyl (1885-1955) showed in his paper *Über die asymptotische Verteilung der Eigenwerte (On the asymptotic distribution of eigenvalues)* [1] that the asymptotic distribution of eigenvalues of the positive Laplacian operator on a bounded domain $\Omega \subset \mathbb{R}^n$ with Dirichlet boundary is

$$N(\lambda) \sim \frac{\omega_n}{(2\pi)^n} \mathrm{vol}(\Omega) \lambda^{n/2} \tag{1}$$

where $N(\lambda)$ denotes the eigenvalue counting function of the operator, ω_n is the volume of the n-dimensional unit sphere and $\mathrm{vol}(\Omega)$ is the volume of the domain Ω. This result became famously known as Weyls law. Over the years that followed, mathematicians and physicists generalized this result to other types of mathematical structures and operators, as well as improving the remainder estimates for these. Today, Weyls law has become an umbrella term for the asymptotic distribution of eigenvalues for all types of structures and operators. One of these structures which we will be looking at is known as a *quantum graph.*

A quantum graph basically consists of two things:

1. A metric graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$, consisting of a finite set of vertices $\mathcal{V}$ and edges $\mathcal{E}$ along with a set of intervals $[0, \ell_e] \in \mathcal{I}$ for all $e \in \mathcal{E}$, where $0 < \ell_e \leq \infty$ is a positive number assigned each edge $e \in \mathcal{E}$.

2. The assignment of a differential operator acting on functions defined on the edges of the graph which satisfy some local self-adjoint vertex conditions at every vertex $v \in \mathcal{V}$.

The theory of quantum graphs is a relatively new area in mathematics and most of the progress has been made in the last few decades, even though some works that could be classified as quantum graphs appeared at least as early as in the 1930s. The reason of the growth in recent years is due to the numerous applications in physics, chemistry as well as engineering of which quantum graph theory offers a simplified model when dealing with propagation of waves in very thin branching structures. In this thesis we will only deal with compact metric graphs, which is to say, the edges are all of finite length, and with the operator known as the Hamiltonian $\mathcal{L}$ acting as the negative second-order differential operator on functions defined on the edges and satisfying some local self-adjoint vertex conditions. Our main goal is to show that for a quantum graph endowed with arbitrary self-adjoint vertex conditions which corresponds to a nonnegative self-adjoint matrix Λ_v for all $v \in \mathcal{V}$, the eigenvalue counting function follows the asymptotic law:

$$N_\Gamma(k) \sim \frac{L}{\pi} k \text{ as } k \to \infty \tag{2}$$

where L is the sum of the lengths of all the edges and $N_\Gamma(k)$ is the eigenvalue counting function of the Hamiltonian operator, or simply, the eigenvalues of the graph Γ. In addition, we will also look at a less general case, namely a graph consisting of only Dirichlet and Kirchoff conditions at each vertex and show that that $N_\Gamma(k)$ follows

$$N_\Gamma(k) = \frac{L}{\pi}k + \mathcal{O}(1) \tag{3}$$

where the remainder term is bounded above and below by some constants independent of k. Both of these results are known as Weyl's law in each respective case.

Since the theory of quantum graphs is heavily grounded in functional analysis, the first chapter aim to introduce the basic but necessary concepts in functional analysis in Hilbert spaces. The next chapter involves quantum graphs, and the first three sections are dedicated to define what quantum graphs are and how all self-adjoint realizations of the Hamiltonian arise in terms of the vertex conditions. In addition, we will compute the spectrum of the Hamiltonian on the trivial graph with the Dirichlet vertex conditions. The following three sections of Chapter 2 concern the sesquilinear form of the Hamiltonian and the introduction of the extended δ-type of vertex conditions and its interlacing properties. The two final sections of Chapter 2 is dedicated to the proof of Weyl's law in each respective case.

Chapter 1

Background

In this chapter we will introduce some selected topics from the theory of Hilbert spaces. To put it more concretely, we will begin Section 1.1 with the definition of a Hilbert space and give two important examples in the form of the Lebesgue and Sobolev spaces. In addition to this we will also mention Riesz's representation theorem which will be useful in the definition of the adjoint operator. In Section 1.2 we will look at the so-called self-adjoint operators. In Section 1.3 sesquilinear forms are the main topic and we'll see a relationship between semi-bounded sesquilinear forms and their corresponding self-adjoint operators. Chapter 1 closes with Section 1.4 in which we look at the spectral properties of self-adjoint operators.

1.1 Hilbert spaces and Riesz's theorem

We begin by recalling some basic facts and notions from linear algebra and analysis.

Definition 1.1.1. *An inner product $\langle\cdot,\cdot\rangle : V \times V \longrightarrow \mathbb{C}$ is a map over a vector space V to the field of scalars $\mathbb{C}$ such that for all vectors $u, v, w \in V$ and scalars $\alpha \in \mathbb{C}$*

- $\langle u, v\rangle = \overline{\langle v, u\rangle}$
- $\langle \alpha u, v\rangle = \alpha\langle u, v\rangle$ *and* $\langle u + v, w\rangle = \langle u, v\rangle + \langle w, v\rangle$
- $\langle u, u\rangle \geq 0$ *and* $\langle u, u\rangle = 0$ *if and only if* $u = 0$

An inner product induces a norm given by $\|\cdot\| := \sqrt{\langle\cdot,\cdot\rangle}$. A vector space which is complete by the norm induced by an inner product is called a Hilbert space. If $\langle u, v\rangle = 0$ then we say that u is orthogonal to v and we denote it by $u \perp v$.

We will throughout the text denote a general complex Hilbert space by $\mathcal{H}$ and its elements by u, v, w. Let I be a finite index set, then

$$\bigoplus_{i\in I} \mathcal{H}_i = \Big\{ (u_i)_{i\in I} \mid u_i \in \mathcal{H}_i, \Big\} \tag{1.1}$$

is a Hilbert space with an inner product defined by

$$\langle (u_i)_{i\in I}, (v_i)_{i\in I}\rangle = \sum_{i=1}^{\infty} \langle u_i, v_i\rangle_{\mathcal{H}_i}. \tag{1.2}$$

As a first example of a Hilbert space which will reappear throughout the text, we choose the Lebesgue space $L^2(a,b)$ of square-integrable complex-valued functions on a real interval (a,b).

Example 1.1.1. ***(The** $L^2(a,b)$**-space)*** *We denote by $C(a,b)$ the space of continuous complex-valued functions on (a,b). Then*

$$\langle f,g\rangle = \int_a^b f(x)\overline{g(x)}dx \tag{1.3}$$

can be shown to be an inner product on $C[a,b]$. The integral is taken in the sense of Lebesgue and the bar over g denotes complex conjugation. The induced norm is given by

$$\|f\|_{L^2} = \Big(\int_a^b |f(x)|^2 dx\Big)^{\frac{1}{2}}. \tag{1.4}$$

We define the Lebesgue space of square-integrable complex-valued functions on the real interval (a,b) as

$$L^2(a,b) = \overline{\left\{ f\in C[a,b] \mid \|f\|_{L^2} = \left(\int_a^b |f(x)|^2 dx\right)^{\frac{1}{2}} < \infty \right\}} \tag{1.5}$$

where the overline denotes the closure of the set with respect to the L^2-norm (1.4) . In addition, $L^2(a,b)$ can be shown to be a Hilbert space with the inner product (1.3), however, the proof of completeness requires a bit of measure theory and therefore will be omitted, see [3] page 97.

Another example of a Hilbert space is the one-dimensional Sobolev space $H^k(a,b)$. This time we consider functions $f\in L^2(a,b)$ along with their k-th weak derivative D^kf. The Sobolev space is then defined as the space of functions in $L^2(a,b)$ for which D^kf belongs to $L^2(a,b)$.

Example 1.1.2. ***(The** $H^k(a,b)$**-space)*** *We begin by defining $L^1(a,b)$ similarly as $L^2(a,b)$ by*

$$L^1(a,b) = \overline{\left\{ f\in C[a,b] \mid \|f\|_{L^1} = \int_a^b |f(x)|dx < \infty \right\}} \tag{1.6}$$

where the closure is taken with the L^1-norm. We define the subspace $L^1_{loc}(a,b)$ of $L^1(a,b)$ as

$$L^1_{loc}(a,b) = \left\{ f\in L^1(a,b) \mid \int_a^b f(x)dx < \infty \text{ and } f \text{ locally integrable.} \right\} \tag{1.7}$$

where "locally integrable" means that that the integral over $|f|$ over any compact subset of its domain (a,b) is finite. Given an $f\in L^1_{loc}(a,b)$, if there exists a function $g\in L^1_{loc}(a,b)$ with the property that

$$\int_a^b f(x)\phi'(x)dx = -\int_a^b g(x)\phi(x)dx \tag{1.8}$$

for all $\phi \in C^\infty_{comp}(a,b)$*, then* g *is said to be the weak derivative of* f*. The subscript "comp" refers to that* ϕ *has compact support, which is to say (informally) that* ϕ *vanishes outside the interval* (a,b) *and in some small neighborhood around the boundary points. We define the Sobolev space* $H^k(a,b)$ *as*

$$H^k(a,b) = \{f \in L^2(a,b) \mid D^i f \in L^2(a,b),\ i = 1,2,\ldots,k\} \tag{1.9}$$

where $D^k f$ *denotes the* k*-th weak derivative of* f*. With the inner product given by*

$$\langle f,g\rangle_{H^k} = \sum_{i=0}^{k} \langle D^i f, D^i g\rangle_{L^2} \tag{1.10}$$

H^k *is a Hilbert space, where the subscript* L^2 *refers to standard integral product given in (1.3). Moreover,* H^k *is dense in* $L^2(a,b)$ *by using the well-known fact that* $\overline{C^\infty_{comp}(a,b)} = L^2(a,b)$ *and by the following inclusion*

$$C^\infty_{comp}(a,b) \subset H^k(a,b) \subseteq L^2(a,b) \tag{1.11}$$

for $k = 0,1,\ldots$*.*

We will from here on assume that all derivates are taken in the weak sense. Before ending this section we will briefly mention functionals and the Riesz representation theorem which will be helpful when defining the adjoint operator in the next section.

Definition 1.1.2. *A functional* ϕ *refers to a mapping* $\phi : V \to \mathbb{C}$ *from an inner product space* V *to its field of scalars* $\mathbb{C}$*. It is called bounded if there exists a positive real number* $m > 0$ *such that* $|\phi(u)| \leq m\|u\|_V$ *for all* $u \in V$*.*

It turns out that every bounded functional on a Hilbert space $\mathcal{H}$ can be written as a function in terms of the inner product and some unique element in $\mathcal{H}$. In other words, there exists a bijection $y \mapsto \langle y, \cdot\rangle$ between elements $y \in \mathcal{H}$ and the space of bounded linear functionals on $\mathcal{H}$. It should be mentioned that in general, finding this unique element v explicitly is no easy task.

Theorem 1.1.1. ***(Riesz's representation theorem)*** *Let* $\phi : \mathcal{H} \longrightarrow \mathbb{C}$ *be a bounded, linear*[1] *functional defined on a Hilbert space* $\mathcal{H}$*. Then there exists a unique element* $v \in \mathcal{H}$ *such that* $\phi(u) = \langle u, v\rangle$ *for all* $u \in \mathcal{H}$ *and* $\|\phi\| = \|v\|$*.*

Remark. *The norm of a functional* $\|\phi\|$ *is the same as the operator norm given in Definition 1.2.1.*

Proof. See [3] page 206. □

1.2 The adjoint operator

Now we will take a look at operators defined in Hilbert spaces. A certain kind of operators are of interest to us due to their spectral properties as we'll see in Section 2.4; these are known as self-adjoint operators. Since operators can be thought of as a generalization of matrices in finite dimensions, we would like to extend some of the concepts known from linear algebra into infinite dimensional spaces (such as the function spaces L^2 or H^k). Two well-known notions are the Hermitian matrix and the conjugate transpose of a matrix. These are the finite dimensional equivalent of the self-adjoint and adjoint operator respectively.

[1] $\phi(u+v) = \phi(u) + \phi(v)$ and $\phi(\alpha u) = \alpha\phi(u)$ for all $u, v \in \mathcal{H}$ and $\alpha \in \mathbb{C}$.

Definition 1.2.1. *A densely defined linear operator T is a linear mapping*

$$T \ : \mathcal{D}(T) \to \mathcal{H}$$

from a dense linear subspace $\mathcal{D}(T)$ of $\mathcal{H}$ called the domain of T into $\mathcal{H}$. We define the norm of an operator as

$$\|T\| = \sup_{u \in \mathcal{D}(T),\ u \neq 0} \frac{\|Tu\|}{\|u\|} = \sup_{u \in \mathcal{D}(T),\ \|u\|=1} \|Tu\|. \tag{1.12}$$

We call T semi-bounded if $\langle Tu, u\rangle \geq m\|u\|^2$ for all $u \in \mathcal{D}(T)$ and $m \in \mathbb{R}$. We call T bounded if $\|T\|$ is finite.

Suppose we have a densely defined operator T and we consider the inner product $\langle Tu, v\rangle$. We can think of the adjoint operator T^* as the operator which "switches place" in the inner product and preserves the value, i.e $\langle Tu, v\rangle = \langle u, T^*v\rangle$ for all $u, v \in \mathcal{H}$. The existence of such operator is not guaranteed due to the fact that T might be unbounded, and hence can't be defined on the whole Hilbert space [2]. Instead we would like to find a set in which we can define our T^* on. Let

$$\Omega(T) = \left\{ v \in \mathcal{H} \mid \sup_{u \in \mathcal{D}(T),\ \|u\|=1} |\langle Tu, v\rangle| < \infty \right\} \tag{1.13}$$

then for each $v \in \Omega(T)$ we define the functional $\phi_v(u) = \langle Tu, v\rangle$ which is clearly bounded (and hence continuous). We can uniquely extend[3] this functional to one which is both bounded and defined on all of $\mathcal{H}$; we denote this extension by $\tilde{f}_v(u)$. By Riesz's theorem (Theorem 1.1.2) we can then find an unique element $v^* \in \mathcal{H}$ such that $\tilde{f}_v(u) = \langle u, v^*\rangle$ for all $u \in \mathcal{H}$. The adjoint operator is then defined by $T^*v = v^*$ with the domain given by $\mathcal{D}(T^*) = \Omega(T)$. We summarize this in the definition below.

Definition 1.2.2. *Let $T : \mathcal{D}(T) \longrightarrow \mathcal{H}$ be a densely defined linear operator in $\mathcal{H}$. The adjoint operator $T^* : D(T^*) \longrightarrow \mathcal{H}$ is defined as follows. The domain $\mathcal{D}(T^*)$ of T^* consists of all $v \in \mathcal{H}$ such that that there exists a $v^* \in \mathcal{H}$ satisfying*

$$\langle Tu, v\rangle = \langle u, v^*\rangle \text{ for all } u \in \mathcal{D}(T).$$

Then the adjoint operator is defined as $T^*v = v^*$.

It should be mentioned that the requirement for T to be densely defined is due to the fact that the mapping $T^*v = v^*$ is not unique otherwise. If $\overline{\mathcal{D}(T)} \neq \mathcal{H}$, then $\overline{\mathcal{D}(T)}^{\perp} = \{u \in \mathcal{H} \mid \langle u, w\rangle = 0 \text{ for all } w \in \overline{\mathcal{D}(T)}\} \neq \{0\}$ and we can find a non-zero vector $u_0 \in \overline{\mathcal{D}(T)}^{\perp}$ such that $\langle w, u_0\rangle = 0$ for all $w \in \mathcal{D}(T)$. But then

$$\langle u, v^*\rangle = \langle u, v^*\rangle + \langle u, u_0\rangle = \langle u, v^* + u_0\rangle \tag{1.14}$$

which implies non-uniqueness. Assume instead $\overline{\mathcal{D}(T)} = \mathcal{H}$ then $\overline{\mathcal{D}(T)}^{\perp} = \{0\}$ and so if $\langle u, u_0\rangle = 0$ holds for all $u \in \mathcal{D}(T)$ then $u_0 = 0$. This shows that $v^* + u_0 = v^*$ in (1.9) and therefore v^* is unique.

[2]This is due to the Hellinger-Toeplitz theorem which states that an everywhere-defined symmetric operator is bounded, see [2] page 525.

[3]This is due to the Hahn-Banach theorem, see [3] page 150.

Definition 1.2.3. *A densely defined linear operator $T : \mathcal{D}(T) \longrightarrow \mathcal{H}$ is said to be symmetric if for all $u, v \in \mathcal{D}(T)$*

$$\langle Tu, v\rangle = \langle u, Tv\rangle. \tag{1.15}$$

If $\mathcal{D}(T_1) \subset \mathcal{D}(T_2)$ and $T_1 u = T_2 u$ holds for all $u \in \mathcal{D}(T_1)$ for two operators T_1, T_2 we call T_2 an extension of T_1 and denote it by $T_1 \subset T_2$.

An operator is symmetric if and only if $T \subset T^*$. Indeed, by the definition of the adjoint $\langle Tu, v\rangle = \langle u, T^* v\rangle$ holds for all $u \in \mathcal{D}(T), v \in \mathcal{D}(T^*)$. If $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ then $T^* v = Tv$ for all $v \in \mathcal{D}(T)$ and it follows that T is symmetric. Conversely, if T is symmetric then $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in \mathcal{D}(T)$. Then $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ and $Tu = T^* u$ for all $u \in \mathcal{D}(T)$, hence by definition $T \subset T^*$.

Example 1.2.1. *Consider the operator $T : \mathcal{D}(T) \to L^2(a, b)$ defined by $Tf = -\frac{d^2 f}{dx^2}$. We would like to show that T can be made symmetric by choosing a suitable domain. We will show this by the use of partial integration.*

$$\langle Tf, g\rangle = \langle -f'', g\rangle = \int_a^b (-f''(x))\overline{g(x)}dx \tag{1.16}$$

$$= \left[(-f'(x))\overline{g(x)}\right]_a^b + \int_a^b f'(x)\overline{g'(x)}dx \tag{1.17}$$

$$= \left[(-f'(x))\overline{g(x)}\right]_a^b + \left[f(x)\overline{g'(x)}\right]_a^b - \int_a^b f(x)\overline{g''(x)}dx \tag{1.18}$$

$$= \left[f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right]_a^b + \langle f, -g''\rangle \tag{1.19}$$

For T to be symmetric we require that

$$\left[f(x)\overline{g'(x)} - f'(x)\overline{g(x)}\right]_a^b = 0. \tag{1.20}$$

This can happen in several different ways. If for example T was to be defined on $C^\infty_{comp}[a, b]$ then (1.20) vanishes and the operator is symmetric. Furthermore, $C^\infty_{comp}[a, b]$ is also dense in $L^2[a, b]$ as we've already pointed out in Example 1.1.1.

Since $\mathcal{D}(T) \subset \mathcal{D}(T^*)$ always holds for symmetric operators. One might be interested in when those domains coincide, i.e $T = T^*$. This leads us to the definition of the self-adjoint operator.

Definition 1.2.4. *An operator T is called self-adjoint if T is symmetric and $\mathcal{D}(T) = \mathcal{D}(T^*)$.*

Explicitly computing the adjoint and its domain is often a laborious task which makes the work of checking if an operator is self-adjoint quite difficult. However, as we will see in the next example, T is not self-adjoint if we can find an element which is in $\mathcal{D}(T^*)$ but not in $\mathcal{D}(T)$.

Example 1.2.2. *For the sake of simplicity we will consider $Tf = -\frac{d^2 f}{dx}$ defined on $C^\infty_{comp}[-1, 1]$. If we can find a $g \in \mathcal{D}(T^*)$ but $g \notin C^\infty_{comp}[-1, 1]$ then we're done. Let*

$$g(x) = \begin{cases} -x^3, & -1 \le x < 0 \\ x^3, & 0 \le x \le 1 \end{cases} \qquad g^*(x) = \begin{cases} -6x, & -1 \le x < 0 \\ 6x, & 0 \le x \le 1 \end{cases}$$

then for $f \in C^{\infty}_{comp}[-1,1]$ and $g \in L^2(-1,1)$ we have

$$\langle f, g^* \rangle = \int_{-1}^{1} f(x)\overline{g^*(x)}dx = \int_{-1}^{0} f(x)(-6x)dx + \int_{0}^{1} f(x)(6x)dx \tag{1.21}$$

$$= 3f(-1) + f'(-1) + 3f(1) - f'(1) + \int_{-1}^{1} f''(x)\overline{g(x)}dx \tag{1.22}$$

$$= \langle Tf, g \rangle \tag{1.23}$$

and so $g \in \mathcal{D}(T^*)$ with $T^*g = g^*$. But g is obviously not in $C^{\infty}_{comp}[-1,1]$ and so T cannot be self-adjoint.

It can however be shown that $Tf = -\frac{d^2 f}{dx^2}$ is self-adjoint on the domain given by $H^2_0(a,b) = \{f \in H^2(a,b) \mid f(a) = 0, f(b) = 0\}$.

1.3 Sesquilinear forms

In this section we will take a brief look at sesquilinear forms. We will see that sesquilinear forms have a close connection to operators, and certain properties of the sesquilinear form carry over to the corresponding operator. The reason why we bother ourselves with working with the sesquilinear forms is that these are often easier to work with than the operator itself, and the domain of the form is larger and less "sensitive" to changes (in the sense of varying boundary conditions, for example).

Definition 1.3.1. *A map $\mathfrak{s} : \mathcal{D}(\mathfrak{s}) \times \mathcal{D}(\mathfrak{s}) \to \mathbb{C}$ is called a sesquilinear form on $\mathcal{H}$ if*

$$\mathfrak{s}[u+v, w] = \mathfrak{s}[u,w] + \mathfrak{s}[v,w] \quad \text{and} \quad \mathfrak{s}[\alpha u, v] = \alpha \mathfrak{s}[u,v]$$
$$\mathfrak{s}[u, v+w] = \mathfrak{s}[u,v] + \mathfrak{s}[u,w] \quad \text{and} \quad \mathfrak{s}[u, \alpha v] = \overline{\alpha} \mathfrak{s}[u,v]$$

for all $u, v \in \mathcal{D}(s)$ and $\alpha \in \mathbb{C}$. The domain $\mathcal{D}(\mathfrak{s})$ of $\mathfrak{s}$ is a linear subspace of $\mathcal{H}$ and $\mathfrak{s}$ is called densely defined if $\mathcal{D}(\mathfrak{s})$ is dense in $\mathcal{H}$. Furthermore, we call

- *$\mathfrak{s}$ symmetric if $\mathfrak{s}[u,v] = \overline{\mathfrak{s}[v,u]}$ for all $u, v \in \mathcal{D}(\mathfrak{s})$*
- *$\mathfrak{s}$ semi-bounded if there exists a constant $\alpha \in \mathbb{R}$ such that $\mathfrak{s}[u,u] \geq \alpha \|u\|^2$ for all $u \in \mathcal{D}(\mathfrak{s})$. We then call α a lower bound of $\mathfrak{s}$.*

We define the quadratic form $\mathfrak{q}$ as $\mathfrak{q}[u] = \mathfrak{s}[u,u]$ with $\mathcal{D}(\mathfrak{q}) = \mathcal{D}(\mathfrak{s})$. If $\mathfrak{s}$ is semi-bounded we call $\mathfrak{s}$ closed if $\mathcal{D}(\mathfrak{s})$ is complete with the norm induced by the inner product defined by $\langle u, v \rangle_{\mathfrak{s}} = \mathfrak{s}[u,v] + (1-\alpha)\langle u, v \rangle$ where α is a lower bound of $\mathfrak{s}$.

An already familiar sesquilinear form is the inner product (as in Definition 1.1.1). However, a more interesting example is shown below where the sesquilinear form of (1.20) is closely related to the operator $Tf = -\frac{d^2 f}{dx^2}$ previously defined in Example 1.2.1.

Example 1.3.1. *Consider the densely defined sesquilinear form*

$$\mathfrak{s}[f,g] = \int_a^b f'(x)\overline{g'(x)}dx \tag{1.24}$$

on the form domain $f, g \in \mathcal{D}(\mathfrak{s}) = H_0^1(a,b) = \{f \in H^1(a,b) \mid f(a) = 0,\ f(b) = 0\}$ in $L^2(a,b)$. To show that this is indeed a sesquilinear form follows directly from the summation rule of integrals and complex conjugation. This form can easily seen be symmetric

$$\overline{\mathfrak{s}[g,f]} = \overline{\int_a^b g'(x)\overline{f'(x)}dx} = \int_a^b \overline{g'(x)}f'(x)dx = \mathfrak{s}[f,g]$$

and lower semibounded

$$|\mathfrak{s}[f,f]| = \Big| \int_a^b f'(x)\overline{f'(x)}dx \Big| = \int_a^b |f'(x)|^2 dx \geq 0.$$

It can also be shown to be closed. Indeed, with $\alpha = 0$ as in Definition 1.3.1 we get

$$\langle f, g\rangle_{\mathfrak{s}} = \int_a^b f'(x)\overline{g'(x)}dx + \int_a^b f(x)\overline{g(x)}dx. \tag{1.25}$$

As we have mentioned in Example 1.1.2 in Section 1.1, the Sobolev space equipped with the above inner product is a Hilbert space (and hence complete) thus $\mathfrak{s}$ is closed.

The following important theorem which shows the correspondance between certain sesquilinear forms and self-adjoint operators will conclude this section.

Theorem 1.3.1. *Suppose $\mathfrak{s}$ is a sesquilinear form which is densely defined, symmetric, closed and semi-bounded by some $m \in \mathbb{R}$ in $\mathcal{H}$. Then there exists a unique self-adjoint operator T in $\mathcal{H}$ with $\mathcal{D}(T) \subset \mathcal{D}(\mathfrak{s})$ corresponding to $\mathfrak{s}$ such that*

$$\mathfrak{s}[u,v] = \langle Tu, v\rangle \quad \textit{for} \quad u \in \mathcal{D}(T), v \in \mathcal{D}(\mathfrak{s}). \tag{1.26}$$

Proof. See [2] page 225. □

1.4 Spectral properties of self-adjoint operators

In Section 1.2 we said that all finite-dimensional linear operators can be seen as matrices. From linear algebra we are familiar with the problem of finding the eigenvalue $\lambda \in \mathbb{C}$ and the corresponding eigenvector $u \neq 0$ such that

$$Au = \lambda u$$

for some $n \times n$-matrix A with coefficients in $\mathbb{C}^n$. We know that for a $n \times n$-matrix there are at least 1 and at most n distinct eigenvalues. Finding the eigenvalues is usually done by computing λ for which $\det(A - \lambda I) = 0$. Taking the step to infinite-dimensional spaces we are interested in when

$$Tu = \lambda u \tag{1.27}$$

for some operator T defined on $\mathcal{H}$ and non-zero $u \in \mathcal{H}$. Instead of solving for which $\lambda \in \mathbb{C}$ $\det(A - \lambda I) = 0$, we are now interested in the properties of the resolvent operator $R_\lambda = (T - \lambda I)^{-1}$. Basically, the eigenvalues λ of T is the set for which R_λ doesn't exist.

Definition 1.4.1. *Let $T : \mathcal{D}(T) \to \mathcal{H}$ be a closed*[4]*, linear operator from its domain $\mathcal{D}(T)$ into a complex Hilbert space $\mathcal{H}$. For every $\lambda \in \mathbb{C}$ we associate λ with the operator $T_\lambda = T - \lambda I$ where I denotes the identity operator in $\mathcal{D}(T)$. The inverse of T_λ (if it exists) is called the resolvent operator, which we denote by $R_\lambda(T) = (T - \lambda I)^{-1}$.*

- *The resolvent set $\rho(T)$ is the collection of all $\lambda \in \mathbb{C}$ such that R_λ exists, is bounded and defined on all of $\mathcal{H}$.*
- *The spectrum of T is defined as $\sigma(T) = \mathbb{C} \setminus \rho(T)$.*

The spectrum can then be decomposed in the following way:

- *The point spectrum is defined as $\sigma_p(T) = \{\lambda \in \mathbb{C} \mid \ker(T - \lambda I) \neq \{0\}\}$. Then $\lambda \in \sigma_p(T)$ is called an eigenvalue with multiplicity $\dim(\ker(T - \lambda I))$ and the corresponding eigenvector are all the non-zero elements of $\ker(T - \lambda I)$.*
- *The discrete spectrum $\sigma_d(T)$ is the set of all isolated*[5] *eigenvalues with finite multiplicity.*
- *The continuous spectrum $\sigma_c(T)$ is the set for which $\ker(T - \lambda I) = \{0\}$ and $Ran(T - \lambda I) \neq \mathcal{H}$ but $\overline{Ran(T - \lambda I)} = \mathcal{H}$.*
- *The residual spectrum $\sigma_r(T)$ is the set of all λ for which $T_\lambda = T - \lambda I$ has a bounded inverse not defined on all of $\mathcal{H}$.*
- *The essential spectrum $\sigma_{ess}(T)$ is defined as $\sigma_{ess} = \sigma(T) \setminus \sigma_d(T)$.*

The next two theorems demonstrate some nice spectral properties of self-adjoint operators.

Theorem 1.4.1. *The eigenvalues of a self-adjoint operator are real and the eigenvectors corresponding to distinct eigenvalues are orthogonal.*

Proof. Let λ be an eigenvalue. Then $Tu = \lambda u,\ u \neq 0$ and $\langle Tu, u\rangle = \lambda\langle u, u\rangle$. Since T is self-adjoint, $\langle Tu, u\rangle$ is real and $\langle u, u\rangle > 0$, therefore we conclude that λ must be real. Suppose $Tv = \mu v$ and $v \neq 0,\ \mu \neq \lambda$ then

$$\langle Tu, v\rangle - \langle u, Tv\rangle = \langle \lambda u, v\rangle - \langle u, \mu v\rangle = (\lambda - \mu)\langle u, v\rangle = 0$$

and since $\lambda \neq \mu$ we have $\langle u, v\rangle = 0$, hence $u \perp v$. □

Theorem 1.4.2. *The spectrum of a self-adjoint operator is real.*

Proof. We begin with a lemma whose proof can be found in [3] that states that if for some λ and $m > 0$

$$\|(T - \lambda I)u\| \geq m\|u\| \tag{1.28}$$

[4]The operator T is called closed if the set $\{\langle u, Tu\rangle \mid u \in \mathcal{D}(T)\}$ is a closed subset of $\mathcal{H} \bigoplus \mathcal{H}$.

[5]By isolated we mean that there exists a neighborhood around the eigenvalue λ of which there are no other points in the spectrum.

holds for all $u \in \mathcal{H}$ then $\lambda \in \rho(T)$. Now suppose $\lambda = a + bi,\ b \neq 0$ and $a, b \in \mathbb{R}$. The idea of the proof is to use the above lemma to show that $\lambda \in \rho(T)$. Let $v = (T - \lambda I)u$, then

$$\langle v, u\rangle = \langle Tu, u\rangle - \lambda\langle u, u\rangle$$
$$\langle u, v\rangle = \langle Tu, u\rangle - \overline{\lambda}\langle u, u\rangle$$

which follows from definition of an inner product and that $\langle Tu, u\rangle \in \mathbb{R}$ for all $u \in \mathcal{H}$ since T is self-adjoint. Now we can estimate

$$\langle v, u\rangle - \langle u, v\rangle = (\lambda - \overline{\lambda})\langle u, u\rangle = 2bi\|x\|^2 \tag{1.29}$$

with

$$2|b|\|u\|^2 \leq |\langle v, u\rangle| + |\langle u, v\rangle| = 2|\langle u, v\rangle| \leq 2\|u\|\|v\|. \tag{1.30}$$

where the last inequality follows from the Cauchy-Schwartz inequality. By applying (1.28) on (1.30)

$$|b|\|u\| \leq \|(T - \lambda I)u\| \tag{1.31}$$

we get $\lambda \in \rho(T)$, or equivalently, $\lambda \notin \sigma(T)$ since $b \neq 0$ and (1.28) holds for all $u \in \mathcal{H}$. □

The final theorem which concludes this section and chapter is the Min-Max principle. This will give us a characterization of the eigenvalues of T in terms of the sesquilinear form. We will assume $\sigma_{ess}(T) = \emptyset$ for the operator T which corresponds to $\mathfrak{s}$. Since $\mathfrak{s}$ is bounded, so is T and the eigenvalues can be numbered from below as $\lambda_1 \leq \lambda_2 \leq \ldots$ (with possibility of multiplicity).

Theorem 1.4.3. ***(The Min-Max principle)*** *Let the assumptions from Theorem 1.3.1 hold along with the additional assumption that $\sigma_{ess}(T) = \emptyset$. Then the eigenvalues $\lambda_1, \lambda_2, \ldots$ of T can be numbered as*

$$\lambda_1 \leq \lambda_2 \leq \ldots \tag{1.32}$$

counted with multiplicty. Each λ_i can then be written as

$$\lambda_i(T) = \min_{\substack{V \text{ subspace of } \mathcal{D}(\mathfrak{s}) \\ dim(V) = i}} \max_{\substack{u \in V \\ \|u\| = 1}} \mathfrak{s}[u, u] \tag{1.33}$$

where $i \in \mathbb{N}$.

Proof. See [2] page 265. □

Chapter 2

Quantum graphs

Our main goal in this chapter is to show that for a quantum graph equipped with the Hamiltonian operator with arbitrary self-adjoint vertex conditions which give rise to a nonnegative self-adjoint matrix Λ_v at each vertex $v \in \mathcal{V}$, the eigenvalue counting function of the graph (or equivalently, the Hamiltonian) follows the asymptotic law

$$N_\Gamma(k) \sim \frac{L}{\pi}k \text{ as } k \to \infty. \tag{2.1}$$

where $N_\Gamma(k) = \#\{\lambda \in \sigma(\Gamma) \mid \lambda \leq k^2\}$ denotes the eigenvalue counting function on Γ and $\sigma(\Gamma)$ denotes the spectrum of the graph (or equivalently, the spectrum of the Hamiltonian on Γ). Moreover, in the less general case where the graphs vertex conditions consists purely of Dirichlet and Kirchoff conditions, we will be able to derive a remainder term of constant order, i.e

$$N_\Gamma(k) = \frac{L}{\pi}k + \mathcal{O}(1). \tag{2.2}$$

The first two sections are devoted to the introduction of what quantum graphs are, along with the computation of the spectrum of a simple graph consisting of two vertices and an edge with Dirichlet conditions at both vertices. The following two sections aim to introduce how all self-adjoint realizations of the Hamiltonian occur in terms of the vertex conditions along with the description of the sesquilinear form of the Hamiltonian. More concretely, we will show that the given sesquilinear form satisfies the requirements of Theorem 1.3.1, therefore there exists a unique self-adjoint operator corresponding to this form. This operator will be shown to be the Hamiltonian. In addition, the spectrum of the Hamiltonian is purely discrete (or equivalently, the essential spectrum is empty) and hence the Min-max theorem (Theorem 1.4.3) holds. Section 2.5 aims to describe a certain kind of vertex conditions known as the extended δ-type, of which both the Dirichlet and Kirchoff conditions are a special case of. In Section 2.6 we will see what happens with the eigenvalues of a graph when changing the parameter in the extended δ-type of conditions, this so-called interlacing property will be the cornerstone in the proof of Weyl's law when considering a graph with purely Dirichlet and Kirchoff conditions. The final section of this chapter is devoted to Weyl's law in the more general case.

This chapter is very much based on the works by Gregory Berkolaiko and Peter Kuchment in their book *Introduction to Quantum Graphs*, see [5]. An easy-going introduction into the field of quantum graph with plenty of examples can be found in Gregory Berkolaiko's paper *An elementary introduction to Quantum Graphs*, see [6].

2.1 Introduction

We begin with the definition of a metric graph.

Definition 2.1.1. *A metric graph $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ is a set of vertices $\mathcal{V}$ and edges $\mathcal{E}$ such that each edge $e \in \mathcal{E}$ is assigned a positive number ℓ_e called the length of the edge. Each edge then corresponds to an interval $[0, \ell_e] \in \mathcal{I}$. The graph is compact if there are finitely many edges, each with finite length.*

Along this interval (or edge) we have the coordinate $x_e \in [0, \ell_i]$. This gives a natural orientation of the edges, however, this orientation can be made arbitrarily and have no bearing on the resulting theory. Edges who share a common vertex are called incident and the degree d_v of a vertex v is the number of edges connected to it. The two vertices connected by an edge are called edge ends, and since we associate each edge with an interval the edge ends are mapped to the end points on the interval. Since each edge is a positive interval we can then define a space of functions living on these edges, these will be taken to be the familiar L^2-and H^k spaces as in Example 1.1.1 and 1.1.2 respectively.

Definition 2.1.2. *The Lebesque space $\left(L^2(\Gamma), \|f\|^2_{L^2(\Gamma)}\right)$ and Sobolev space $\left(\tilde{H}^k(\Gamma), \|f\|^2_{H^k(\Gamma)}\right)$ of a metric graph Γ is respectively defined as*

$$L^2(\Gamma) = \bigoplus_{e \in \mathcal{E}} L^2(e) \qquad \|f\|^2_{L^2(\Gamma)} = \sum_{e \in \mathcal{E}} \|f\|^2_{L^2(e)}, \tag{2.3}$$

$$\tilde{H}^k(\Gamma) = \bigoplus_{e \in \mathcal{E}} H^k(e) \qquad \|f\|^2_{\tilde{H}^k(\Gamma)} = \sum_{e \in \mathcal{E}} \|f\|^2_{H^k(e)}. \tag{2.4}$$

Furthermore, we define the space $H^1(\Gamma)$ as

$$H^1(\Gamma) = \{f \in \tilde{H}^1(\Gamma) \mid f \text{ is continuous } \}. \tag{2.5}$$

We usually refer to $L^2(\Gamma)$ as the space of the graph. If Γ is compact, an element $f \in L^2(\Gamma)$ is a vector $f = (f_1, f_2, \ldots, f_{|\mathcal{E}|})$ where $f_e : L^2[0, \ell_e] \to \mathbb{C}$. In the definition of $H^1(\Gamma)$ we say that f is continuous, this should be interpreted as; for all edges e incident to a vertex v, $f_e(v)$ assumes the same value. In other words, $f(v)$ is uniquely defined.

Making this graph quantum is done by assigning an operator to it. In this thesis we will be working with an operator known as the Hamiltonian, which we denote by $\mathcal{L}$.

Definition 2.1.3. *The Hamiltonian operator $\mathcal{L} : \mathcal{D}(\mathcal{L}) \to L^2(\Gamma)$ on a graph $\Gamma = (\mathcal{V}, \mathcal{E})$ is the operator which acts as*

$$f(x_e) \mapsto -\frac{d^2 f}{dx_e^2}. \tag{2.6}$$

on each edge $e \in \mathcal{E}$. The domain $\mathcal{D}(\mathcal{L})$ of the operator consists of functions $f \in \tilde{H}^2(\Gamma)$ which satisfy some local self-adjoint conditions at the vertices.

This operator can be shown to be bounded from below (See Section 2.4) and as we've seen in Section 1.2, for operators of this type we need to find a suitable dense subspace. Since the operator acts as the negative second-derivative along the edges, it is natural to assume that $f_e \in H^2(e)$ on each edge, or equivalently, $f \in \tilde{H}^2(\Gamma)$. This is however not enough to make the Hamiltonian self-adjoint, but if one chooses certain kind of vertex conditions then $\mathcal{L}$ can be proven to be self-adjoint. Again, since the operator acts as the negative second-derivative, the boundary conditions, or perhaps rather the vertex conditions may only involve the values of $f_e(x)$ and $f_e'(x)$ at the edge ends. The derivative f_e' of f_e at some vertex v is taken in the outgoing direction (i.e away from the vertex). Before ending this section we summarize the definition of a quantum graph below.

Definition 2.1.4. *A quantum graph is a compact metric graph* $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ *equipped with the Hamiltonian operator* $\mathcal{L}$ *acting as the negative second-derivative on the functions along the edges. The operator domain consists of functions from* $\tilde{H}^2(\Gamma)$ *which satisfy some local self-adjoint matching conditions at the vertices.*

Before looking more closely at how one can choose these matching conditions, we will in the following section compute the eigenvalues of the trivial graph with the Dirichlet conditions.

2.2 The Trivial Graph

In this section we will compute the eigenvalues of the Hamiltonian on a graph with two vertices and one edge, which we recognize as simply the interval $[0, \ell]$, see Figure 2.1. The eigenvalues of the Hamiltonian acting on functions f defined on $[0, \ell]$ is given by the second-order linear differential equation

$$-f'' = k^2 f. \tag{2.7}$$

We count the eigenvalues of the graph in terms of k, which can easily be related back to the "true" eigenvalue λ by $\lambda = k^2$. The solution to (2.7) is given by

$$f(x) = A\cos(kx) + B\sin(kx) \tag{2.8}$$

where the constants A, B is determined by the vertex (boundary) conditions. We will look at two types of boundary conditions, namely the Dirichlet and the Neumann conditions. The Dirichlet and Neumann conditions at a vertex v are defined as

$$\textbf{Dirichlet}: f_e(v) = 0 \text{ for all edges } e \text{ incident to } v. \tag{2.9}$$
$$\textbf{Neumann}: f_e'(v) = 0, \text{for all edges } e \text{ incident to } v, \text{ where the} \tag{2.10}$$
$$\text{direction of the derivative is taken from the vertex} \tag{2.11}$$
$$\text{into the edge.} \tag{2.12}$$

In the case of the trivial graph $[0, \ell]$, the Dirichlet conditions at both vertices are equivalent to $f(0) = f(\ell) = 0$ and the Neumann conditions are equivalent to $f'(0) = 0,\ -f'(L) = 0$. We don't have to bother looking for any negative eigenvalues. Indeed, consider the inner product of $-f''$ with f. By partial integration and using that $f(0) = f(\ell) = 0$ in the Dirichlet case, or $f'(0) = 0,\ -f'(L) = 0$ in the Neumann

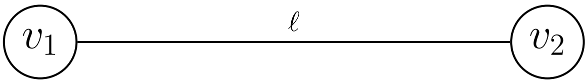

Figure 2.1: *A graph with 2 vertices and 1 edge with length ℓ.*

case, we get

$$\begin{aligned}\langle -f'', f\rangle &= \int_0^\ell -f''(x)\overline{f(x)}\;dx = \big[-f'(x)\overline{f(x)}\big]_0^\ell - \int_0^\ell (-f'(x))\overline{f'(x)}\;dx \\ &= \int_0^\ell |f'(x)|^2\;dx \geq 0. \end{aligned} \tag{2.13}$$

Now suppose λ is an eigenvalue. Then

$$\langle -f'', f\rangle = \langle \lambda f, f\rangle = \lambda\langle f, f\rangle \tag{2.14}$$

and so we see that (2.13) is clearly non-negative and for (2.14) to be non-negative $\lambda > 0$ since $\langle f, f\rangle \geq 0$. We begin by trying to find the positive eigenvalues in the Dirichlet case. Assume $\lambda > 0$, then solving (2.7) using (2.8) and (2.9) we get

$$f(0) = A\cos(k\cdot 0) + B\sin(k\cdot 0) = A = 0$$

Since $A = 0$ and $f\ell) = 0$ we can solve for the eigenvalues (in terms of k)

$$f(\ell) = B\sin(k\ell) = 0 \Longrightarrow k = \frac{\pi n}{\ell},\; n = 1, 2, 3\ldots$$

Note that we're not interested in the trivial solution $f(x) \equiv 0$ and so $B \neq 0$. Then each positive eigenvalue can be written as $\lambda_n = (\frac{\pi n}{\ell})^2$ with the corresponding eigenfunctions $f_n(x) = \sin(\frac{\pi n x}{\ell}),\; n \in \mathbb{N}$. In the case of $\lambda = 0$, we get $-f'' = 0$ which is solved by $f(x) = Ax + B$, with the vertex conditions at the endpoints we arrive at the trivial solution $f(x) = 0$. In conclusion, the only eigenvalues of Γ with the Dirichlet conditions imposed on both vertices are $\lambda_n = \big(\frac{\pi n}{\ell}\big)^2,\; n \in \mathbb{N}$, with corresponding eigenfunctions $f_n(x) = \sin(\frac{\pi n x}{\ell})$. In addition, the eigenfunctions are orthogonal which is shown by the following computation with $n_i \geq 1, i = 1, 2$ and constants $c_i = \frac{\pi n_i}{\ell}$ we have

$$\begin{aligned}\int_0^\ell \sin(c_1 x)\sin(c_2 x)dx &= \int_0^\ell \frac{\cos((c_1 - c_2)x) - \cos((c_1 + c_2)x)}{2}dx \\ &= \left[\frac{\sin((c_1 - c_2)x)}{c_1 - c_2} - \frac{\sin((c_1 + c_2)x)}{c_1 + c_2}\right]_0^\ell \\ &= 0.\end{aligned}$$

Thus all eigenvalues are real and the corresponding eigenfunctions are orthogonal. As we will see later, these Dirichlet conditions actually give rise to a self-adjoint operator, and so these properties are to be expected in accordance to Theorem 1.4.1 and Theorem 1.4.2. Moreover, due to Theorem 2.4.4, the spectrum of the self-adjoint Hamiltonian is purely discrete and so in this example, $\sigma(\Gamma) = \{\big(\frac{\pi n}{\ell}\big)^2,\; n \in \mathbb{N}\}$. The eigenvalue counting function of this graph can then be written as

$$N_\ell(k) = \#\{\lambda \in \sigma(\Gamma) | \lambda \leq k^2\} = \left\lfloor \frac{k\ell}{\pi} \right\rfloor \tag{2.15}$$

where the brackets denotes the lower integer part function. If we instead consider a graph $\Gamma_D = (\mathcal{V}, \mathcal{E})$ with Dirichlet conditions at every vertex, then this a completely decoupled graph. If we consider one Dirichlet interval at first, the eigenfunction of this interval can be extended to the whole graph by setting it identical to zero on the rest of the intervals (and so the Dirichlet conditions are trivially fulfilled). The union of the spectra of these intervals are then contained in the spectra of the graph. Conversely, restricting an eigenfuncton of Γ_D to any interval gives an eigenfunction of this interval. hence the following equality hold

$$\sigma(\Gamma_D) = \bigcup_{e \in \mathcal{E}} \left\{ \left(\frac{\pi n}{\ell_e}\right)^2, \ n \in \mathbb{N} \right\} \tag{2.16}$$

and the corresponding counting function $N_{\Gamma_D}(k)$ is just the sum of all individual counting functions on the edges, that is

$$N_{\Gamma_D}(k) = \sum_{e \in \mathcal{E}} N_{\ell_e}(k). \tag{2.17}$$

Computing the spectrum of the Neumann conditions is very similar to what we have already done. We have $f'(0) = 0$ which implies $B = 0$ in (2.7) since $k > 0$. To avoid the trivial solution, $A \neq 0$, which together with $-f'(\ell) = 0$ implies that $\sin(k\ell) = 0$, hence the solutions are $k = \frac{\pi n}{\ell}$, $n \in \mathbb{N}$ and the eigenvalues on a graph with Neumann conditions at both vertices are exactly the same as in the Dirichlet case, namely $\lambda_n = \left(\frac{\pi n}{\ell}\right)^2$, $n \in \mathbb{N}$ with the corresponding eigenfunctions $f_n(x) = \cos(\frac{\pi n x}{\ell})$. If we consider a graph consisting of Neumann conditions at every vertex, the spectrum of such a graph is, just as in the Dirichlet case, the union of the spectra of each edge (the Neumann conditions are yet another example of a decoupling condition).

2.3 Vertex conditions

As we have already seen in the previous section, with the Dirichlet boundary conditions the Hamiltonian is seemingly a self-adjoint operator. In this section we will give a description how all self-adjoint realizations of the Hamiltonian arise. We will see that this can be done in two (equivalent) ways, one in terms of two matrices A_v, B_v and the other in terms of three projectors. It should be noted that these conditions are local, that is, we are considering one vertex v at a time. The Hamiltonian act as a negative second derivative on each edge and so we need have two conditions per edge, or d_v conditions per vertex. These conditions involve the values of f and f' at v. Now, consider a vertex v with degree d_v and functions $f_1, \ldots, f_{d_v}$ on the edges incident to v. We may then define the column vectors $F(v), F'(v)$ as

$$F(v) = \begin{bmatrix} f_1(v) \\ f_2(v) \\ \vdots \\ f_{d_v}(v) \end{bmatrix} \qquad F'(v) = \begin{bmatrix} f_1'(v) \\ f_2'(v) \\ \vdots \\ f_{d_v}'(v) \end{bmatrix}. \tag{2.18}$$

The homogeneous conditions for which $F(v), F'(v)$ must satisfy at a vertex v can be written using two $d_v \times d_v$-matrices A_v, B_v such that

$$A_v F(v) + B_v F'(v) = 0 \tag{2.19}$$

and to ensure d_v independent conditions, we require the $d_v \times 2d_v$-matrix $(A_v \ B_v)$ to be of maximal rank for all $v \in \mathcal{V}$.

Theorem 2.3.1. *Let $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a compact metric graph. The Hamiltonian $\mathcal{L}$ acting on functions $f \in \tilde{H}^2(\Gamma)$ which satisfy some local vertex conditions involving the values of the function and their derivatives at the vertices is self-adjoint if and only if the vertex conditions can be written in any of the two equivalent forms.*

1. *There exists $d_v \times d_v$-matrices A_v and B_v such that the $d_v \times 2d_v$-matrix $(A_v \ B_v)$ has maximal rank and $A_v B_v^*$ is self-adjoint for every vertex $v \in \mathcal{V}$ with degree d_v. The boundary values of f at v should also satisfy $A_v F(v) + B_v F'(v) = 0$.*

2. *There exist three mutually orthogonal projectors $P_{D,v}, P_{N,v}$ and $P_{R,v} = \mathbb{I} - P_{D,v} - P_{N,v}$ acting in $\mathbb{C}^{d_v}$ and a self-adjoint operator Λ_v acting in $P_{R,v}\mathbb{C}^{d_v}$ for each vertex $v \in \mathcal{V}$ such that the boundary values of f at v satisfy*

$$\begin{cases} P_{D,v}F(v) = 0 \\ P_{N,v}F'(v) = 0 \\ P_{R,v}F'(v) = \Lambda_v P_{R,v}F(v). \end{cases} \tag{2.20}$$

Proof. See [5]. □

We will in Section 2.5 see how the matrices and projectors can be chosen when we consider a certain type of conditions called the extended δ-type vertex conditions, of which the Dirichlet conditions in Section 2.2 are a special case. Before doing that, we will take a look at the sesquilinear form of the Hamiltonian. In order to do this, writing the vertex conditions in terms of the projectors as in (2.20) is the most suitable choice.

2.4 Sesquilinear form of the Hamiltonian

Our main goal in this section is to show that the sesquilinear form in Definition 2.4.1 is densely defined, symmetric, bounded from below and closed. Then due to Theorem 1.3.1 there exists a unique self-adjoint operator corresponding to that form. This corresponding self-adjoint operator will be then proven to be the Hamiltonian operator.

Definition 2.4.1. *Let $\mathfrak{s}$ denote the sesquilinear form defined as*

$$\mathfrak{s}[f,g] = \sum_{e \in \mathcal{E}} \int_e f'(x)\overline{g'(x)}dx + \sum_{v \in \mathcal{V}} \langle \Lambda_v P_{R,v}F(v), P_{R,v}G(v) \rangle \tag{2.21}$$

Here $\langle , \rangle$ denotes the standard inner product in $P_{R,v}\mathbb{C}^{d_v}$. The domain $\mathcal{D}(\mathfrak{s})$ consists of functions belonging to $H^1(e)$ on each edge with the added condition that $P_{D,v}F(v) = 0$ for all $v \in \mathcal{V}$. The corresponding quadratic form is then

$$\mathfrak{q}[f] = \sum_{e\in\mathcal{E}} \int_e |f'(x)|^2 dx + \sum_{v\in\mathcal{V}} \langle \Lambda_v P_{R,v} F(v), P_{R,v} F(v)\rangle \tag{2.22}$$

for $f \in \mathcal{D}(\mathfrak{s})$.

Showing that (2.21) satisfies Definition 1.3.1 of a sesquilinear form is straightforward and therefore will be omitted. In order to show that $\mathfrak{s}$ is bounded from below we need an estimate in the form of the below lemma.

Lemma 2.4.1. *Let* $f \in H^1[0,\ell]$ *then for any* $0 < \delta \le \ell$,

$$|f(0)|^2 \le \frac{2}{\delta}\|f\|_{L^2[0,\ell]} + \delta\|f'\|^2_{L^2[0,\ell]}. \tag{2.23}$$

Proof. Since $f \in H^1[0,\ell]$, f is absolutely continuous[1] and can be written on the form

$$f(x) = f(0) + \int_0^x f'(t)dt \tag{2.24}$$

for all $x \in [0,\ell]$. We denote the indicator function by $\mathbf{1}_{[0,\ell]}$, then using Cauchy-Schwartz we have

$$\left|\int_0^x f'(t)dt\right|^2 = \left|\int_0^x \mathbf{1}_{[0,\ell]} f'(t)dt\right|^2 \le \|\mathbf{1}_{[0,x]}\|^2_{L^2[0,\ell]}\|f'\|^2_{L^2[0,\ell]} = x\|f'\|^2_{L^2[0,\ell]}. \tag{2.25}$$

By taking the $L^2(0,\delta)$-norm on (2.25), we get

$$\left\|\int_0^x f'(t)dt\right\|^2_{L^2[0,\delta]} \le \|x\|^2_{L^2[0,\delta]}\|f'\|^2_{L^2[0,\ell]} = \frac{\delta^2}{2}\|f'\|^2_{L^2[0,\ell]}. \tag{2.26}$$

Solving for $f(0)$ in (2.23) and taking the $L^2[0,\delta]$-norm on both sides and using the standard inequality $(a+b)^2 \le 2a^2 + 2b^2$, we end up with

$$\|f(0)\|^2_{L^2[0,\delta]} = \delta|f(0)|^2 \le 2\|f\|^2_{L^2[0,\delta]} + \delta^2\|f'\|_{L^2[0,\ell]} \tag{2.27}$$

and dividing by δ finishes the proof. □

Theorem 2.4.2. *The sesquilinear form* $\mathfrak{s}$ *defined in Defintion 2.4.1 is densely defined, symmetric, semi-bounded from below and closed.*

Proof. The form is densely defined since $\bigoplus_{e\in\mathcal{E}} C^\infty_{comp}(e) \subset \mathcal{D}(\mathfrak{s})$ is dense in $L^2(\Gamma)$. Symmetry is shown quite easily by using the standard properties of inner products

[1] See [8] page 31 for proof of this claim.

and the fact that Λ_v is self-adjoint. The computation is straight-forward as can be seen below

$$\overline{\mathfrak{s}[g,f]} = \overline{\sum_{e\in E}\int_0^{L_e} g'(x)\overline{f'(x)}dx + \sum_{v\in\mathcal{V}}\langle \Lambda_v P_{R,v}F_v, P_{R,v}G_v\rangle} \tag{2.28}$$

$$= \sum_{e\in E}\int_0^{L_e} f'(x)\overline{g'(x)}dx + \sum_{v\in\mathcal{V}}\langle P_{R,v}F, \Lambda_v P_{R,v}G_v\rangle \tag{2.29}$$

$$= \sum_{e\in E}\int_0^{L_e} f'(x)\overline{g'(x)}dx + \sum_{v\in\mathcal{V}}\langle \Lambda_v P_{R,v}F_v, P_{R,v}G_v\rangle \tag{2.30}$$

$$= \mathfrak{s}[f,g]. \tag{2.31}$$

for all $f,g \in \mathcal{D}(\mathfrak{s})$. Next we will show that $\mathfrak{s}$ is bounded from below. Since Λ_v is self-adjoint, the eigenvalues of Λ_v are real and with

$$\lambda_{max} = \max_{v\in\mathcal{V}}\{|\lambda| \ : \ \lambda \in \sigma(\Lambda_v)\} \tag{2.32}$$

the following inequality holds

$$\sum_{v\in\mathcal{V}}\langle \Lambda_v P_{R,v}F_v, P_{R,v}F_v\rangle \leq \lambda_{max}\sum_{v\in\mathcal{V}}|P_{R,v}F_v|^2 \leq \lambda_{max}\sum_{v\in\mathcal{V}}|F_v|^2 \tag{2.33}$$

since $F_v = P_{R,v}F_v + P_{D,v}F_v + P_{N,v}F_v$ and $P_{R,v}, P_{D,v}, P_{N,v}$ are mutually orthogonal, then $|F_v|^2 = |P_{R,v}F_v|^2 + |P_{D,v}F_v + P_{N,v}F_v|^2$ and $|F|^2 \geq |P_{R,v}F_v|^2$. Let $\mathfrak{q}[f]$ denote the quadratic form as in Definition 2.4.1, then

$$\mathfrak{q}[f] = \|f'\|^2_{L^2(\Gamma)} + \sum_{v\in\mathcal{V}}\langle \Lambda_v P_{R,v}F_v, P_{R,v}F_v\rangle \tag{2.34}$$

$$\geq \|f'\|^2_{L^2(\Gamma)} - \lambda_{max}\sum_{v\in\mathcal{V}}|F_v|^2 \tag{2.35}$$

$$\geq \|f'\|^2_{L^2(\Gamma)} - 2\lambda_{max}\sum_{e\in\mathcal{E}}\left(\frac{2}{\delta}\|f\|^2_{L^2(e)} + \delta\|f'\|^2_{L^2(e)}\right) \tag{2.36}$$

$$= (1-2\delta\lambda_{max})\|f'\|^2_{L^2(\Gamma)} - \frac{4\lambda_{max}}{\delta}\|f\|^2_{L^2(\Gamma)} \tag{2.37}$$

where we applied Lemma 2.4.1 at all the vertices with the same parameter δ chosen such that $\ell_{min} \geq \delta > 0$. In addition, if we choose $\delta \leq \frac{1}{2\lambda_{max}}$ then $1 - 2\lambda_{max}\delta \geq 0$ and we can disregard the $\|f'\|^2_{L^2(\Gamma)}$-term. That is, by choosing $\delta \leq \min\{\ell_{min}, \frac{1}{2\lambda_{max}}\}$ we can find a constant $c > 0$ such that

$$\mathfrak{q}(f) \geq -c\|f\|^2_{L^2(\Gamma)} \tag{2.38}$$

holds. Let c_0 denote the optimal bound. To show closedness of the form, we need to show that $\mathcal{D}(\mathfrak{q})$ is complete with the norm

$$\|f\|_{\mathfrak{q}} := \sqrt{\mathfrak{q}[f] + (1+c_0)\|f\|^2_{L^2(\Gamma)}}. \tag{2.39}$$

If we can show that there exists some $\alpha, \beta > 0$ such that

$$\alpha\|f\|_{H^1(\Gamma)} \leq \|\cdot\|_{\mathfrak{q}} \leq \beta\|f\|_{H^1(\Gamma)} \tag{2.40}$$

then the two norms are equivalent and $(\mathcal{D}(\mathfrak{q}), \|\cdot\|_{\mathfrak{q}})$ is just $H^1(\Gamma)$, which is known to be a Hilbert space (and hence complete). It is easy to see that we can find such α, β. Indeed, by squaring the norm and using the already familiar inequalities we get

$$\mathfrak{q}[f] + (1+c_0)\|f\|^2_{L^2(\Gamma)} \leq \|f'\|^2_{L^2(\Gamma)} + 2\lambda_{max}\Big(\frac{2}{\delta}\|f\|^2_{L^2(\Gamma)} + \delta\|f\|^2_{L^2(\Gamma)}\Big) \tag{2.41}$$
$$+ (1+c_0)\|f\|^2_{L^2(\Gamma)} \tag{2.42}$$
$$\leq \beta^2(\|f\|^2_{L^2(\Gamma)} + \|f'\|^2_{L^2(\Gamma)}) \tag{2.43}$$
$$= \beta^2\|f\|^2_{H^1(\Gamma)} \tag{2.44}$$

which holds if we choose β large enough. With $c = c_0 + 1$ and putting together (2.37) and (2.39) we get

$$(1 - 2\delta\lambda_{max})\|f'\|^2_{L^2(\Gamma)} - \frac{4\lambda_{max}}{\delta}\|f\|^2_{L^2(\Gamma)} + c\|f\|^2_{L^2} > 0 \tag{2.45}$$

and we can write (2.45) as some $\alpha^2\|f\|^2_{H^1(\Gamma)} > 0$ with $\alpha > 0$, then

$$\mathfrak{q}[f] + (1+c_0)\|f\|^2_{L^2(\Gamma)} \geq (1 - 2\delta\lambda_{max})\|f'\|^2_{L^2(\Gamma)} - \frac{4\lambda_{max}}{\delta}\|f\|^2_{L^2(\Gamma)} + (1+c_0)\|f\|^2_{L^2}$$
$$\geq \alpha^2\|f\|^2_{H^1(\Gamma)}$$

and hence the two norms are equivalent. Showing that $\mathcal{D}(\mathfrak{q})$ is complete in the norm $\|\cdot\|_{\mathfrak{q}}$ is then equivalent to showing that $\mathcal{D}(\mathfrak{q})$ is a closed subspace of $H^1(\Gamma)$. Since $H^1(\Gamma)$ is complete with $\|\cdot\|_{H^1(\Gamma)}$ norm, every convergent sequence in $\mathcal{D}(\mathfrak{q})$ has a limit in $H^1(\Gamma)$, moreover, due to Lemma 2.4.1, the limit function itself belongs to $\mathcal{D}(\mathfrak{q})$. □

Since (2.18) satisfy the requirements of Theorem 1.3.1, we will now show that the corresponding operator is actually the Hamiltonian.

Theorem 2.4.3. *The unique self-adjoint operator corresponding to the sesquilinear form $\mathfrak{s}$ in Definition 2.4.1 is the Hamiltonian operator $\mathcal{L}$.*

Proof. By Theorem 1.3.1, the corresponding operator $\mathcal{G}$ satisfies $\mathfrak{s}[f,g] = \langle \mathcal{G}f, g\rangle$ for all $f \in \mathcal{D}(\mathcal{G})$, $g \in \mathcal{D}(\mathfrak{s})$ and $\mathcal{D}(\mathcal{G}) \subset \mathcal{D}(\mathfrak{s}) = \{f \in \tilde{H}^1(\Gamma) \mid P_{D,v}F_v = 0, \forall v \in \mathcal{V}\}$. We will begin by showing that the operator acts as the negative second derivative along the edges. Pick any $f \in \mathcal{D}(\mathcal{G})$, then we can find an $h \in L^2(\Gamma)$ such that

$$\mathfrak{s}[f,g] = \langle h, g\rangle \tag{2.46}$$

for all $g \in \tilde{H}^1(\Gamma)$. If we choose our g such that $g_e \in C^\infty_{comp}(e)$ on each edge, then $G_v = 0$ for all $v \in \mathcal{V}$ and $\sum_{v\in\mathcal{V}}\langle \Lambda_v P_{R,v}F_v, P_{R,v}G_v\rangle = 0$. Equation (2.46) can then be written as

$$\sum_{e\in\mathcal{E}}\int_e f'(x)\overline{g'(x)}dx = \sum_{e\in\mathcal{E}}\int_e h(x)\overline{g(x)}dx \tag{2.47}$$

and by partial integration we get

$$-\sum_{e\in\mathcal{E}}\int_e f''(x)\overline{g(x)}dx = \sum_{e\in\mathcal{E}}\int_e h(x)\overline{g(x)}dx. \tag{2.48}$$

since g_e vanishes on the boundary. Then by matching respective edge, we see that $h_e(x) = \mathcal{G}f_e = -f''(x_e)$ and $f_e(x) \in H^2(e)$ since $f''(x_e) \in L^2(e)$. Next, we'd like to show that the functions $f \in \mathcal{D}(\mathcal{G})$ satisfy the self-adjoint vertex conditions as in the second part of Theorem 2.3.1. The first condition $P_{R,v}F = 0$ holds trivially since $f \in \mathcal{D}(\mathcal{G}) \subset \mathcal{D}(\mathfrak{s})$. Now, pick a function $g \in \mathcal{D}(\mathfrak{s})$ which is non-zero in a small neighborhood around a single vertex, then using partial integration again in (2.46), we can cancel the integral terms and be left with

$$\langle \Lambda_v P_{R,v} F_v, G_v \rangle = \langle F'_v, G_v \rangle \tag{2.49}$$

Since G can be chosen arbitrarily and $P_{D,v}G = 0$, we get

$$\begin{aligned}(\Lambda_v P_{R,v} F_v - F'_v) &\in (\ker(P_{D,v}))^{\perp} = \operatorname{ran}(P_{D,v}) \qquad (2.50)\\ &= \ker(\mathbb{I} - P_{D,v}) = \ker(P_{N,v} + P_{R,v}) \qquad (2.51)\end{aligned}$$

thus

$$(P_{N,v} + P_{R,v})(\Lambda_v P_{R,v} F_v - F'_v) = 0 \tag{2.52}$$

which reduces to

$$\Lambda_v P_{R,v} F_v - P_{N,v} F'_v - P_{R,v} F'_v = 0 \tag{2.53}$$

and by applying $P_{N,v}$ to both sides, we see that $P_{N,v}F'_v = 0$ and $\Lambda_v P_{R,v} F_v = P_{R,v} F'_v$. Conversely, we'd like to show that if $f \in \widetilde{H}^2(\Gamma)$ and satisfy $P_{D,v}F_v = 0,\ P_{N,v}F'_v = 0$ and $\Lambda_v P_{R,v} F_v = P_{R,v} F'_v$ then f belongs to the domain of $\mathcal{G}$. For any $g \in \mathcal{D}(\mathfrak{s})$,

$$\begin{aligned}\langle -f'', g \rangle &= -\sum_{e \in \mathcal{E}} \int_e (-f'(x))\overline{g'(x)}dx + \sum_{v \in \mathcal{V}} \langle F', G \rangle \qquad (2.54)\\ &= \sum_{e \in \mathcal{E}} \int_e f'(x)\overline{g'(x)}dx \qquad (2.55)\\ &+ \sum_{v \in \mathcal{V}} \langle P_{D,v}F' + P_{N,v}F' + P_{R,v}F', P_{D,v}G + P_{N,v}G + P_{R,v}G \rangle \qquad (2.56)\end{aligned}$$

which simplifies to

$$\sum_{e \in \mathcal{E}} \int_e f'(x)\overline{g'(x)}dx + \sum_{v \in \mathcal{V}} \langle \Lambda_v P_{R,v} F_v, P_{R,v} G_v \rangle \tag{2.57}$$

using that $P_{N,v}F'_v = 0,\ P_{R,v}F'_v = \Lambda_v P_{R,v} F_v,\ P_{D,v}G_v = 0$ and orthogonality of the projectors. □

Our final theorem in this section states that the spectrum of the graph (or equivalently, the spectrum of the Hamiltonian) is purely discrete. In other words, $\sigma_{ess}(\mathcal{L}) = \emptyset$ and the Min-max theorem (Theorem 1.4.3) holds.

Theorem 2.4.4. *The spectrum of $\mathcal{L}$ is purely discrete.*

Proof. See [5]. □

2.5 The extended δ-type vertex conditions

Our focus in this section is to describe a certain kind of vertex conditions known as the extended δ-type and derive its sesquilinear form. The δ-type of vertex conditions can be summarized as

Definition 2.5.1. *The δ-type of vertex conditions at a vertex $v \in \mathcal{V}$ are defined as;*

$$\begin{cases} f \text{ is continuous on } \Gamma, \\ \sum_{i=1}^{d_v} \frac{df_i}{dx}(v) = \alpha_v f(v) \end{cases} \tag{2.58}$$

for $-\infty < \alpha_v < \infty$ with the direction of the derivative is outgoing; from the vertex into the edge. With $\alpha_v = 0$ we get the Kirchoff conditions.

The corresponding projectors as in Theorem 2.3.1 at a vertex v with degree d_v endowed with the δ-type of vertex conditions are

$$P_{D,v} = \begin{bmatrix} \frac{d_v-1}{d_v} & -\frac{1}{d_v} & \dots & -\frac{1}{d_v} \\ -\frac{1}{d_v} & \frac{d_v-1}{d_v} & \dots & -\frac{1}{d_v} \\ \dots & \dots & \dots & \dots \\ -\frac{1}{d_v} & -\frac{1}{d_v} & \dots & \frac{d_v-1}{d_v} \end{bmatrix}, \qquad P_{R,v} = \frac{1}{d_v} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \dots & \dots & \dots & \dots \\ 1 & 1 & \dots & 1 \end{bmatrix} \tag{2.59}$$

with $P_{N,v} = 0$ and $\Lambda_v = \frac{\alpha_v}{d_v}$. With a simple computation we get the contribution to the quadratic form at the vertex v endowed with the δ-type condition as

$$\langle \Lambda_v P_{R,v} F_v, P_{R,v} F_v \rangle = \frac{\alpha_v}{d_v} \langle P_{R,v} F_v, P_{R,v} F_v \rangle = \frac{\alpha_v}{d_v} \langle F_v, F_v \rangle = \alpha_v |f(v)|^2 \tag{2.60}$$

and of course the contribution vanish with the Kirchoff conditions since $\alpha_v = 0$. The form domain of a graph purely consisting of δ-type conditions at all vertices is simply $H^1(\Gamma)$. The extended δ-type of conditions is to allow $\alpha_v = \infty$. By dividing (2.58) with α_v and taking the limit, we simply get the condition

$$\begin{cases} f \text{ is continuous on } \Gamma, \\ f(v) = 0 \end{cases} \tag{2.61}$$

which we simply recognize as the Dirichlet conditions. The domain of this form is quite not the same as for the δ-type. Instead, for functions on the edges connected to a vertex with $\alpha_v = \infty$, we set $f(v) = 0$. Similarly, as in the Kirchoff case, the Dirichlet condition will give no contribution to the quadratic form since $F(v) = 0$.

2.6 Eigenvalue interlacing

Now we will look at a certain kind of interlacing property of quantum graphs which will later be proven to be useful in the proof of Weyl's law in Section 2.7. The idea is to look what happens with the eigenvalues of a graph when changing the parameter α in the δ-type of vertex conditions at a single vertex v. Note that we do not assume δ-type of vertex conditions at the other vertices, these can be made arbitrary with the only condition that the Hamiltonian is still self-adjoint.

Theorem 2.6.1. *Let $\Gamma_\alpha = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a metric graph equipped with the Hamiltonian operator. Let v be a distinguished vertex of Γ_α endowed with the δ-type of condition with some parameter $0 < \alpha \leq \infty$. The remaining vertices are endowed with arbitrary self-adjoint conditions. Let $\Gamma_{\alpha'}$ be the same graph obtained by changing α to α' where $-\infty < \alpha \leq \alpha' \leq \infty$. Then the following chain of inequalities hold*

$$\lambda_n(\Gamma_\alpha) \leq \lambda_n(\Gamma_{\alpha'}) \leq \lambda_{n+1}(\Gamma_\alpha). \tag{2.62}$$

Proof. We begin by first assuming that $-\infty < \alpha \leq \alpha' < \infty$. The quadratic forms of each respective graph Γ_α, $\Gamma_{\alpha'}$ is

$$\mathfrak{q}_\alpha[f] = \sum_{e\in\mathcal{E}} \int_e f'(x)\overline{g'(x)}dx + \alpha|f(v)|^2 + \sum_{v\in\mathcal{V}'} \langle \Lambda_v P_{R,v}F_v, P_{R,v}F_v\rangle \tag{2.63}$$

$$\mathfrak{q}_{\alpha'}[f] = \sum_{e\in\mathcal{E}} \int_e f'(x)\overline{g'(x)}dx + \alpha'|f(v)|^2 + \sum_{v\in\mathcal{V}'} \langle \Lambda_v P_{R,v}F_v, P_{R,v}F_v\rangle \tag{2.64}$$

where $\mathcal{V}'$ denotes the set $\mathcal{V}$ with our distinguished δ-type vertex v removed and $f \in H^1(\Gamma_\alpha)$. Since we have shown that the sesquilinear form satisfies the requirements of the Min-max theorem (Theorem 1.4.3) in Section 2.4, the eigenvalues of the graph can then be written as

$$\lambda_n = \min_{\substack{V \text{ subspace of } \mathcal{D}(\mathfrak{q}) \\ dim(V) = n}} \max_{\substack{f\in V \\ \|f\|=1}} \mathfrak{q}[f]. \tag{2.65}$$

As a direct consequence we get the first inequality

$$\lambda_n(\Gamma_\alpha) \leq \lambda_n(\Gamma_{\alpha'}) \tag{2.66}$$

since $\mathfrak{q}_\alpha \leq \mathfrak{q}_{\alpha'}$ and $\mathcal{D}(\mathfrak{q}_\alpha) = \mathcal{D}(\mathfrak{q}_{\alpha'})$. Let Γ_∞ denote the same graph as Γ_α but with $\alpha = \infty$. The corresponding form is

$$\mathfrak{q}_\infty[f] = \sum_{e\in\mathcal{E}} \int_e f'(x)\overline{g'(x)}dx + \sum_{v\in\mathcal{V}'} \langle \Lambda_v P_{R,v}F_v, P_{R,v}F_v\rangle \tag{2.67}$$

with the domain $\mathcal{D}(\mathfrak{q}_\infty)$ consisting of all $f \in \mathcal{D}(\mathfrak{q}_\alpha)$ with $f(v) = 0$ at the particular δ-type vertex v. Again, by the Min-max theorem,

$$\lambda_n(\Gamma_{\alpha'}) \leq \lambda_n(\Gamma_\infty). \tag{2.68}$$

holds true since on the domain of $\mathcal{D}(\mathfrak{q}_\infty)$, $\mathfrak{q}_\infty = \mathfrak{q}_\alpha$, and minimizing over a smaller domain yields a bigger result. The final inequality we would like to show is

$$\lambda_n(\Gamma_\infty) \leq \lambda_{n+1}(\Gamma_\alpha). \tag{2.69}$$

Let $\mathfrak{q}_\alpha$ obtain its minimum on a subspace U which is spanned by the first $n+1$ eigenvectors with $\dim(U) = n+1$. Then we can find a subspace U_∞ with $\dim(U_\infty) = n$ such that $U_\infty \subset U$ and $U_\infty \subset \mathcal{D}(\mathfrak{q}_\infty)$. Then

$$\lambda_n(\Gamma_\infty) = \min_{\substack{V \text{ subspace of } \mathcal{D}(\mathfrak{q}) \\ dim(V) = n}} \max_{\substack{f\in V \\ \|f\|=1}} \mathfrak{q}_\infty[f] \leq \max_{\substack{f\in U_\infty \\ \|f\|=1}} \mathfrak{q}_\infty[f] \tag{2.70}$$

since the values of the quadratic form agree on the subspaces V and U_∞. Next,

$$\max_{\substack{f\in U_\infty\\ \|f\|=1}} \mathfrak{q}_\infty[f] = \max_{\substack{f\in U_\infty\\ \|f\|=1}} \mathfrak{q}_\alpha[f] \tag{2.71}$$

$$\leq \max_{\substack{f\in U\\ \|f\|=1}} \mathfrak{q}_\alpha[f] = \lambda_{n+1}(\Gamma_\alpha), \tag{2.72}$$

where the first equality follows from the fact that we consider functions f for which $f(v) = 0$ and $\mathfrak{q}_\alpha = \mathfrak{q}_\infty$ on U_∞. The second inequality follows easily since we just increase the domain over which we're maximizing the quadratic form. Putting the chain of inequalities together we arrive at

$$\lambda_n(\Gamma_\alpha) \leq \lambda_n(\Gamma_{\alpha'}) \leq \lambda_n(\Gamma_\infty) \leq \lambda_{n+1}(\Gamma_\alpha) \tag{2.73}$$

which is equivalent to the statement of the theorem. □

2.7 Weyl's Law for Kirchoff and Dirichlet vertex conditions

Now we are finally ready for the proof of Weyl's law. We will assume that every vertex of the graph is either endowed with Kirchoff or Dirichlet vertex conditions (we allow the graph to be mixed). The idea of the proof is simple; we want to bound the eigenvalue counting function $N_\Gamma(k)$ on Γ by an already known counting function plus or minus some constant.

Theorem 2.7.1. *Let $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a metric graph equipped with the Hamiltonian operator. At each vertex the conditions are either Dirichlet or Kirchoff. We denote the length of the graph as $L = \ell_1 + \ell_2 + \ldots + \ell_{|\mathcal{E}|}$ where ℓ_i corresponds to the length of edge $e_i \in \mathcal{E}$. Then the eigenvalue counting function $N_\Gamma(k) = \#\{\lambda \in \sigma(\Gamma) \mid \lambda \leq k^2\}$ can be written as*

$$N_\Gamma(k) = \frac{L}{\pi}k + \mathcal{O}(1) \tag{2.74}$$

where the remainder term is bounded above and below by constants independent of k.

Proof. As we have shown in Section 2.2, the eigenvalue counting function of an edge with Dirichlet conditions at both vertices can be written as

$$N_e(k) = \left\lfloor \frac{k\ell}{\pi} \right\rfloor \tag{2.75}$$

which we clearly can bound by

$$\frac{k\ell}{\pi} - 1 \leq N_e(k) \leq \frac{k\ell}{\pi}. \tag{2.76}$$

We adopt the previous notation from Theorem 2.6.2 where Γ_0 denotes the graph with a distinguished vertex v endowed with the Kirchoff condition and Γ_∞ denotes the graph Γ_0 obtained by changing the vertex condition on v to Dirichlet. The chain of inequalities given by Theorem 2.6.2 implies that the eigenvalue counting function N_{Γ_0} can be bounded by

$$N_{\Gamma_\infty}(k) \leq N_{\Gamma_0}(k) \leq N_{\Gamma_\infty}(k) + 1. \tag{2.77}$$

Since our original graph Γ is endowed with either the Kirchoff or Dirichlet conditions at every $v \in \mathcal{V}$, we can apply Theorem 2.6.2 a maximum of $|\mathcal{V}|$ times (the maximum corresponds to a graph with Kirchoff conditions at every vertex). We can bound $N_\Gamma(k)$ by

$$N_{\Gamma_D}(k) \leq N_\Gamma(k) \leq N_{\Gamma_D}(k) + |\mathcal{V}|. \tag{2.78}$$

From Section 2.2 we know that the counting function of $N_{\Gamma_D}(k)$ is

$$N_{\Gamma_D}(k) = \sum_{i=1}^{|\mathcal{E}|} N_{\ell_e}(k) = \left\lfloor \frac{k\ell_1}{\pi} \right\rfloor + \left\lfloor \frac{k\ell_2}{\pi} \right\rfloor + \ldots + \left\lfloor \frac{k\ell_{|\mathcal{E}|}}{\pi} \right\rfloor \tag{2.79}$$

which is bounded from above by

$$N_{\Gamma_D}(k) \leq \frac{\ell_1 + \ell_2 + \ldots + \ell_{|\mathcal{E}|}}{\pi} k \tag{2.80}$$

and below by

$$N_{\Gamma_D}(k) \geq \frac{\ell_1 + \ell_2 + \ldots + \ell_{|\mathcal{E}|}}{\pi} k - |\mathcal{E}| \tag{2.81}$$

by using (2.76) on each term of (2.79). By putting together (2.78) along with (2.80) and (2.81) we arrive at the desired conclusion

$$\frac{L}{\pi} k - |\mathcal{E}| \leq N_\Gamma(k) \leq \frac{L}{\pi} k + |\mathcal{V}| \tag{2.82}$$

with $L := \ell_1 + \ell_2 + \ldots + \ell_{|\mathcal{E}|}$. □

2.8 Weyl's Law for nonnegative Λ_v-matrices

In this final section we will consider a graph endowed with arbitrary self-adjoint conditions at the vertices for which the corresponding self-adjoint matrix Λ_v is nonnegative. We will show that the eigenvalue counting function of said graph Γ follows the asymptotic law

$$N_\Gamma(k) \sim \frac{L}{\pi} k \text{ as } k \to \infty. \tag{2.83}$$

Theorem 2.8.1. *Let $\Gamma = (\mathcal{V}, \mathcal{E}, \mathcal{I})$ be a metric graph equipped with the Hamiltonian operator. The vertices are all endowed with arbitrary vertex conditions which give rise to a nonnegative self-adjoint matrix Λ_v at a vertex v. We denote the length of the graph as $L := \ell_1 + \ell_2 + \ldots + \ell_{\mathcal{E}}$ where ℓ_i corresponds to the length of the edge $e_i \in \mathcal{E}$. Then the eigenvalue counting function $N_\Gamma(k) = \#\{\lambda \in \sigma(\Gamma) \mid \lambda \leq k^2\}$ of the graph obeys the following asymptotic law*

$$N_\Gamma(k) \sim \frac{L}{\pi} k \text{ as } k \to \infty. \tag{2.84}$$

Proof. Let $N_{\Gamma_D}(k)$ denote the eigenvalue counting function of a quantum graph with Dirichlet conditions at every vertex. Let $N_{\Gamma_N}(k)$ be defined similarly, but with the Neumann conditions at every vertex instead. Finally, let $N_\Gamma(k)$ denote the counting function for a graph with arbitrary self-adjoint vertex conditions which give rise to a nonnegative self-adjoint matrix Λ_v. Since we know from Section 2.2 and Section 2.7 that $N_{\Gamma_D}(k)$ and $N_{\Gamma_N}(k)$ follows the same asymptotic law, the assertion of the theorem follows if we can bound $N_\Gamma(k)$ above and below by $N_{\Gamma_N}(k)$ and $N_{\Gamma_D}(k)$.

In the case of Dirichlet graph, $P_{D,v} = \mathbb{I}$ and $F_v = 0$ for all $v \in \mathcal{V}$. With $f \in \mathcal{D}(\mathfrak{q}_D)$ the quadratic forms of $\mathfrak{q}_D[f]$ and $\mathfrak{q}[f]$ coincide since

$$\langle \Lambda_v P_{R,v} F_v, P_{R,v} F_v \rangle = 0 \tag{2.85}$$

for all $v \in \mathcal{V}$. Since $\mathcal{D}(\mathfrak{q}_D) \subseteq \mathcal{D}(\mathfrak{q})$ we get as a direct consequence of the Min-max theorem (Theorem 1.4.3) the following inequality:

$$\lambda_n(\Gamma_D) \geq \lambda_n(\Gamma) \tag{2.86}$$

for all $n \in \mathbb{N}$, hence

$$N_\Gamma(k) \geq N_{\Gamma_D}(k). \tag{2.87}$$

Conversely, the Neumann graph Γ_N is completely decoupled with Neumann conditions at every vertex. The Neumann conditions at a single vertex v in terms of the projectors are $P_{N,v} = \mathbb{I},\ P_{R,v} = P_{D,v} = 0$. Since $\mathrm{Ran}(P_{R,v}) = \{0\}$ for all $v \in \mathcal{V}$, the corresponding self-adjoint operator Λ_v acts in $\{0\}$ for all $v \in \mathcal{V}$ and hence there is no contribution of $\langle \Lambda_v P_{R,v} F_v, P_{R,v} F_v \rangle$ to the quadratic form $\mathfrak{q}_N$. Since the graph is completely decoupled, there is no continuity requirement at the junctions and the form domain consists of all $f \in \tilde{H}^1(\Gamma)$. The domain of $\mathfrak{q}$ is then a subset of $\mathfrak{q}_N$ and $\mathfrak{q} \geq \mathfrak{q}_N$ on $\mathcal{D}(\mathfrak{q})$ since Λ_v is nonnegative. Then again by the Min-max theorem

$$\lambda_n(\Gamma) \geq \lambda_n(\Gamma_N) \tag{2.88}$$

for all $n \in \mathbb{N}$, hence

$$N_{\Gamma_N}(k) \geq N_\Gamma(k). \tag{2.89}$$

Since $N_{\Gamma_N}(k),\ N_{\Gamma_D}(k)$ is already known from Section 2.2 and Section 2.7, combining (2.87) and (2.89) we get

$$\frac{L}{\pi}k \sim N_{\Gamma_D}(k) \leq N_\Gamma(k) \leq N_{\Gamma_N}(k) \sim \frac{L}{\pi}k. \tag{2.90}$$

which proves the assertion of the theorem. □

Since we have only shown Weyl's law in two special cases, it is then natural to ask if there is some general form of Weyl's law which works for all types of vertex conditions? This is indeed the case, in a recent paper, see [7], the authors Almasa Odžak and Lamija Šćeta showed that for a general compact graph with arbitrary self-adjoint conditions the counting function $N(k)$ satisfy

$$N(k) = \frac{L}{\pi}k + \mathcal{O}(k^{\frac{2}{3}}). \tag{2.91}$$

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