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Abstract

The text is a brief exploration of commutative algebra in the context of using commutative algebra and its mathematical neighbors (category theory, homological algebra, topology) to mainly prove theorems about flatness. It starts with a quick recap of the relevant ring theory and moves on to construct the prime spectrum of a ring. Proving some elementary theorems in the process. Then central concepts like modules and exact sequences are introduced, as well as flatness after introducing the tensor product. After some elementary Theorems on flatness the Tor-functors are constructed and some more advanced proofs are proven with the help of them and direct limits. The end of the paper discusses the role of localization in flatness as well as how flatness relates to the prime spectrum.

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1 Introduction

This project arose from trying to come to grips with commutative algebra as a nexus of different mathematical fields and techniques. While the sheer size of the subject overwhelmed me at first I eventually found that through focusing my attention on flat modules and the spectrum of a ring I could quickly get to the point where I could bring to bear techniques from category theory, homological algebra and topology. Subjects like category theory and homological algebra seem supremely sublime to me. They are the best illustrators I have found of Grothendieck's notion of submersing a tough nut in liquid until it opens almost all by itself rather than bringing forth the hammer and smashing it open.

Since a personal focus was the technical aspects of the subject I have aimed to prove most theorems that are introduced except if the proof itself would require an unnecessary expansion of the subject or is deemed otherwise uninteresting. Most theorems have also been proved without a model proof which means mistakes are mine alone. However, I have tried to be sufficiently systematic in developing machinery to avoid glaring holes.

The text itself will start with some concepts from category theory and a subsequent quick development of the needed ring theory. Then the Spectrum of a ring is introduced and some of its basic properties are proved. In particular it is proved that it is an example of a functor. Then modules are treated with the aim of introducing the tensor product together with the notion of flatness (as specific exact functors). Here the development of general theorems on flatness also starts, along with some additional machinery. The work reaches its technical peak in the subsequent development of the Tor-functors and their application to flatness, especially the theorem that combines direct limits and Tor. One application is a partial converse to an earlier theorem, showing that modules over PIDs are flat iff torsionfree. The work is then rounded off with a section on localization and its connection to flatness as well as a section on a few connections between the prime spectrum and flatness.

Rings are always commutative with identity, generally denoted by R. Ideals will usually be denoted by gothic letters like $\mathfrak{a}, \mathfrak{p}$. The equals sign $=$ stands for both things being equal and things being isomorphic. Some familiarity with basic ring theory is assumed, though some basic theorems are stated for completeness. Some topology is also used but not developed so familiarity is assumed there as well.

The main reference for commutative algebra has been Atiyah-MacDonald, but I have also used Matsumura's Commutative Ring Theory as well as a result from Dummit and Foote. The basic concepts from category theory are from Mac Lane and the algebraically flavored direct limit along with most of the related algebraic results is from Matsumura. For the homological algebra I have used Weibel and Matsumura.

2 Category Theory

Category theory is one of the unifying ideas in contemporary mathematics, it allows us to treat aspects of large systems of mathematics all at once and it also allows us to move between these systems in a precise way. Fields that seemed disparate can thus be connected by a stroke of the pen.

A **Category** consists of a class of objects $\mathscr{C}$ and for every two objects A, B in the category a set $\operatorname{Hom}(A, B)$ of morphisms that fulfills these conditions: for every $f \in \operatorname{Hom}(A, B)$ and $g \in \operatorname{Hom}(B, C)$ there exists an $h \in \operatorname{Hom}(A, C)$ such that the following diagram commutes.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "h"] & B \arrow[d, "g"] \\ & C \end{tikzcd}$$

h is then the composition of f, g: $h = g \circ f$. Also in every $\operatorname{Hom}(A, A)$ there exists an identity element 1_A such that for every $f \in \operatorname{Hom}(A, B)$ we have $f \circ 1_A = 1_B \circ f = f$ Which simply means that we can always compose morphisms where domain and codomain lines up and that we have left and right identities. Furthermore composition must be associative whenever defined.

Category theory can thus be characterized as an abstract algebra of functions, and not just functions in the usual sense since something like an ordered set can be made a category by letting the morphisms indicate the order relations.

Category theory would be nothing without its power of movement between categories, something which is achieved through functors. A (covariant) **functor** $\mathscr{F}$ from $\mathscr{C}$ to $\mathscr{D}$ maps objects and morphisms in such a way that for $A, B, C \in \mathscr{C}$ and $f \in \operatorname{Hom}(A, B), g \in \operatorname{Hom}(B, C), h = g \circ f$ the following diagram commutes.

$$\begin{tikzcd} \mathscr{F}(A) \arrow[r, "\mathscr{F}(f)"] \arrow[dr, "\mathscr{F}(h)"] & \mathscr{F}(B) \arrow[d, "\mathscr{F}(g)"] \\ & \mathscr{F}(C) \end{tikzcd}$$

Functors thus preserves the algebraic structure and creates images of categories in other categories, allowing us to embed parts of a field of mathematics more or less faithfully (depending on the functor) in some other field.

Direct Limits

We shall have occasion to use the notion of direct limit so we give a treatment of it here.

A **directed set** is a partially ordered set Λ such that for each $a, b \in \Lambda$ there exists a c such that $a \leq c, b \leq c$. Furthermore for each $a \in \Lambda$ there exists a set M_a and for $a \leq b$ there exists a function $f_{ba} : M_a \to M_b$ which satisfies for $a \leq b \leq c, f_{cb} \circ f_{ba} = f_{ca}$ and $f_{aa} = identity$. This a direct system indexed by Λ and can be denoted $\{M_\lambda, f_{\mu\lambda}\}$. These properties also imply that one may view the direct system as the directed set together with a functor into the relevant category.

A map ϕ of direct systems $\mathscr{D} = \{M_\lambda, f_{\mu\lambda}\}, \mathscr{D}' = \{M'_\lambda, f'_{\mu\lambda}\}$ indexed by the same set Λ is a collection of functions $\phi_\lambda : M_\lambda \to M'_\lambda$ such that $\phi_\mu f_{\mu\lambda} = f'_{\mu\lambda}\phi_\lambda$.

The direct limit is characterized in terms of a universal mapping property, as is often the case when working around category theory. The direct limit $\varinjlim M_\lambda$ of a direct system $\mathscr{D}$ is a set M_∞ and a map $\psi : \mathscr{D} \to M_\infty$ with the universal property that for any map to sets $\varphi : \mathscr{D} \to X$ there exists a unique map $h : M_\infty \to X$ such that $\varphi_\lambda = h\psi_\lambda$.

The first time one usually meets this kind of universal property is in the isomorphism theorem for groups, where maps factor uniquely through G/N provided that the kernel lines up. But universal properties are ubiquitous throughout mathematics when one looks for them.

For our purposes we can even explicitly construct the direct limit: take the disjoint union of the direct system $\sqcup_\lambda M_\lambda$ and form the equivalence relation generated by the condition $x \equiv y$ if $x \in M_\lambda, y \in M_\mu$ and there exists a v such that $f_{v\lambda}(x) = f_{v\mu}(x)$. Then $M_\infty = \sqcup_\lambda M_\lambda / \equiv$.

3 Rings and Ideals

We will here state some of the needed theorems in elementerary commutative ring theory. Then we will begin treating the prime spectrum of a ring with these algebraic tools.

Theorem 3.1 *For ideals* $\mathfrak{b} \subset R$ *with* $\mathfrak{a} \subset \mathfrak{b}$ *there is a bijective order-preserving correspondence with ideals* $\overline{\mathfrak{b}}$ *in* $R/\mathfrak{a}$ *such that* $\mathfrak{b} = \phi^{-1}(\overline{\mathfrak{b}})$*,* ϕ *being the projection* $R \to R/\mathfrak{a}$*.*

Proof: [1, Ch 1 Thm 1] ■

This is analogous to the case of groups and to further extend the analogy we can note that the kernel of a homomorphism ϕ is an ideal and that homomorphisms f factor uniquely through $R/\mathrm{Ker}(f)$. This is an elementary but fundamental result since the notion of ideals is crucial for rings and this theorem tells us precisely what happens to the structure of ideals when taking quotients.

Theorem 3.2 *Every nonzero ring has a maximal ideal.*
Proof: [1, Ch 1 Thm 3] ■

This statement is proven with Zorn's lemma, we will need it for our own purposes later so we state it here:

Theorem 3.3 *Let Σ be a partially ordered set. If every chain in Σ has a maximal element then Σ itself has maximal elements (elements that are not smaller than any other under the ordering).*

One cannot reasonably wish for a better result since chains have very simple orderings and some properties behave well when taking for example the union of a chain.

Corollary 3.4 *Any ideal $\mathfrak{a} \neq (1)$ is contained in a maximal ideal.*
Proof: Follows from 3.1 and 3.2 applied to $R/\mathfrak{a}$ ■

Corollary 3.5 *Any nonunit x is contained in a maximal ideal.*
Proof: Consider 3.4 and the principal ideal (x). ■

This result allows us to constrain the structure of ideals, or knowing the structure, constrain the elements we have. An example of the constraining is that a local ring cannot have an idempotent element e. Both e and $1-e$ are in maximal ideals but together they generate the whole ring so cannot be in the same ideal, thus there are at least two maximals in a ring with idempotents.
The set of nilpotent elements forms an ideal $\mathfrak{N}$ called the nilradical of R

Theorem 3.6 *$R/\mathfrak{N}$ has no nilpotents $\neq 0$ and $\mathfrak{N}$ is the intersection of all prime ideals of R.*
Proof: [1, Ch 1 Thm 8] ■

The first part is simple and the second uses Zorn's lemma again, which is an interesting asymmetry.

The **radical** of an ideal $\mathfrak{a}$ is the set $r(\mathfrak{a}) = \{x \in R | x^n \in \mathfrak{a} \text{ for some } n\}$ and is itself an ideal.

Theorem 3.7 *The radical of an ideal $\mathfrak{a}$ is the intersection of all prime ideals containing $\mathfrak{a}$.*
Proof: Consider $R/\mathfrak{a}$. By the definition of the radical it is clear that the nilradical in $R/\mathfrak{a}$ corresponds to the radical of $\mathfrak{a}$. By 3.6 the nilradical

is the intersection of all prime ideals in $R/\mathfrak{a}$ and so by 3.1 the radical of $\mathfrak{a}$ corresponds to the intersection of every prime ideal containing $\mathfrak{a}$. ■

Prime Spectrum

We now begin a foray into the correspondence between algebra and geometry by constructing the prime spectrum of a ring and investigating its properties under the Zariski topology. A general reference for the needed topology is [6], here we will simply assume what is needed since this is not primarily a work on topology.

Let $X = \mathrm{Spec}(R) = \{\mathfrak{p}|\mathfrak{p} \text{ is prime in } R\}$ and let E be any subset of R. Then

$$V(E) = \{\mathfrak{p} \in X|E \subset \mathfrak{p}\}.$$

In other words, $V(E)$ is defined as the set of prime ideals containing E.

Theorem 3.8
i) if $\mathfrak{a} = RE$ then $V(E) = V(\mathfrak{a}) = V(r(\mathfrak{a}))$.
ii) $V(0) = X, V(1) = \emptyset$
iii) if (E_i) is any family of subsets then $V(\cup E_i) = \cap V(E_i)$
iv) $V(\mathfrak{a} \cap \mathfrak{b}) = V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$ for ideals $\mathfrak{a}, \mathfrak{b}$.

Proof: i) If $E \subset \mathfrak{p}$ then by definition of ideal we must still have $RE \subset \mathfrak{p}$ so that $V(E) \subset V(\mathfrak{a})$. We also have that if $x^n \in \mathfrak{p}, \mathfrak{p}$ prime then x or x^{n-1} is in $\mathfrak{p}$ from which we conclude x is in $\mathfrak{p}$ by repeating as many times as necessary. From this we get $V(\mathfrak{a}) \subset V(r(\mathfrak{a}))$. Since $\mathfrak{a} \subset r(\mathfrak{a})$ we also get $V(r(\mathfrak{a})) \subset V(\mathfrak{a})$ and for the same reason $V(\mathfrak{a}) \subset V(E)$.
ii) 0 is contained in any ideal and thus in any prime ideal. $V(1) = V(R) = 0$ by i).
iii) If $\mathfrak{p} \in V(\cup E_i)$ then $\cup E_i \subset \mathfrak{p}$ so that every $E_i \subset \mathfrak{p}$ which means $\mathfrak{p} \in V(E_i)$ for every E_i and thus $\mathfrak{p} \in \cap V(E_i)$. Conversely if $\mathfrak{p} \in \cap V(E_i)$ then $\mathfrak{p}$ contains every E_i so that $\mathfrak{p} \in V(\cup E_i)$.
iv) if $\mathfrak{a} \subset \mathfrak{p} \vee \mathfrak{b} \subset \mathfrak{p}$ then $\mathfrak{a}\mathfrak{b} \subset \mathfrak{p}$ so $V(\mathfrak{a}) \cup V(\mathfrak{b}) \subset V(\mathfrak{a} \cap \mathfrak{b} \subset V(\mathfrak{a}\mathfrak{b})$ (last one for free). Also if $\mathfrak{p} \notin V(\mathfrak{a}) \cup V(\mathfrak{b})$ then there exist $a \in \mathfrak{a}$ and $b \in \mathfrak{b}$ such that $a, b \notin \mathfrak{p}$ so that $ab \notin \mathfrak{p}$ and $V(\mathfrak{a}\mathfrak{b}) \subset V(\mathfrak{a}) \cup V(\mathfrak{b})$. ■

Setting the $V(E)$ as the closed sets then Theorem 3.8 shows that this defines a topology on the prime spectrum of the ring. We thus link our algebraic object (the ring) to a geometric one (a topology) wherein the important prime ideals and their relational structure is fully reflected. This topology on the spectrum is called the Zariski topology.

So far we have defined all the closed sets, but it is often more convenient to work with a basis of open sets where one obtains all the open sets one needs in a hopefully simple form. From the closed sets we can see that all the open sets are of the form $X_E = V(E)^c$, but if we index E so that $E = \{f_i\}_{i \in I}, f_i \in R$ we can see from Theorem 3.8 iii) that

$$X_E = (\cap V(f_i))^c = \cup X_{f_i}$$

and thus

Theorem 3.9 *The open sets $X_f = V(f)^c$ for $f \in R$ form a basis of open sets for the Zariski topology.* ■

The X_f are precisely the prime ideals not containing f. Now we establish some elementary properties for these open sets.

Theorem 3.10
i) $X_f \cap X_g = X_{fg}$
ii) $X_f = \emptyset \iff f$ is nilpotent
iii) $X_f = X \iff f$ is a unit
iv) $X_f = X_g \iff r((f)) = r((g))$
v) For every open cover of X_f there is a finite subcover (it is quasi-compact)
vi) An X_E is quasi-compact iff it can be expressed as a finite union of X_f

Proof: i) from Theorem 3.8 i) and iv) we get $X_f \cap X_g = V((f)(g))^c = V(fg)^c = X_{fg}$.
ii) If f is nilpotent it is in every prime by Theorem 3.6 so $X_f = V(f)^c = \emptyset$. If $V_f = \emptyset$ then f must be in every prime and again by Theorem 3.6 must be nilpotent.
iii) If f is a unit it is contained in no prime and so $X_f = V(f)^c = X$. Conversely if $X_f = X$ then f can not be contained in any prime and by Corollary 3.4 must be a unit.
iv) If $r((f)) = r((g))$ then by Theorem 3.7 i $V(f) = V(g)$ so that $X_f = X_g$. Conversely assume $X_f = X_g$. Then $V((f)) = V((g))$ so that $V(r(f)) = V(r(g))$ which implies that a prime contains $r(f)$ iff it contains $r(g)$. By Theorem 3.7 then $r(f) = r(g)$.
v) We need only consider basis sets since any open cover can be refined to one of only basis sets. Suppose we have an open cover of X_f by $X_{g_i}, i \in I$. Let $E = \{g_i\}_{i \in I}$ then we have $X_f \subset X_E \implies V(E) \subset V((f)) \implies (f) \subset RE$. By iv) this means that f^n is an element of the generated ideal of E for some n which in turn means it can be expressed as a linear sum of some g_{i_j}. Then those g_{i_j} generate an ideal containing (f^n) as well and so $V((f)) = V((f^n)) \supset \cup V(\sum g_{i_j})$ and $X_f \subset \cup X_{g_{i_j}}$.
vi) If: any cover of X_E is a cover for each X_f and each X_f has a finite subcover resulting in a finite subcover of X_E. only if: immediate since if it has no expression as a finite union of X_f we need only cover by X_f. ■

The crux of the proof in Theorem 3.10 v) is that ring addition is a binary operation and thus infinite sums in the ring operation are meaningless, all sums are finite, and a finite sum corresponds to a finite union. This can look strange at first since this seems to exclude power series rings and infinite direct products, but it does not, of course. Power series and infinite direct products have as elements formal infinite sums, i.e. representations by infinite sums. They are not infinite sums in the ring operation.

To distinguish between primes as members of Spec and as ideals over the ring we will sometimes denote points in Spec by x, y etc. and the corresponding ideal by $\mathfrak{p}_x, \mathfrak{p}_y$ etc.. We now explore the spectrum and some related topology further.

Theorem 3.11
i) $\{x\}$ *is closed in Spec iff* $\mathfrak{p}_x$ *is maximal.*
ii) $\overline{\{x\}} = V(\mathfrak{p}_x)$ *(closure) .*
iii) $y \in \overline{\{x\}} \iff \mathfrak{p}_x \subset \mathfrak{p}_y$.
iv) For every pair of points x, y *in Spec there is an open neighborhood containing one but not the other.*

Proof: i) If $\mathfrak{p}_x$ is maximal then $V(\mathfrak{p}_x) = \{x\}$ so the point is closed. If $\{x\}$ is closed then there exists a set E such that $V(E) = \{x\}$. Now set $\mathfrak{a}$ as the ideal generated by E, then it is clear by Theorem 3.8 i) that $\mathfrak{a}$ is only contained in one prime ideal and since $\mathfrak{a}$ must also be contained in $\mathfrak{p}_x$ it shows that $\mathfrak{p}_x$ is maximal.
ii) Certainly $\overline{\{x\}} \subset V(\mathfrak{p}_x)$ since $V(\mathfrak{p}_x)$ is closed, $x \in V(\mathfrak{p}_x)$ and the closure of a set is the intersection of all closed sets containing it. Now if $y \notin \overline{\{x\}}$ then there must exist a closed set containing x but not y. This closed set is $V(\mathfrak{a})$ for some ideal $\mathfrak{a}$ and from this we get $y \notin V(\mathfrak{a})$ so that $\mathfrak{a} \not\subset \mathfrak{p}_y$ and also $\mathfrak{a} \subset \mathfrak{p}_x$ which leads to $\mathfrak{p}_x \not\subset \mathfrak{p}_y$ and $y \notin V(\mathfrak{p}_x)$.
iii) By ii) if $y \in \overline{\{x\}}$ then $y \in V(\mathfrak{p}_x)$ so that $\mathfrak{p}_x \subset \mathfrak{p}_y$. Conversely if $\mathfrak{p}_x \subset \mathfrak{p}_y$ then again $y \in V(\mathfrak{p}_x = \overline{\{x\}}$.
iv) If $\mathfrak{p}_x \subset \mathfrak{p}_y$ (strict inclusion) then $X_{\mathfrak{p}_y}$ contains x but not y. If $\mathfrak{p}_x \not\subset \mathfrak{p}_y$ then $X_{\mathfrak{p}_x}$ contains y but not x. ■

A topological space X is irreducible if it is nonempty and every pair of nonempty open subsets has nonempty intersection.

Theorem 3.12 $X = $ *Spec*(R) *is irreducible iff* $\mathfrak{N}$ *is prime.*

Proof: Of course we need only consider basis sets. By Theorem 3.9 ii) we also need only consider $X_f, f \notin \mathfrak{N}$. If $\mathfrak{N}$ is prime then for $f, g \notin \mathfrak{N}$ we get $\mathfrak{N} \in X_f \cap X_g$ so that all these intersections are nonempty. If $\mathfrak{N}$ is nonprime then clearly the ring has zerodivisors that are not nilpotent. Pick f, g to be a pair of such zerodivisors fulfilling $fg = 0$. Then by Theorem 3.8 i) and ii) $X_f \cap X_g = X_{fg} = \emptyset$ so that X is not irreducible.

So far we have already established some connections between rings and

topological spaces but we can do better since we can make the construction of the spectrum into a functor.
Let $\phi : A \to B$ be a ring homomorphism and $\mathfrak{p} \subset B$ be prime. Then one needs little more than the definition of homomorphism to prove that $\phi^{-1}(\mathfrak{p})$ is prime as well. From this we see that we can get a map
$\phi^* : \mathrm{Spec}(B) \to \mathrm{Spec}(A)$ by sending primes to their inverse so that each homomorphism of rings induces a function on a topological space. These are the right ingredients for a functor, we just need to prove that the functions have the right properties. ■

Theorem 3.13
i) ϕ^* is continuous.
ii) If $\psi : B \to C$ is another ring homomorphism then $(\psi \circ \phi)^* = \phi^* \circ \psi^*$.
Proof. i) Of course we need only prove that the preimage of a basis set is open. Let $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$. If $y \in Y$ and $\phi(f) \notin \mathfrak{p}_y$ then $f \notin \phi^{-1}(\mathfrak{p}_y)$ so that $\phi^*(\mathfrak{p}_y) \in X_f$ indicating $Y_{\phi(f)} \subset \phi^{*-1}(X_f)$. Now if $\phi(f) \in \mathfrak{p}_x$ then $f \in \phi^{-1}(\mathfrak{p}_x)$ so that $\phi^*(V(\phi(f))) \subset V(f)$ and $V(\phi(f)) \cap \phi^{*-1}(X_f) = \emptyset$ showing the reverse inclusion, impying

$$\phi^{*-1}(X_f) = Y_{\phi(f)}.$$

ii) If $(\psi \circ \phi)^*(x) = y$ then $\mathfrak{p}_y = (\psi \circ \phi)^{-1}(\mathfrak{p}_x) = \phi^{-1}(\psi^{-1}(\mathfrak{p}_x))$ so that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$ by the definition of the map. ■

The construction of the spectrum is, in other words, a contravariant functor from the category of commutative rings to the category of topological spaces, the morphisms being ring homomorphisms and continuous functions respectively.

4 Modules and exact sequences

Modules are structures defined over a ring and gives us another set of rich categories to facilitate our study of commutative algebra and algebraic geometry. It also gives us the chance to treat seemingly very different objects (ideals, quotients, free factors etc.) in the same framework.

An R-module M is an abelian group such that R acts linearly on it by multiplication, i.e. for $r, s \in R, x, y \in M$ we have

$$r(x + y) = rx + ry$$

$$(r + s)x = rx + sx$$

$$(rs)x = r(sx)$$

$$1x = x.$$

Modules are a direct analogy to vector spaces but over arbitrary rings, indeed vector fields are just modules over fields.

A **module homomorphism** f is a group homomorphism that also preserves the module structure, i.e. $f(rx) = rf(x)$. A **submodule** M' of M is a subgroup that is also closed under the action of R, much like how ideals in rings behave. Indeed ideals are submodules of R as a module over itself. As with groups and rings modules also have quotients and isomorphism theorems which work in essentially the same ways, as well as having direct sums and product that work like those for groups.

A **free module** is a module which is a direct sum of copies of R. Vector spaces are always in this form but this is not in general for rings. Free modules are, however, still very useful. Partly this is because of results like the next one which make free modules building blocks in the theory of modules.

Theorem 4.1 *Any module M is the quotient of some free module.*

Proof: If $\{x_i\}_{i\in I}$ is a set of generators for M, then form the free module $F = \oplus_{i\in I} R$. Now let (a_i) be a sequence of elements in R indexed by I such that only finitely many of the elements are non-zero, i.e. let it be an element of F. Then define the module morphism $f : F \to M$ by $f((a_i)) = a_1x_1 + ...a_ix_i + ...$. It is easy to see that it is by construction indeed a module homomorphism and that it is surjective on M. It is also well-defined since only finitely many of the a_i can be non-zero. Thus M can always be expressed as the quotient of a free module. ■

This goes back to the earlier point that sums are always finite, thus we can form direct sums in a free way and quotient out redundant terms to get any module. An extreme application of this line of thinking will be illustrated with the construction of the tensor product.

Exact Sequences and Exact Functors

A diagram of modules of the form

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is called a **short exact sequence** if every image is the kernel of the next map. We see that f is injective iff $0 \to M' \xrightarrow{f} M$ is exact and g is surjective iff $M \xrightarrow{g} M'' \to 0$ is exact.

A functor $\mathscr{F}$ is called exact if for any exact sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

the sequence

$$0 \to \mathscr{F}(M') \xrightarrow{\mathscr{F}(f)} \mathscr{F}(M) \xrightarrow{\mathscr{F}(g)} \mathscr{F}(M'') \to 0$$

is also exact. Respectively it is right-exact or left-exact if it preserves exactness in sequences of the form $M \xrightarrow{g} M'' \to 0$ or $0 \to M' \xrightarrow{f} M$. An exact functor is both right and left exact. Exactness is thus a constraint on what a functor can do with with the morphisms, preserving injectivity and surjectivity. Exactness is also the defining character of flat modules, one of our main objects of study.

A sequence

$$0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is called **split exact** if there exists an isomorphism $M = M' \oplus M''$

Theorem 4.2 *A sequence is split exact iff there exists an injective homomorphism $i : M'' \to M$ such that $gi = id$.*

Proof: If it is split then we can simply take i to be the coordinate injection into M. If i exists then for any equivalence class to $\ker(g)$ there exists a unique $x \in M''$ such that $i(x)$ is a representative, if not it would be impossible for gi to be the identity. This means that every element in M can be written uniquely (by exactness of f and uniqueness of x) as a sum $i(x) + f(y), x \in M'', y \in M'$, therefore $M = M' \oplus M''$. ∎

The Tensor Product and Flatness

A function f is bilinear if for modules M, N, K and $f : M \times N \to K$ is R-linear in both coordinates. So that $f(x, y)$ is linear in $x \in M$ when keeping $y \in N$ constant and vice versa.

The tensor product of modules is a natural and interesting concept with a monstrous construction. The construction itself is not vital so we give only the main idea: for two modules M, N, take the cartesian product $C = M \times N$. Then form a free module which has as basis every element of C (yes you heard me), then quotient it with the least submodule such that elements are in the same equivalence class iff one looks like the other under bilinearity.

Basically: Take every formal combination of elements in the cartesian product and force bilinearity on them. It is a perfect illustration of simplistic brutality in mathematics. Tensor products are always taken over a specific ring since the ring plays a part in how bilinearity looks. The tensor product is denoted $M \otimes_R N$ but generally we will drop the R Since we generally do not change rings. Tensor elements can be denoted $x \otimes y$ though that is ambiguous since its meaning depends on the underlying modules. The forced bilinearity results in tensor elements obeying rules one would expect:

$$rx \otimes y = x \otimes ry = r(x \otimes y)$$

$$(x + y) \otimes z = x \otimes z + y \otimes z$$

$$x \otimes (y+z) = x \otimes y + x \otimes z$$

The tensor product also has a universal mapping property: there exists a bilinear map $g : M \times N \to M \otimes_R N$ such that for every bilinear map $f : M \times N \to K$ there exists a unique linear map $h : M \otimes N \to K$ such that $hg = f$.

Remarkably then, we find that multilinear algebra is in a sense a special case of linear algebra, which seems highly counterintuitive. There are a number of identities that the tensor product obeys and these will make it easier to work with. For every module M we have.

$$R \otimes_R M = M \tag{1}$$

This follows from the fact that by bilinearity $r \otimes m = 1 \otimes rm$ so that every element is uniquely determined by a pair $(1, m)$, or simply m. The tensor product is both commutative and associative:

$$M \otimes N = N \otimes M \tag{2}$$

$$(M \otimes N) \otimes K = M \otimes (N \otimes K) \tag{3}$$

For any index set Λ,

$$(\oplus_{\lambda \in \Lambda} M_\lambda) \otimes N = \oplus_{\lambda \in \Lambda}(M_\lambda \otimes N) \tag{4}$$

and we also have

$$R/I \otimes M = M/IM \tag{5}$$

These isomorphisms are all natural. The way to prove these identities is to obtain maps both ways through the universal mapping property and to show that they are inverses.

An interesting observation is that the tensor product also inherits a natural module structure from the modules in the product, and that homomorphisms $f : M \to M'$ get a natural map $f \otimes 1 : M \otimes N \to M' \otimes N$ between their respective products which also plays nice with composition. Therefore for each module N we can form a functor $- \otimes N$ which sends any module M to its tensor product $M \otimes N$ and any homomorphism f to $f \otimes 1$. This gets us many functors in the category of modules. The study of the exactness of these is precisely the study of flatness.

Flatness

The tensor functors have shared generic exactness, expressed in the next theorem.

Theorem 4.3 *For every N and exact sequence*

$$M' \to M \to M'' \to 0$$

The sequence

$$M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact (tensor functors are always right-exact)

Proof: [1, Ch 2, Thm 18] ■

They are, however, not exact in general. If one is exact, we say that the module is **flat**. Theorem 4.2 tells us that a module is flat iff its tensor always preserves injectivity of maps. One example of a flat R-module is R itself, since by equation (1) we have $R \otimes_R M = R$, so that tensoring an exact sequence leaves it essentially unchanged. Some rings have the special property that every one of their modules is flat, these rings are called **absolutely flat**.

Theorem 4.4 *A direct sum of modules $M = \oplus M_i$ is flat iff every module is flat.*

Proof: Suppose the direct sum is not flat, then there exists some exact sequence of modules

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

such that $f \otimes 1 : A \otimes M \to B \otimes M$ is not injective. Then by equation (4) there exists an element $0 \neq x \in \oplus_i A \otimes M_i$. So if x is non-zero at index i we see that the induced sequence

$$0 \to A \otimes M_i \to B \otimes M_i$$

is not exact and M_i is not flat for this i.

Conversely assume that some M_i is not flat. Then there exists an exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

such that for some $x \in A \otimes M_i$, $f \otimes 1(x) = 0$. Then M cannot preserve injectivity since if we take the element that is x at i and zero everywhere else in $A \otimes M$ this nonzero element must again map to 0. ■

Corollary 4.5 *Free modules are flat*

Proof: follows from the above remarks on flatness of R and Theorem 4.4. ■

Theorem 4.6 *If a module M is flat over an integral domain R, then it is torsion-free.*

Proof: Assume M is not torsion-free, then it has an element x such that $rx = 0$ for some $r \in R$. Since multiplication by an element of R is a module homomorphism we can form the exact sequence

$$0 \to R \xrightarrow{r} R \to R/(r) \to 0.$$

If we now tensor by M we can see that the map $r \otimes 1$ sends the nonzero element $1 \otimes x$ to the element $r \otimes x = 1 \otimes xr = 0$ so that injectivity is not preserved. ■

These theorems are already quite powerful, with just the structure theorem of finitely generated modules over PIDs we can completely characterize finitely generated flat modules over PIDs. However, we will have to wait on the Tor-functors to get a converse for Theorem 4.6.

Theorem 4.7 *let M be a finitely generated module over a PID. Then M is torsion-free iff M is free.*

Proof: [5, Ch 12, Thm 5] ■

Finitely generated flat modules over PIDs are thus the free modules. Nothing more, nothing less.

Exactness of Direct Limits

Our introduction earlier of the direct limit was not for naught, we will now see how it can increase our flexibility when working with modules.

Theorem 4.8 *Every module M is the direct limit of its finitely generated submodules ordered by inclusion, the maps being injections.*

Proof: It is clear that every $x \in M$ is contained in some finitely generated submodule, for example $x \in Rx$. If we let the direct system be indexed by Λ then for every $x \in M_\lambda$, x is sent to its equivalence class which is the class of elements that are eventually injected to the same element, thus the elements of M_∞ have a natural correspondence with M.

Somewhat more rigorously: If we let each M_λ inject directly into M we get a map from the direct system into modules which induces a homomorphism $h : M_\infty \to M$ which is precisely the correspondence above. Thus h is an isomorphism since it is a bijective homomorphism. ■

Generally for direct systems of modules the map $x \mapsto x_\infty = \lim x$ is a module homomorphism as it turned out to be above by its equivalence to injection. The next theorems in conjunction with the previous theorem will allow us to reduce some problems over all modules to problems over finite modules.

Theorem 4.9 Let $\mathscr{F}$ be a direct system indexed by Λ and N a module. Then

$$(\lim_{\rightarrow} M_\lambda) \otimes N = \lim_{\rightarrow}(M_\lambda \otimes N)$$

Proof: let $T_\infty = \lim\limits_{\rightarrow}(M_\lambda \otimes N)$ and $M_\infty = \lim\limits_{\rightarrow} M_\lambda$. Then we let $f : M_\lambda \to M_\infty$ be the map $x \mapsto \lim x$. Then we get an induced map $f \otimes 1 : M_\lambda \otimes N \to M_\infty \otimes N$ of the other direct system yielding us a unique homomorphism $h : T_\infty \to M_\infty \otimes N$ so that $h(\lim(x \otimes y)) = x_\infty \otimes y$.

For each $y \in N$ we can also define $g_{\lambda,y} : M_\lambda \to T_\infty$ by $x \mapsto \lim(x \otimes y)$ and get a unique morphism $g_y : M_\infty \to T_\infty$ for each y. For $x_\infty \in M_\infty$ we have $g(x_\infty, y) = g_y(x_\infty) = \lim(x \otimes y)$ and since lim is a homomorphism and the tensor terms are bilinear in x and y g is a bilinear map which by the universal property of the tensor product induces a homomorphism $l : M_\infty \otimes N \to T_\infty$ such that $l(x_\infty \otimes y) = \lim(x \otimes y)$. h and l are obviously inverses which proves the theorem. ■

Theorem 4.10 *If we have an exact sequence of three direct systems of modules (indexed by Λ), meaning that every sequence*

$$M'_\lambda \xrightarrow{\phi_\lambda} M_\lambda \xrightarrow{\psi_\lambda} M''_\lambda$$

is exact, then

$$M'_\infty \xrightarrow{\phi_\infty} M_\infty \xrightarrow{\psi_\infty} M''_\infty$$

is exact. Or: direct limit is an exact functor.

Proof: $\psi_\infty \phi_\infty = 0$ is immediate since it must identify each element with the same equivalence class as 0. Now if $\psi_\infty(y_\infty) = 0$ then it is the limit of some $\psi_\lambda(y)$ so that $\lim(\psi_\lambda(y)) = 0$. But since it is in the same equivalence class as 0 it must be identified with 0 somewhere along the way, so that there exists a $\mu \geq \lambda$ such that $f''_{\mu\lambda}(\psi_\lambda(y)) = 0$. Since we have morphisms of direct systems we get

$$f''_{\mu\lambda}(\psi_\lambda(y)) = \psi_\mu(f_{\mu\lambda}(y)) = 0$$

and since we have exactness at μ there exists an $x \in M'_\mu$ such that $\phi_\mu(x) = f_{\mu\lambda}(y)$. Taking limits on both sides we get $y_\infty = \phi_\infty(x_\infty)$ which shows the reverse inclusion. ■

The general technique in the proof seems to be quite common in algebra: if something happens at infinity it must happen finitely. The proof that a ring is noetherian iff its ideals are finitely generated employs the same line of thinking.

Similar to how we can create direct systems by taking submodules and letting the maps be injections we can create direct systems by quotienting by submodules and letting the maps be projections. For example, take a module M and a submodule N and take all finite submodules N_λ of N. Then if $N_\mu \subset N_\lambda$ there is a canonical projection $p_{\mu\lambda} : M/N_\mu \to M/N_\lambda$ which clearly satisfies $p_{\mu\lambda}p_{\lambda\nu} = p_{\mu\nu}$. It is similarly easy to check the other conditions which shows that it is a direct system. Of course one need not take all finitely generated submodules to create a direct system this way, it is simply one option.

5 The Tor-functors

A **complex** is a sequence of objects (of modules for our purposes) and morphisms

$$\cdots \to M_n \xrightarrow{d_n} M_{n-1} \xrightarrow{d_{n-1}} M_{n-2} \to \ldots$$

such that $d_{n-1}d_n = 0$ for all n, we denote it by with a lower dot: $M_\bullet$. Notice that exactness is a special case of this condition. Any exact sequence can also be made into a complex by extending zeros out to the sides and indexing.

A morphism of complexes $f : A_\bullet \to B_\bullet$ is a set of object (module) morphisms f_n such that the diagram commutes

$$\begin{array}{ccccccccc}
\ldots & \longrightarrow & A_n & \xrightarrow{d_n} & A_{n-1} & \xrightarrow{d_{n-1}} & A_{n-2} & \longrightarrow & \ldots \\
 & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \downarrow{\scriptstyle f_{n-2}} & & \\
\ldots & \longrightarrow & B_n & \xrightarrow{d'_n} & B_{n-1} & \xrightarrow{d'_{n-1}} & B_{n-1} & \longrightarrow & \ldots
\end{array}$$

A sequence

$$A_\bullet \to B_\bullet \to C_\bullet$$

of complexes is exact iff every sequence

$$A_n \to B_n \to C_n$$

is exact. Exactness for complexes is in other words just a straightforward piggyback on the exactness of elements in the complexes.

To each pair d_{n-1}, d_n in a complex there is associated the module $H_n(A) = \ker(d_{n-1})/\mathrm{im}(d_n)$, these are the homology modules of the complex. The next theorem is the true heart of the black magic that is homological algebra:

Theorem 5.1 *Given an exact sequence of complexes*

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0$$

There exists an induced long exact sequence

$$\ldots \to H_{n+1}(C_\bullet) \xrightarrow{\delta_{n+1}} H_n(A_\bullet) \to H_n(B_\bullet) \to H_n(C_\bullet) \xrightarrow{\delta_n} H_{n-1}(A_\bullet) \to \ldots$$

of the homology modules. ■

The proof is long and involved and so will not be given here. The construction of the δ-connectors happens by starting with an element in the kernel and then chasing it through preimages and images until one ends up in $A_.$. It is straightforward and not very illuminating beyond the first steps. After constructing the Tor-functors this theorem will be put to good use in proving flatness theorems. But to get there we need a bit more theory.

A module P is called projective if for every diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "p"] & M' \arrow[r] & 0 \end{tikzcd}$$

with exact lower row there exists a lift $h : P \to M$ such that $ph = f$.

Theorem 5.2 *Free modules F are projective.*

Proof: let $f : F \to M'$ be a homomorphism and $p : M \to M'$ be a surjection. Now let $x_{ii\in I}$ be the basis elements for F. For every x_i there exists a y_i such that $f(x_i) = p(y_i)$ since p is surjective, so define h by $h(x_i) = y_i$. Then $h(rx_n + x_m) = rh(x_n) + h(x_m)$ since $f(rx_n + x_m) = rf(x_n) + f(x_m) = rp(y_n) + p(y_m)$, and linearity on F then follows from linearity on basis. ■

Corollary 5.3 *Every module is the quotient of a projective module.*

Proof: We have shown this for free modules which are also projective.■

This is the piece we need to construct the Tor-functors, the following theorem and corollary will let us use the black magic of homological algebra efficiently.

Theorem 5.4 *A module P is projective iff it is the summand of a free module.*

Proof: We know by 4.1 that there exists a free module F such that P is the quotient of F. Let p be the quotient map, then we get an exact sequence

$$0 \to \ker(p) \xrightarrow{i} F \xrightarrow{p} P \to 0.$$

Now consider $id : P \to P$, this lifts to F by $h : P \to F$ such that $ph = id$, thus the sequence splits and P is a direct summand of F.

Now if $F = M \oplus P$ then let $i : P \to F$ be the injection and $p : F \to P$ the projection. Then for a diagram

$$\begin{tikzcd} & P \arrow[d, "f"] & \\ M \arrow[r, "g"] & M' \arrow[r] & 0 \end{tikzcd}$$

we can form the map $fp : F \to M'$ and since F is free and therefore projective there is a map $h : F \to M$ such that $gh = fp$. But then $ghi = fpi = f$ so that hi is a lift from P. ■

Corollary 5.5 *Projective modules are flat*

Proof: The theorem above tells us that projective modules are direct summands of flat modules, and so flat by Theorem 4.4. ■

Onto construction. Start with a module M. By Theorem 5.3 M is the quotient of a projective module P_0. The kernel of this quotient map is then also a module and is the quotient of some projective module P_1 and continuing in this way yields a long exact sequence

$$... \to P_1 \to P_0 \to M \to 0$$

which is called a **projective resolution** of M. Removing M at the end creates a complex of projective modules. We can tensor this complex with any other module N and by doing so we create a new complex

$$T_\bullet = ... \to N \otimes P_1 \to N \otimes P_0 \to 0.$$

Then $\mathrm{Tor}_n(M, N) = H_n(T_\bullet)$. Technically important is that this is independent of the specific projective resolution and that we also have $\mathrm{Tor}_n(M, N) = \mathrm{Tor}_n(N, M)$. For proofs of this see [5, Ch 17, Prop 14] and [7, Ch 6] (the discussion below Theorem 1.5a). By construction we have $\mathrm{Tor}_0(M, N) = M \otimes N$. The Tor-functors are in a sense a measure of the deviation from flatness. One indication of this we will soon see.

Consider now a module N with associated complex $T_\bullet$ and an exact sequence

$$0 \to M \to M' \to M'' \to 0.$$

From the complex and the exact sequence we can get an induced exact sequence of complexes

$$0 \to M \otimes T_\bullet \to M' \otimes T_\bullet \to M'' \otimes T_\bullet \to 0.$$

This is exact since every projective module is flat and the induced sequences of modules looks like

$$0 \to M \otimes T_n \to M' \otimes T_n \to M'' \otimes T_n \to 0.$$

The homology modules are of course the Tor-modules and so from every tensoring by N of an exact sequence we get an induced long exact sequence

$$... \to \mathrm{Tor}_n(N, M) \to \mathrm{Tor}_n(N, M') \to \mathrm{Tor}_n(N, M'') \to \mathrm{Tor}_{n-1}(N, M) \to ...$$

$$... \to \mathrm{Tor}_1(N, M'') \to N \otimes M \to N \otimes M' \to N \otimes M'' \to 0$$

by Theorem 5.1. We can see that for the construction itself we really only needed Theorem 5.2 or Corollary 5.3 which enables us to do the resolution. However, what we are really after is the long exact sequence and for that we need a result like Corollary 5.5 so that we are actually guaranteed to get an induced exact sequence of complexes from an exact sequence of modules.

The Tor-functors and the associated homology is a powerful piece of technology which will now help us prove further theorems about flatness.

Theorem 5.6 *The following are equivalent for modules M:*
i) M is flat
ii) $Tor_n(M, N) = 0$ for all N and $n > 0$
iii) $Tor_1(M, N) = 0$ for all N

Proof: i) $\implies$ ii) is immediate since if M is flat exactness will be preserved and all homology modules will be 0. ii) $\implies$ iii) needs nothing further. iii) $\implies$ i): from the homology exact sequence theorem and the definition of the Tor-functors every exact sequence

$$0 \to N \to N' \to N'' \to 0$$

induces a long exact sequence

$$... \to \mathrm{Tor}_1(M, N') \to \mathrm{Tor}_1(N'', M) \to M \otimes N \to M \otimes N' \to M \otimes N'' \to 0.$$

This shows that if $\mathrm{Tor}_1(M, N) = 0$ for every N it will preserve exactness when tensoring, thus it is flat. ■

As a first step we have changed the character of the problem. Instead of asking if $- \otimes M$ preserves exactness we can now ask whether $\mathrm{Tor}_1(M, N)$ is zero for all N. We still have to contend with an awful many modules, but we shall soon see that this situation can be improved considerably.

Theorem 5.7 *If*

$$0 \to N \to N' \to N'' \to 0$$

is an exact sequence and N'' is flat then $Tor_n(M, N) = Tor_n(M, N')$ for all n, M.

Proof: Since $\mathrm{Tor}_n(M, N'') = 0$ for all n, M the long exact sequence induced by M will look like

$$... \to 0 \to \mathrm{Tor}_n(M, N) \to \mathrm{Tor}_n(M, N') \to 0 \to ...$$

and the theorem follows. ■

Corollary 5.8 *If*

$$0 \to N \to N' \to N'' \to 0$$

is exact and N'' flat then N is flat iff N' is flat.

Proof: follows directly from the above two. ■

Theorem 5.9 *N is flat iff $Tor_1(R/\mathfrak{a}, N) = 0$ for all finitely generated ideals $\mathfrak{a}$.*

Proof: Only if is immediate. If we take in steps. Step 1: we prove a slightly weaker version: N is flat if $\mathrm{Tor}_1(M, N) = 0$ for each finitely generated module M. Let

$$0 \to M'' \to M' \to M \to 0$$

be any exact sequence and let M_λ be the finitely generated submodules of M. Then for each λ we get a new exact sequence

$$0 \to M''_\lambda \to M'_\lambda \to M_\lambda \to 0$$

where $M''_\lambda = M''$ for every λ and M'_λ is the preimage of M_λ. Thus we get a short exact sequence of direct systems. Since $\mathrm{Tor}_1(M_\lambda, N) = 0$ for each λ tensoring with N preserves exactness for each of the sequences and thus the exactness of direct systems. Now since tensoring commutes with direct limits and the direct limit is exact it follows that

$$0 \to M'' \otimes N \to M' \otimes N \to M \otimes N \to 0$$

is exact and that N is flat.

Step 2: Assume $\mathrm{Tor}_1(M, N) = 0$ for every M generated by only one element. Then if M is a finitely generated submodule and $x_1, ..., x_n$ a set of generators we set M_i to be the submodule generated by the first i generators. Then for each M_i we can create the exact sequence

$$0 \to M_{i-1} \to M_i \to M_i/M_{i-1} \to 0.$$

Since $\mathrm{Tor}(M_i/M_{i-1}, N) = 0$ by assumption we know by the above that $\mathrm{Tor}(M_i, N) = \mathrm{Tor}_1(M_{i-1}, N)$ and since $\mathrm{Tor}_1(M_1, N) = 0$ we get by induction that $\mathrm{Tor}_1(M, N) = 0$. If M is generated by one element x then of course $M = R/\mathfrak{a}$ where $\mathfrak{a} = \{r|rx = 0\}$.

Step 3: Now if M is of the form $R/\mathfrak{a}$ and if $\mathfrak{a}$ is not finitely generated we can form an exact sequence of direct systems

$$0 \to \mathfrak{a}_\lambda \to R \to R/\mathfrak{a}_\lambda$$

where $\mathfrak{a}_\lambda$ are the finitely generated submodules to $\mathfrak{a}$ (finite subideals). Then by the same reduction as above we see that it is enough that $\mathrm{Tor}_1(R/\mathfrak{a}) = 0$ holds for finitely generated ideals $\mathfrak{a}$ and the theorem follows. ■

Corollary 5.10 *A module M is flat iff $\mathfrak{a} \otimes M = \mathfrak{a}M$ for all finitely generated ideals $\mathfrak{a}$.*

Proof: Apply the above theorem to the exact sequence

$$0 \to \mathfrak{a} \to R \to R/\mathfrak{a} \to 0$$

tensored with M remembering the tensor identities.

After going on a journey through homology and direct limits we have managed to reduce the problem of the flatness of a module to a computation on a very special class of modules. From this it easily follows that

Corollary 5.11 *Fields are absolutely flat*

Proof: Fields only have the ideals 0 and R, both trivially fulfilling the above criterion. ■

One can prove this much easier by obtaining the result that every vector space is a free module but now we get it for free anyway. This is not all we can do with this reduction though, now we can prove the promised converse to theorem 4.6 for PIDs.

Theorem 5.12 *If a module M is torsion-free over a PID R then it is flat.*

Proof: For any $r \in R$ consider the exact sequence

$$0 \to R \xrightarrow{r} R \to R/(r) \to 0$$

and then for M derived long exact sequence

$$... \to \mathrm{Tor}_1(M, R/(r)) \to M \otimes R \xrightarrow{1 \otimes r} M \otimes R \to M \otimes R/(r) \to 0.$$

Then we can note that the kernel of the map $1 \otimes r$ are precisely the elements $m \otimes 1$ such that $m \otimes r = rm \otimes r' = 0$ i.e. it is isomorphic to all m that are killed by r. Thus if M has no torsion elements all $\mathrm{Tor}_1(M, R/(r))$ will be zero. Since these are all finite ideals in a PID Theorem 5.9 implies that M is flat. ■

Theorem 5.13 *The following are equivalent:*
i) R is absolutely flat
ii) Every principal ideal is idempotent
iii) Every finitely generated ideal is a summand of R

Proof: i) $\implies$ ii) Consider the exact sequence

$$0 \to (x) \to R \to R/(x)$$

for some $x \in R$. Then if we tensor with $R/(x)$ we get the exact sequence

$$0 \to (x) \otimes R/(x) \to R \otimes R/(x) \to R/(x) \otimes R/(x)$$

which by tensor identities becomes the exact sequence

$$0 \to (x)/(x^2) \to R/(x) \to R/(x).$$

The last one because $R/(x) \otimes R/(x) = (R/(x))/((x)R/(x)) = R/(x)$. This implies $(x)/(x^2) = 0$ so that $(x) = (x^2)$ and $x = rx^2$ for some $r \in R$. Then rx is idempotent and generates the ideal.

ii) $\implies$ iii) If every principal ideal is idempotent then every non-unit is either idempotent or a unit multiple of an idempotent. Thus if $\mathfrak{a}$ is generated by two idempotent elements f, e then $e + f - ef$ is idempotent and $ef(e + f - ef) = ef, (1-e)(e+f-ef) = f - ef$ and $(1-f)(e+f-ef) = e - ef$ which shows that $(e, f) = (e + f - ef)$ and that by induction every finitely generated ideal is principal and idempotent. And if e is idempotent any $r \in R$ can be written uniquely as $re + r(1-e)$ since $e(1-e) = 0$. Thus $R = (e) \oplus (1-e)$.

iii) $\implies$ i) If every finitely generated ideal $\mathfrak{a}$ is a direct summand then $R = A \oplus \mathfrak{a}$ and $R/\mathfrak{a} = A$ is also a direct summand, so both are flat since R is flat. Thus by Theorem 4.4 R is absolutely flat. ■

This shows that also direct sums of fields are absolutely flat (every principal ideal will be idempotent)

Corollary 5.14 *Absolutely flat local rings are fields*

Proof: If R is local, absolutely flat and has an x that is not a unit, then it has an idempotent e. Then e and $1-e$ cannot be in the same proper ideal and so must belong to distinct maximal ideals. ■

6 Localization

Localization is an important tool in commutative algebra and algebraic geometry. It can reduce many problems of arbitrary rings to problems of local rings and it corresponds to geometrically focusing in on an open set or around a point. As such localization will often allow one to treat simpler rings and carry the results over to the desired case, analogously to how direct limits allowed us to sometimes only treat finite modules and then generalize.

A **multiplicative set** S is a set that is closed under internal multiplication, i.e. $x, y \in S \implies xy \in S$. If one has a multiplicative set one may form the ring $S^{-1}R$ of elements of the form $(r, s), r \in R, s \in S$, often denoted r/s, such that $(r_1, s_1) \equiv (r_2, s_2)$ if $(r_1 s_2 - r_2 s_1)u = 0$ for some $u \in S$. This definition serves to rid us of the troubles that inverting zero-divisors might bring. Otherwise the operations work as with ordinary fractions: $x/s + y/t = (xt + ys)/st, (x/s)(y/t) = (xy)/(st)$. An important thing to note here is that in ideal $\mathfrak{p}$ is prime iff $R \setminus \mathfrak{p}$ is a multiplicative set. This means that for every prime ideal we can take its localization $R_{\mathfrak{p}}$ which is $(R \setminus \mathfrak{p})^{-1}R$. Localizing for modules of R works the same as localizing for R. Localization has a weak analogous theorem to Theorem 3.1.

Theorem 6.1 *The prime ideals* $\mathfrak{q}$ *of* $S^{-1}R$ *is in bijective correspondence with the prime ideals* $\mathfrak{p}$ *of* R *with* $S \cap \mathfrak{p} = \emptyset$*.*

Proof: [1, Ch 3, Thm 11] ■

Another fact is very important: $R_\mathfrak{p}$ is a local ring for every ideal $\mathfrak{p}$. This is of course because every prime ideal $\mathfrak{q} \in R_\mathfrak{p}$ corresponds to a prime ideal $\mathfrak{p}' \in R$ with $\mathfrak{p}' \subset \mathfrak{p}$ by the above theorem. The ideal generated by the image of $\mathfrak{p}$ is for this reason the only maximal in $R_\mathfrak{p}$.

The Local Character of Flatness

What does it mean for a property in algebra to be local? The previous discussion provides a hint: a property P is local if R or M has P iff $R_\mathfrak{p}$ or $M_\mathfrak{p}$ has P for every prime $\mathfrak{p}$. In other words, it is properties that are invariant when traversing to and from a ring and its local rings derived from its prime ideals. Unsurpisingly, as it is the section title, flatness is a local property.

Theorem 6.2 *The following are equivalent: i)* M *is flat as an* R*-module*
ii) $M_\mathfrak{p}$ *is flat as an* $R_\mathfrak{p}$*-module for every prime* $\mathfrak{p}$
iii) $M_\mathfrak{m}$ *is flat as an* $R_\mathfrak{m}$*-module for every maximal ideal* $\mathfrak{m}$*.*

Proof: [1, Ch 3, Thm 10] ■

This means that studying flatness in the context of local rings is in a sense really enough, even if it might not be practicable in every case. This theorem also illustrates that often one need not check for every prime ideal, considering only the maximal ideals can yield enough information.

Lemma 6.3 *If* R *is absolutely flat then every prime ideal is maximal*

Proof: If R is absolutely flat then every nonunit is associate to an idempotent. If $\mathfrak{p}$ is prime then for every idempotent e, $\mathfrak{p}$ must contain either e or $1 - e$, but not both since then it is not a proper ideal. As a consequence every prime is maximal since any $x \notin \mathfrak{p}$ is either a unit or associate to $1 - e$ with $e \in \mathfrak{p}$. ■

Theorem 6.4 *A ring* R *is absolutely flat iff* $R_\mathfrak{m}$ *is a field for every maximal ideal* $\mathfrak{m}$*.*

Proof: If follows from Theorem 6.2 and Corollary 5.11, since every $R_\mathfrak{m}$ is absolutely flat it follows that every $M_\mathfrak{m}$ is flat and thus that every M is flat.

Only if: Since every prime is maximal by Lemma 6.3 it follows from Theorem 6.1 that $R_\mathfrak{m}$ has only one prime ideal and by Theorem 3.6 this prime ideal is the nilradical. But if $R_\mathfrak{m}$ has a nilpotent r/s then r is a nilpotent in R, which shows that the nilradical is zero by Theorem 5.13 ii). And since it is the only prime ideal $R_{\mathfrak{m}}$ is a field by Theorem 3.5. ■

It is interesting that so often the maximal ideals themselves hold all the relevant information so that one need not go to every prime ideal.

7 Spec and Flatness

Flatness is a wonderfully algebraic notion, arising out of the unreasonable construction of the tensor product and being workable by the esoteric tools of homological algebra and category theory. But we have also established a foothold in geometry (or topology) by constructing the spectrum of a ring.

Theorem 7.1 *The following are equivalent statements:*
i) $R/\mathfrak{N}$ is absolutely flat
ii) Every prime ideal of R is maximal
iii) Every point of Spec(R) is closed
iv) Spec(R) is Hausdorff

Proof: Some are immediate, like ii) iff iii) ($\Longrightarrow$ follows from the definition of V and converse from Theorem 3.11 ii)) and iv) $\Longrightarrow$ iii) (standard topology). i) $\Longrightarrow$ ii) is just Lemma 6.3 together with Theorems 3.1, 3.6 since they imply that the prime structure is undisturbed when quotienting by nilradical. Left then are ii) $\Longrightarrow$ i) and ii) $\Longrightarrow$ iv).

Assume that $R/\mathfrak{N}$ is not absolutely flat. Then there exists a maximal ideal $\mathfrak{m} \in R/\mathfrak{N}$ such that $(R/\mathfrak{N})_{\mathfrak{m}}$ is not a field. This ring is local without nilpotent elements so it has one maximal ideal (which is nonzero by assumption) and the intersection of all prime ideals is zero. Hence there are primes apart from the unique maximal and so by Theorem 6.1 $R/\mathfrak{N}$ has nonmaximal primes.

Now assume Spec(R) is not Hausdorff. Then there exist points x, y in Spec(R) such that for every $f \in p_x, f \notin p_y$ and $g \in p_y, g \notin p_x$ we have $X_f \cap X_g \neq \emptyset$. This implies that for all such pairs we have $(X_f \cap X_g)^c = V(f) \cup V(g) = V(f+g) \neq V(1)$. Thus $p_x \cup p_y$ does not generate the whole ring and so at least one of the primes is not maximal. ■

The topologist's favorite, the Hausdorff space, arises only on the spectrum under the algebraic condition that the ring is absolutely flat after quotienting out the nilradical. We see also that if all primes are maximal then it is essentially only the nilpotent elements that stand in the way of absolute flatness.

One thing we have not given much attention yet is when a module M is itself a ring. If we have a homomorphism of rings $f : A \to B$ then we can give B an A-module structure simply by $ab = f(a)b$ for $a \in A, b \in B$. If B is flat as an A-module with this structure we call f a flat morphism. We should expect this to have consequences, and indeed it does.

Theorem 7.2 *Let $f : A \to B$ be flat. Then the following are equivalent:*
i) $f^{-1}(Bf(\mathfrak{a})) = \mathfrak{a}$ for all ideals $\mathfrak{a}$ in A.
ii) the map $f^* : Spec(B) \to Spec(A)$ is surjective.
iii) For every maximal ideal $\mathfrak{m}$ in A we have $Bf(\mathfrak{m}) \neq B$.
iv) If M is a non-zero A-module then $B \otimes_A M \neq 0$.
v) The mapping $x \mapsto 1 \otimes x$ from M to $B \otimes M$ is injective for all A-modules M.

Proof: For i) $\Longrightarrow$ ii) we need to show that every prime in A also is a preimage of a prime in B. Let $\mathfrak{p} \subset A$ be prime. Now let S be the multiplicative set $f(A - \mathfrak{p})$ in B. Then $Bf(\mathfrak{p}) \cap S = \emptyset$ since $f^{-1}(Bf(\mathfrak{p})) = \mathfrak{p}$. So localizing at S shows that $Bf(\mathfrak{p})$ corresponds to a proper ideal of $S^{-1}B$ (by 6.1). By 3.4 this is contained in a maximal ideal $\mathfrak{m}$ which can be pulled back to a prime $\mathfrak{q}$ in B. Of course this means $S \cap \mathfrak{q} = \emptyset$ so $f^{-1}(\mathfrak{q}) = \mathfrak{p}$ since it must pull back to a prime not containing an element of $A - \mathfrak{p}$ and $f(\mathfrak{p}) \subset \mathfrak{q}$.

ii) $\Longrightarrow$ iii) Quite immediate. If $f(\mathfrak{m})$ contains a unit then it will not be the preimage of any proper ideal and thus f^* cannot be surjective.

iii) $\Longrightarrow$ iv) Let x be a non-zero element of M. Then $M' = Ax = A/\mathfrak{a}$ for some ideal $\mathfrak{a}$. From tensor identities we know that $M' \otimes_A B = B/(Bf(\mathfrak{a}))$ and since no image of a maximal ideal contains a unit neither will the image of $\mathfrak{a}$ so $B/(Bf(\mathfrak{a})) \neq 0$. Since M' is a submodule it can be injected into M and since B is flat tensoring will preserve injectivity, thus $M \otimes_A B \neq 0$.

iv) $\Longrightarrow$ v) Assume first that M is a B-module with the A-module structure $ax = f(a)x$. If the map is not injective then some $1 \otimes x = 0$ for some non-zero $x \in M$. But then $1 \otimes x = 1 \otimes ax = 1 \otimes 0$ for some $a \in A$ with $f(a) = 1$ but this is impossible with their module structures, so the map is injective. Assume then that M is any A-module. Then $B \otimes_A M$ is a B-module and $a(b \otimes x) = f(a)b \otimes x$ so it fulfills the former assumptions. So let M' be the kernel of the map from M to $B \otimes_A M$, then by flatness of B the sequence

$$0 \to B \otimes_A M' \to B \otimes_A M \to B \otimes_A (B \otimes_A M)$$

is exact. And since the last map is injective $B \otimes_A M' = 0$ and so $M' = 0$.

v) $\Longrightarrow$ i) As a special case of v) the map f must be injective since $a \mapsto 1 \otimes a = f(a) \otimes 1$ is injective. Letting $M = A/\mathfrak{a}$ we also get an injective map $h : A/\mathfrak{a} \to B \otimes A/\mathfrak{a}$. Furthermore $h(\bar{x}) = 1 \otimes \bar{x} = f(x) \otimes 1$ for all $x \in A$ projected to $\bar{x} \in A/\mathfrak{a}$. This means that the induced injective map $h' : A/\mathfrak{a} \to B/B\mathfrak{a}$ is a ring homomorphism with $h'(\bar{a}\bar{b}) = \overline{f(a)f(b)}$. Thus $(\phi \circ f)^{-1}(0) = \mathfrak{a}$ with $\phi : B/B\mathfrak{a}$ so that $f^{-1}(B\mathfrak{a}) = \mathfrak{a}$. ■

An f that fulfills any of the conditions is also called a **faithfully flat morphism**. Condition ii) in Theorem 7.2 is actually a generic condition for integral extensions [5, Ch 15, Thm 26] which shows that if the injection from a ring to its integral extension is flat then it is faithfully flat.

We see then that flatness has some versatility as a concept and can show up in many parts of algebra. There is quite a lot more to explore in the commutative case, in particular it seems to have quite a few interactions with dimension theory. In another direction flatness can also be defined for the non-commutative case, providing fresh new pastures for exploration. Our journey, however, ends here.

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