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Kahane's Theorem on divergence of Fourier series

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Abstract

A general approach to problems in mathematical analysis is the use of approximate representations of functions, such as Taylor series. When regarding approximate representations of functions of a periodic nature, in the study we call Fourier analysis, we attempt representations of functions as a sum of trigonometric functions. In this thesis we present one of the problems encountered in Fourier analysis with regard to the pointwise convergence of Fourier series. We prove Kahane's Theorem on divergence of Fourier series of continuous functions on large sets by a topological approach. To do so, we introduce the fundamental tools of topology and review Baire's Category Theorem and the Kuratowski-Ulam Theorem, and the fundamental notions of Fourier analysis such as Fourier series, the Dirichlet Kernel and the Convolution of functions.

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Contents

Introduction

Harmonic analysis began as the study of waves, originating from problems such as the properties of vibrating strings [11, p.1]. As such waves tend to be more complex than a simple sine or cosine function, this eventually gave rise to what we call Fourier analysis - named after Jean-Baptiste Joseph Fourier for his contributions to the study of trigonometric series.

Given a periodic function, the idea behind Fourier analysis is that we want to approximate the function by a series of trigonometric functions - which we call the Fourier series of the function. This approach is similar to how we approximate functions such as sine by a polynomial using Maclaurin's formula or Taylor's formula. Since the function is periodic, the function repeats itself and we can focus on a single period to understand the whole function. These periodic functions may of course be considered as waves, such as light waves or audio waves, and thus we can think of applications of Fourier analysis in branches such as engineering. For instance, we may wish to decompose a wave into a confined expression, which is more manageable to transmit or duplicate, yet where the confined expression creates a good enough reproduction of the original wave.

Naturally, since we are discussing approximations of a function, we are forced to ask ourselves - how well are these functions approximated? In other words, if we have a periodic function f and the associated Fourier series of that function - does the Fourier series of f converge to f?

Carleson proved in 1966 the pointwise convergence of the Fourier series of f almost everywhere if f is a continuous function [3].

In the opposite direction, Jean-Pierre Kahane and Yitzhak Katznelson proved a related theorem in the same year, stating that for every set of measure zero, there exists a continuous function f whose Fourier series will diverge at every point of that set [6].

On the other hand, Kahane later wrote another article [5] on these results - which is the subject of this thesis - that there exists continuous functions f whose Fourier series diverges on large sets.

Intuitively, these statements seem to be contradictory. However, the difference is rather the view of how “large” sets are. Carleson's approach is through measure theory, where the set of points of divergence of the Fourier series of f is what is called a set of "measure zero" - which simply by the name gives us an insight on how to regard the size of the set.

If we however make a topological approach to the problem, we arrive at the conclusion that the set of points of divergence of the Fourier series of a continuous function f might be large. Namely, as stated by Kahane, that there exists a dense G_δ set of continuous functions such that every f in the set have a Fourier series that diverge in a dense G_δ set in $[0, 1]$. Kahane's theorem can be regarded as an extension of the result presented by du Bois-Reymond in 1876, which states that there exists a continuous function f such that the Fourier series of f diverges at a point [2, p.642].

In Chapter 1, we will begin by familiarizing ourselves with topological definitions and view on problems. In essence, the topological view is geometric - where problems are solved rather by the position and distance between objects, sets and points, rather than their value. One of the first instances of solving a mathematical problem topologically was done by Euler in 1736 with the famous problem of the Seven Bridges of Köningsberg [4, p.7].

Next, in Chapter 2, we will introduce the fundamental tools of Fourier series to the reader, such as defining Fourier coefficents, Fourier series, the Dirichlet Kernel and Convolutions.

Along the way we will pick up useful facts given by theorems such as Baire's Category Theorem and the Kuratowski-Ulam Theorem, and also a few technical lemmas which we will use to prove Kahane's Theorem. Propositions may be considered as highly useful facts which may not be obvious to the reader, and are to be reviewed as more general facts regarding topology.

In Chapter 3 we will prove Kahane's theorem after introducing further definitions and Lemmas we will require in the technical parts of the proof of Kahane's theorem.

Lastly, in Chapter 4, we will briefly mention remarks on results obtained in studies to determine which requisites entail the convergence of Fourier series.

Chapter 1

Metric Spaces

Since the topological approach to problems is done by regarding the distance between e.g. two points, we will first introduce how we define the distance - it would be highly inappropriate (not to say confusing and inconsistent) if the distance between two elements can be determined by e.g. more than one value.

For further reading beyond what is mentioned in this chapter, consult Rudin [10], [11] and Oxtoby [9], since we will follow their presentations.

1.1 Metric spaces, sets and their properties

Definition 1. A **metric space** (X, d) is a set X, together with a function $d : X \times X \to \mathbb{R}$ such that for any elements $x, y, z \in X$, d satisfies;

1. $d(x, y) \geq 0$
2. $d(x, y) = 0 \iff x = y$
3. $d(x, y) = d(y, x)$
4. $d(x, z) \leq d(x, y) + d(y, z)$

   Where d is called the distance, or **metric** on X. The 4^{th} requirement is usually called the *triangle inequality.*

The most intuitive metric space is $(\mathbb{R}, d)$, the real number line where for any two real numbers $a, b \in \mathbb{R}$ we have the familiar distance function $d := |a - b|$, arising from the absolute value. Another familiar metric space is $\mathbb{R}^3$, where the distance between any two points in the Euclidean space is the length of the straight line between a and b.

Often we want to work with more than single elements - when working with for instance the real numbers, it will be more practical to instead speak about sets of elements. Thus we continue with a few definitions regarding sets and their topological properties.

Definition 2. Let (X, d) be a metric space. Let $A \subset X$ be non-empty. We say A is an **open subset** of X if for any point $x \in A$, there exists an $\epsilon > 0$ such that for any point $y \in X$ that satisfies $d(x, y) < \epsilon$, then $y \in A$.

Alternatively, for any given point $x \in A$ there exists a small neighbourhood around x with respect to the metric, where all points in that neighbourhood are also points of A.

In this text we will use the convention that in any metric space, $\emptyset$ is an open set.

Example of an open set in $\mathbb{R}$ is the open interval $(0, 2) \subset \mathbb{R}$, which is a subset of $\mathbb{R}$. For an arbitrary point in $(0, 2)$, say our favourite irrational number $\sqrt{2}$, there exists an ϵ such that all real numbers on the number line within that distance are also within the same subset. In this case, any point x that satisfies $d(\sqrt{2}, x) < 2 - \sqrt{2}$.

Definition 3. Let (X, d) be a metric space and $A \subset X$. We say that a set A is **closed** if the complement of A, denoted as A^c, is open.

An alternative definition for a set A to be closed in (X, d) is that if $\{x_n\}_{n=1}^{\infty}$ is a sequence in A converging to some element x_0, then x_0 is also an element of A. We will introduce the notion of convergence with Definition 9.

Definition 4. Let (X, d) be a metric space, and $A \subset X$ be an nonempty subset. We define the **closure** of A, denoted as $\overline{A}$, to be the set of all points $x \in X$ such that, for every $\epsilon > 0$ there is some $y \in A$ such that $d(x, y) < \epsilon$.

Alternatively, the closure of A is the smallest closed set that contains A.

Definition 5. Let (X, d) be a metric space and $A \subset X$. We say A is **dense** in X if for every $x \in X$ and for every $\epsilon > 0$ there exists an element $y \in A$ such that $d(y, x) < \epsilon$.

Or in words, any non-empty open set of X contains at least one point of A. An example of this is that $\mathbb{Q}$ is dense in $\mathbb{R}$. This follows from the fact that between any two real numbers, there exists a rational number - and therefore any open set containing those two real numbers must also contain at least one rational number.
Sometimes it will be more convenient to speak of **nowhere dense** sets instead. This means that for every element in a nowhere dense set, there exists an open set around the element that does not contain another element of the set. This gives the notion of the elements being spread out. An example is that $\mathbb{Z}$ is nowhere dense in $\mathbb{R}$.

Definition 6. Let (X, d) be a metric space. We say that (X, d) is **separable** if there exists an $A \subset X$ such that A is countable and dense.

Equivalently, we say (X, d) is separable if it has a countable base. A base B is a collection of open sets such that every open set in X can be written as a union of elements of B.
An example is that $\mathbb{R}$ is separable, since we know that $\mathbb{Q} \subset \mathbb{R}$ is a dense subset.

Another way to think about separable metric spaces is that since A is countable, it can be described as a sequence: $A = \{x_n\}_{n=1}^{\infty} \subset X$. Since A is dense, it follows that any open subset of X is going to contain some element of the sequence.

Using our previous example, it follows that $\mathbb{R}$ is separable - we already concluded that $\mathbb{Q}$ is dense in $\mathbb{R}$, and that $\mathbb{Q}$ is countable can be shown by constructing an array of a countable number of countable sequences, and constructing a countable sequence from the diagonals as shown in [11, p.29]. As we have already brought up the topic of sequences, we can appropriately move on to the next section, where we will introduce the important notion of Cauchy sequences.

1.2 Cauchy sequences and complete metric spaces

Definition 7. Let $\{x_n\}_{n=1}^{\infty}$ be a real valued sequence. The **superior limit** of $\{x_n\}_{n=1}^{\infty}$ is defined and denoted as

$$\overline{\lim_{n\to\infty}} x_n := \limsup_{n\to\infty} x_n = \inf_{n\geq 0} \sup_{m\geq n} x_m$$

Although sequences may not converge to a limit, they will always have a superior limit in $\mathbb{R} \cup \{+\infty\}$. For instance, the sequence $x_n = \frac{n}{n+1} \cdot \sin(\frac{(2n+1)\pi}{2})$ has no limit, but the superior limit of the sequence is 1.

Definition 8. Let (X, d) be a metric space. We say that the sequence $\{x_n\}_{n=1}^{\infty}$ is a **Cauchy sequence** if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for all $n, m > N$, $d(x_n, x_m) < \epsilon$.

This intuitively tells us that the elements of a Cauchy sequence are closer to each other as the sequence progresses. Note that this means that each element is closer to all other elements as the sequence progresses, and not only close to the previous, or next, element in the sequence.

Definition 9. Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X. We say that the sequence **converges** to a limit in X if there exists an $x \in X$, such that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $d(x_n, x) < \epsilon$ for $n \geq N$.

Definition 10. Given a metric space (X, d), we say that X is **complete** if every Cauchy sequence converges to a limit in X.

Informally, we can consider this as that the metric space contains no "holes", or missing elements.

Remark. Note that while every convergent sequence is a Cauchy sequence in any metric space, not every Cauchy sequence converges. Take the sequence where $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + ... + \frac{1}{n!}$. This sequence $\{x_n\}_{n=1}^{\infty}$, which is a Cauchy sequence in $\mathbb{Q}$ has the limit e, an irrational number which is not in $\mathbb{Q}$. Therefore $\mathbb{Q}$ is not complete.

So far we have only gone through separate facts and definitions about individual metric spaces. But for instance $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is a metric space that we are familiar with. And since we were on the topic of complete metric spaces and Cauchy sequences - consider the Cartesian product of two complete metric spaces. Then we might ask ourselves that if the product spaces are complete, then the produced space should be complete as well.

1.3 Product spaces

Proposition. *If (X, d) and (Y, p) are complete metric spaces, then $(X \times Y, d+p)$ is also complete.*

Proof. The metric on $X \times Y$ will have the property of $\sigma : (X \times Y) \times (X \times Y) \to [0, \infty)$, taking two points in $X \times Y$ to a value in $[0, \infty)$. Let the metric be defined by $\sigma((x_1, y_1), (x_2, y_2)) := d(x_1, x_2) + p(y_1, y_2)$ for two points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$. First we need to ensure that this is a metric.

First we note that since both d and p are metrics, $d \geq 0$ and $p \geq 0$, so it follows from the definition that $\sigma \geq 0$. Furthermore, it follows that $\sigma = 0$ only if both d and p are zero. Since $d(x_1, x_2) = 0$ only when $x_1 = x_2$, and similarly $p(y_1, y_2) = 0$ only when $y_1 = y_2$, we can conclude

$$\sigma((x_1, y_1), (x_2, y_2)) = 0 \Leftrightarrow x_1 = x_2, y_1 = y_2.$$

Similarly, since d and p are both metrics we know $d(x_1, x_2) = d(x_2, x_1)$ and $p(y_1, y_2) = p(y_2, y_1)$. Thus

$$\sigma((x_1, y_1), (x_2, y_2)) = d(x_1, x_2)+p(y_1, y_2) = d(x_2, x_1)+p(y_2, y_1) = \sigma((x_2, y_2), (x_1, y_1)).$$

Lastly we ensure that the triangle inequality holds. We know it holds for d and p:

$$\begin{aligned} \sigma((x_1, y_1), (x_2, y_2)) &= d(x_1, x_2) + p(y_1, y_2) \\ &\leq d(x_1, x_3) + d(x_3, x_2) + p(y_1, y_3) + p(y_3, y_2) \\ &= d(x_1, x_3) + p(y_1, y_3) + d(x_3, x_2) + p(y_3, y_2) \\ &= \sigma((x_1, y_1), (x_3, y_3)) + \sigma((x_3, y_3), (x_2, y_2)). \end{aligned}$$

Thus σ is indeed a metric.
Now we need to show $X \times Y$ is complete. Let $\{(x_n, y_n)\}_{n=1}^{\infty}$ be a Cauchy sequence in $X \times Y$. We need to show that the sequence has a limit in $X \times Y$. Since $\{(x_n, y_n)\}_{n=1}^{\infty}$ is a Cauchy sequence in $X \times Y$, i.e, for every $\delta > 0$, there exists an $N \in \mathbb{N}$ such that $\sigma((x_n, y_n), (x_m, y_m)) < \delta$ for $n, m \geq N$, then

$$\sigma((x_n, y_n), (x_m, y_m)) = d(x_n, x_m) + p(y_n, y_m) < \delta,$$

thus forcing both $\{x_i\}_{i=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ to be Cauchy sequences in X and Y respectively. Since X and Y are both complete, both $\{x_i\}_{i=1}^{\infty}$ and $\{y_k\}_{k=1}^{\infty}$ have limits, x and y respectively. Let $\epsilon > 0$. Then there exists a large enough $N \in \mathbb{N}$ such that both $d(x_n, x) < \frac{\epsilon}{2}$ and $d(y_n, y) < \frac{\epsilon}{2}$ for $n \geq N$. Then

$$\sigma((x_n, y_n), (x, y)) = d(x_n, x) + d(y_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

At this point we should mention the equivalence of metrics. Given the metric spaces (X, d) and (Y, p), we could have equipped $X \times Y$ with other possible metrics, such as $\sigma := \sqrt{d^2 + p^2}$ or $\sigma := \max\{d, p\}$, which would give us the corresponding metric spaces $(X \times Y, \sqrt{d^2 + p^2})$ and $(X \times Y, \max\{d, p\})$, and both would be equivalent to $(X \times Y, d + p)$.

Definition 11. Let (X, d) and (X, d') be metric spaces. We say that d and d' are **equivalent** metrics if

$$\alpha d(x, y) \leq d'(x, y) \leq \beta d(x, y) \quad \forall x, y \in X,$$

where α and β are positive constants.

For our two other proposed metrics for $X \times Y$, we have

$$\frac{d + p}{\sqrt{2}} \leq \sqrt{d^2 + p^2} \leq d + p$$

$$\frac{d + p}{2} \leq \max\{d, p\} \leq d + p$$

thus making all three metrics equivalent. We would like equivalent metrics to retain the properties of a metric space, e.g. if (X, d) is a complete metric space, then (X, d') should also be a complete metric space if d and d' are equivalent metrics.

Proposition. *Let (X, d) and (X, d') be metric spaces, with d and d' are equivalent metrics. If (X, d) is complete, then (X, d') is also complete.*

Proof. Let $\{x_n\}_{n=1}^{\infty}$ be a Cauchy sequence in (X, d'). Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d'), we have that for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$, we have that $d'(x_n, x_m) \leq \epsilon$. Since d and d' are equivalent, there exists an $\alpha > 0$ such that

$$d(x_n, x_m) \leq \frac{1}{\alpha} d'(x_n, x_m) \leq \frac{1}{\alpha}\epsilon.$$

Therefore, $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d). Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in (X, d) and (X, d) is complete, it has a limit, i.e. for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n \geq N$, we have that $d(x_n, x) \leq \epsilon$. Then $\{x_n\}_{n=1}^{\infty}$ converges to x in d', since for the same ϵ and N, and since d and d' are equivalent, we have

$$d'(x_n, x) \leq \beta d(x_n, x) \leq \beta \epsilon$$

for $n \geq N$ and some $\beta > 0$. Thus (X, d') is complete.

□

Due to the subject of this thesis, we should also discuss metric spaces involving continuous functions. The **set of continuous functions over** $[a, b]$ is denoted as $C([a, b])$.

Proposition. *The set of continuous functions equipped with the metric* $d_\infty : C([0,1]) \times C([0,1]) \to \mathbb{R}$ *defined by*

$$d_\infty(f, g) := \sup_{x\in[0,1]} (|f(x) - g(x)|) = \max_{x\in[0,1]} (|f(x) - g(x)|)$$

is a complete metric space.

We will follow the presentation by Rudin of several theorems in [11]. First we show that d_∞ is a metric, and then that the pointwise limit function f exists for a Cauchy sequence $\{f_n\}_{n=1}^\infty$. Then we show that the sequence converges to f in d_∞, and lastly that f is continuous.

Proof. First we ensure that d_∞ is a metric. By definition, for two functions f and g in $C([0,1])$, $d_\infty(f, g) \geq 0$ due to the absolute value. Furthermore, if $d_\infty(f, g) = 0$, then $f(x) = g(x)$ for all x in $[0,1]$. Thus $f = g$. That $d_\infty(f, g) = d_\infty(g, f)$ follows from the definition and properties of the absolute value. Finally,

$$d_\infty(f, g) = \max_{x\in[0,1]} (|f(x) - g(x)|) = \max_{x\in[0,1]} (|f(x) - g(x) + h(x) - h(x)|)$$

$$= \max_{x\in[0,1]} (|f(x) - h(x) + h(x) - g(x)|) \leq \max_{x\in[0,1]} (|f(x) - h(x)| + |h(x) - g(x)|)$$

$$\leq \max_{x\in[0,1]} (|f(x) - h(x)|) + \max_{x\in[0,1]} (|h(x) - g(x)|) = d_\infty(f, h) + d_\infty(h, g).$$

Thus d_∞ is a metric.
Let $\{f_n\}_{n=1}^\infty$ be a Cauchy sequence in $(C([0,1]), d_\infty)$. This means that for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m, n \geq N$, $d_\infty(f_n, f_m) < \epsilon$.

Since $d_\infty(f_m, f_n) \to 0$ when $m, n \to \infty$, for every $x_0 \in [0,1]$, we have that

$$|f_n(x_0) - f_m(x_0)| \leq d_\infty(f_n, f_m) \to 0.$$

Therefore, since $\mathbb{R}$ is complete, $f_n(x_0)$ converges to some real number as $n \to \infty$. We define the function f to be the pointwise limit of f_n by

$$f(x) = \lim_{n\to\infty} f_n(x), \quad x \in [0,1].$$

Now we need to show that f_n converges to f in d_∞. Let $\epsilon > 0$. Since f_n is a Cauchy sequence, there exists an N such that for $n, m \geq N$, then $d_\infty(f_n, f_m) < \epsilon$. Since for every $x \in [0,1]$ we have

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \lim_{m\to\infty} d_\infty(f_n, f_m) \leq \epsilon$$

we deduce that

$$d_\infty(f_n, f) \leq \lim_{m\to\infty} d_\infty(f_n, f_m) \leq \epsilon.$$

Lastly, let $\epsilon > 0$ and choose an x_0 in $[0, 1]$. Note that since every f_n in the sequence is continuous, each f_n will have a limit as x approaches x_0:

$$\lim_{x\to x_0} f_n(x) = A_n.$$

Since $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence, for every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that for $n, m \geq N$, we have $d_\infty(f_n, f_m) < \epsilon$. Therefore,

$$|A_n - A_m| = \lim_{x\to x_0} |f_n(x) - f_m(x)| \leq \lim_{x\to x_0} d_\infty(f_n, f_m) = d_\infty(f_n, f_m) < \epsilon$$

for all $n, m \geq N$. We deduce that $\{A_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$, and since $\mathbb{R}$ is complete, the sequence $\{A_n\}_{n=1}^{\infty}$ has a limit A in $\mathbb{R}$.
Choose an n large enough so that the following are satisfied:

$$d_\infty(f_n, f) \leq \frac{\epsilon}{3}, \quad |A_n - A| \leq \frac{\epsilon}{3}.$$

Now for the same n, since f_n is continuous there exists a $\delta > 0$ such that for all $x \in (x_0 - \delta, x_0 + \delta)$ we have

$$|f_n(x) - A_n| \leq \frac{\epsilon}{3} \quad \forall x \in (x_0 - \delta, x_0 + \delta).$$

Then:

$$|f(x) - A| \leq d_\infty(f_n, f) + |f_n(x) - A_n| + |A_n - A| \leq \epsilon$$

for all x in $(x_0 - \delta, x_0 + \delta)$. This is the same as f is continuous, and we are done. □

1.3.1 Baire's Category Theorem

Theorem 1 (Baire's Category Theorem)**.** *In a complete metric space, any countable intersection of dense open sets is dense.*

We follow the presentation of the proof made by Kahane in [5].

Proof. Let (X, d) be a complete metric space. Suppose $A_1, A_2, ...$ is a sequence, where A_i be open and dense in X. Let B_0 be an arbitrary open ball in X. Since B_0 is open, and A_1 is open and dense, the intersection $B_0 \cap A_1$ is open and non-empty.

Therefore we can say there exists a smaller open ball B_1, such that the closure $\overline{B}_1$ is contained in the intersection $B \cap A_1$.

Using this smaller open ball B_1, the intersection $B_1 \cap A_2$ is again open and non-empty, for which there exists an open ball B_2 such that $\overline{B}_2$ is contained in $B_1 \cap A_2$. This way we construct a decreasing sequence of closed balls $\overline{B}_1, \overline{B}_2, ...$

Since X is complete, and since the sequence of centers of the open nested balls B_i form a Cauchy sequence, the sequence converges to some $x \in X$. Therefore

$x \in B_0, A_1, A_2, ...$, from which we can conclude that $A_1 \cap A_2 \cap ...$ intersects B_0. Since B_0 was an arbitrary open ball, $A_1 \cap A_2 \cap ...$ is dense in X.

□

A logical conclusion from Baire's category theorem is that if there exists a countable intersection of dense open sets which is not dense, then the metric space is not complete. An example is shown by the rationals. As explained, $\mathbb{Q}$ is countable. For each rational number q_i, define $q_i^c = \mathbb{Q} \setminus \{q_i\}$. Each q_i^c is then an open and dense set - but $\bigcap_{i=1}^{\infty} q_i^c$ is not dense, since the intersection of all these sets is empty. Therefore, once again, $\mathbb{Q}$ is not complete.
From this follows another classification of metric spaces.

Definition 12. A **Baire space** is a metric space where every countable intersection of open and dense subsets is dense.

Remark. Complete metric spaces are Baire spaces.

Here we should mention that there exists an explicit definition for sets of the type $A_1 \cap A_2 \cap ...$, and since several of the theorems and definitions use that terminology, we will adopt it as well with the following definition.

Definition 13. Let (X, d) be a metric space with $A \subset X$. We say that A is a G_δ set if A is the countable intersection of open sets. Alternatively, for $A = A_1 \cap A_2 \cap ...$ and any given point $x \in A$, then $\forall i$ there exists an $\epsilon_i > 0$ such that for all $y \in X$, if $d(x, y) < \epsilon_i$, then $y \in A_i$.

Naturally, every open set is itself a G_δ set. An example of a G_δ set is the irrational numbers in $\mathbb{R}$. Let $\mathbb{Q} = \{q_n\}_{n=1}^{\infty}$ be the set of rational numbers, then for every rational number, the complement q_n^c is an open set. Thus it follows that $\bigcap_{n=1}^{\infty} q_n^c$ is the set of irrationals.

For the next theorem we also need to understand the use of **quasi**, meaning *almost*. Mostly we will use this prefix for *quasi all* and *quasi everywhere*, thus meaning *almost all* and *almost everywhere* respectively. This is rather loosely defined, when we for example state "quasi all $x \in X$", we only get the notion that it applies to the elements of X, with exceptions to some extent. However, to be rigorous we will use Kahane's version; if (X, d) is a metric space, a property P holds for every element in a G_δ set of dense open sets, then P holds quasi-everywhere in X.

1.3.2 Kuratowski-Ulam Theorem

Theorem 2 (Kuratowski-Ulam)**.** *Let (X, d) and (Y, p) be Baire spaces. Suppose that (Y, p) is separable, i.e. (Y, p) has a countable base of open sets. Let A be a dense G_δ set in $(X \times Y, \tilde{d} = d + p)$. For every $x \in X$ we define $\pi_x(A) = \{y \in Y : (x, y) \in A\}$. Then $\pi_x(A)$ is a dense G_δ set in Y for quasi all $x \in X$.*

We follow the proof presented by Kahane and Queffélec [7, p.503].

Proof. Since A is a dense G_δ set, we have that

$$A = \bigcap_{n=1}^{\infty} G_n$$

where each G_n is dense in $X \times Y$. And since Y is separable, let $\{\Gamma_i\}_{i=1}^{\infty}$ be a countable base for Y, where every Γ_i is a non-empty open set. We define

$$E_{n,i} = \{x \in X : \pi_x(G_n) \cap \Gamma_i \neq \emptyset\}.$$

Since G_n is dense in $X \times Y$, and Γ_i is an open set in Y, $G_n \cap (X \times \Gamma_i)$ is non-empty. Therefore there exists an $(x_0, y_0) \in G_n \cap (X \times \Gamma_i)$. Then $y_0 \in \Gamma_i$, and $\pi_{x_0}(G_n)$ is non-empty - and therefore $x_0 \in E_{n,i}$, so $E_{n,i}$ is non-empty. We need to show that $E_{n,i}$ is open and dense.

Let $x_0 \in E_{n,i}$. Then by definition of $E_{n,i}$, there exists a $y_0 \in \Gamma_n$ such that $(x_0, y_0) \in G_n$. Since G_n is open, there exists an open ball $B_{(x_0,y_0)}(r)$ with radius $r > 0$ centered on $(x_0, y_0) \in E_{n,i} \times \Gamma_i$ which is contained in G_n. We can create an open square contained in $B_{(x_0,y_0)}(r)$ by picking an $0 < \epsilon < \frac{r}{\sqrt{2}}$ such that

$$B_{x_0}(\epsilon) = \{x \in X : d(x_0, x) < \frac{\epsilon}{2}\}$$

$$B_{y_0}(\epsilon) = \{y \in \Gamma_i : p(y_0, y) < \frac{\epsilon}{2}\}.$$

Then both $B_{x_0}(\epsilon)$ and $B_{y_0}(\epsilon)$ are open, and

$$B_{x_0}(\epsilon) \times B_{y_0}(\epsilon) \subseteq B_{(x_0,y_0)}(r) \subseteq G_n.$$

From this, we deduce that $B_{x_0}(\epsilon) \subseteq E_{n,i}$, and therefore $E_{n,i}$ must be open.

Now take an arbitrary open set D in (X, d). Since G_n is dense in $X \times Y$, we have that $G_n \cap (D \times \Gamma_i)$ is non-empty, i.e. there exists an $(x_0, y_0) \in D \times \Gamma_i$ such that $y_0 \in \Gamma_i \cap \pi_{x_0}(G_n)$. Therefore $x_0 \in D \cap E_{n,i}$, and we deduce that $E_{n,i}$ is dense in X.

Since $n, i \in \mathbb{N}$ were arbitrarily fixed and $E_{n,i}$ is open and dense in a Baire space, we can deduce from Baire's Category Theorem that

$$E = \bigcap_{n,i=1}^{\infty} E_{n,i}$$

is a dense G_δ set in (X, d). Note that for any $x \in E$, we have that $x \in E_{n,i}$ for all $n, i \in \mathbb{N}$, then for every $n \in \mathbb{N}$ we have that $\pi_x(G_n) \cap \Gamma_i$ is non-empty for all $i \in \mathbb{N}$. Since $\{\Gamma_i\}_{n=1}^{\infty}$ is a base for (Y, p), $\pi_x(G_n)$ is dense in (Y, p).

Take $y_0 \in \pi_{x_0}(G_n)$. Since G_n is open, there exists an open ball $B_{(x_0,y_0)}(r)$ with radius $r > 0$ contained in G_n. Then, if $B_{y_0}(r) = \{y \in \Gamma_i : p(y_0, y) < r\}$ we deduce $B_{y_0}(r) \subseteq \pi_{x_0}(G_n)$, so $\pi_{x_0}(G_n)$ is open.

Since for every $x \in E$ we have that $\pi_x(G_n)$ is dense and open, and (Y, p) is a Baire space, it follows from Baire's Category Theorem that

$$\bigcap_{n=1}^{\infty} \pi_x(G_n)$$

is a dense G_δ set in (Y, p). Since $A = \bigcap_{n=1}^{\infty} G_n$ and E is a dense G_δ set in (X, d), we have that $\pi_x(A)$ is a dense G_δ set for quasi all $x \in X$.

□

Chapter 2

Groundwork of Fourier analysis

When we first study mathematics we soon get introduced to Taylor series and Maclaurin series. These series we learn, are a way to approximate functions by a polynomial function over a small interval. For instance:

$$\sin(x) \approx x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...$$

is a useful way to approximate the sine function in a neighbourhood around 0, with applications especially when it comes to programming.

The idea behind Fourier analysis is similar - given a periodic function over an interval, we can approximate the function by a trigonometric series. And then we arrive at the purpose of this thesis - does the sum converge to the original function? We will approach this problem in the next chapter, for now we will begin the study of Fourier series. It is worth mentioning that while we study mainly real-valued functions, the Fourier series - and trigonometric series in general - are not limited to real valued functions, but cover complex valued functions as well.

For further reading beyond what is mentioned in this chapter, consult Stein and Shakarchi [12].

In this thesis our main concern will be functions over the interval $[0, 1]$. We will use the condition that f is continuous throughout this thesis, however it is important to note that many of the following definitions and theorems stated around the subject of Fourier series do not require continuity of functions - but instead, it suffices that f is merely Riemann integrable, or even Lebesgue integrable. However, since our main interest (Kahane's Theorem) concerns only continuous functions, we shall restrict ourselves to this smaller class of functions.

It is also worth noting that our interest is **periodic functions** - i.e, if f is *1-periodic*, then $f(x) = f(x+1)$. Furthermore, we will require for our periodic functions that if f is periodic over a fundamental interval $[a, b)$, then $f(a) = f(b)$. This provides us with the property that if f is 1-periodic and defined over $\mathbb{R}$, then we can take any interval of length 1 from $\mathbb{R}$ and capture the original function [12, p.33].

2.1 Fourier series

Suppose for the remainder of the thesis, that f is a continuous function defined on $[0, 1]$ with $f(0) = f(1)$.

Definition 14. The n^{th} **Fourier coefficient** of f is denoted as $\widehat{f}(n)$ and is given by

$$\widehat{f}(n) := \int_0^1 f(x)e^{-2\pi inx}dx, \quad n \in \mathbb{Z}.$$

Remark. Note that in general, in the study of Fourier analysis we are interested in functions that are not only defined on an interval $[0, 1]$. If we have an interval of $[a, b]$, then $\widehat{f}(n) = \frac{1}{b-a}\int_a^b f(x)e^{\frac{-2\pi inx}{b-a}}dx, \quad n \in \mathbb{Z}$. However, for simplicity we will in this thesis cover the case of functions on $[0, 1]$, without loss of generality.

Definition 15. The **Fourier series** of f is given by

$$\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}.$$

In the case where we work with functions over an interval $[a, b]$, the Fourier series is formally given by $\sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{\frac{2\pi inx}{b-a}}$. We use the following formal notation to indicate the relationship between f and the Fourier series of f:

$$f(x) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n)e^{2\pi inx}$$

Suppose a trigonometric series of a function f has a finite number of non-zero Fourier coefficients, i.e. for some integer N, $\widehat{f}(n) = 0$ for all $n > N$. Then we say that we have a trigonometric polynomial of degree N - the largest n such that $\widehat{f}(n) \neq 0$.

When generally working with series, we sometimes tend to work with partial sums, i.e. given a series $\sum_{i=0}^{\infty} a_i$ we look at $S_N = \sum_{i=0}^{N} a_i$, and see if S_N converges as $N \to \infty$. The idea is the same when working with trigonometric series.
The N^{th} partial sum of the Fourier series of f is given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi inx}, \quad N \in \mathbb{N}$$

and thus we can provide an alternative statement to the purpose of this thesis: does S_N converge to f as $N \to \infty$? To answer such a question, we need to introduce an important family of trigonometric polynomials.

2.2 Dirichlet Kernel

Definition 16. The function

$$D_N(x) := \sum_{n=-N}^{N} e^{2\pi inx}, \quad x \in [0,1), N \in \mathbb{N}$$

is the **Dirichlet Kernel of order N**.

If one compares the definition of the Dirichlet kernel with the general definition for Fourier series, we can conclude that Dirichlet kernel is the Fourier series of a function f, where the Fourier coefficients have the property that $\widehat{f}(n) = 1$ if $-N \leq n \leq N$, and $\widehat{f}(n) = 0$ otherwise.

The Dirichlet kernel is recurring throughout this thesis (and Fourier analysis), thus we will show that there exists a closed form formula for the partial sums. A compact version of this can be found in [12, p.37].

Write

$$\sum_{n=-N}^{N} e^{2\pi inx} = \sum_{n=0}^{N} e^{2\pi inx} + \sum_{n=-N}^{-1} e^{2\pi inx}.$$

From here, we can see that if $x = 0$, every term in the sum will be 1 and we have $2N + 1$ terms. Next, we note that for the first term on the righthand side (when $n \geq 0$):

$$\sum_{n=0}^{N} e^{2\pi inx} = 1 + e^{2\pi ix} + e^{4\pi ix} + ... + e^{N2\pi ix}.$$

Now, by multiplying with $1 - e^{2\pi ix}$ we can cancel a lot of terms:

$$(1 - e^{2\pi ix})(1 + e^{2\pi ix} + e^{4\pi ix} + ... + e^{N2\pi ix}) = 1 - e^{(N+1)2\pi ix}.$$

Therefore, we can conclude for $x \neq 0$:

$$\sum_{n=0}^{N} e^{2\pi inx} = 1 + e^{2\pi ix} + e^{4\pi ix} + ... + e^{N2\pi ix} = \frac{1 - e^{(N+1)2\pi ix}}{1 - e^{2\pi ix}}.$$

Similarly, for the second term when summing for $n < 0$:

$$\sum_{n=-N}^{-1} e^{2\pi inx} = e^{-N2\pi ix} + e^{(N-1)2\pi ix} + ... + e^{-2\pi ix}$$

$$(1 - e^{2\pi ix})(e^{-N2\pi ix} + e^{(N-1)2\pi ix} + ... + e^{-2\pi ix}) = e^{-N2\pi ix} - 1.$$

Thus:

$$\sum_{n=-N}^{-1} e^{2\pi inx} = e^{-N2\pi ix} + e^{(N-1)2\pi ix} + ... + e^{-2\pi ix} = \frac{e^{-N2\pi ix} - 1}{1 - e^{2\pi ix}}.$$

So for our original series:

$$\sum_{n=-N}^{N} e^{2\pi inx} = \frac{1 - e^{(N+1)2\pi ix}}{1 - e^{2\pi ix}} + \frac{e^{-N2\pi ix} - 1}{1 - e^{2\pi ix}}.$$

After setting both fractions on the same denominator and simplifying:

$$\sum_{n=-N}^{N} e^{2\pi inx} = \frac{e^{-N2\pi ix} - e^{(N+1)2\pi ix}}{1 - e^{2\pi ix}} = \frac{e^{-\frac{1}{2}2\pi ix}}{e^{-\frac{1}{2}2\pi ix}} \frac{e^{-N2\pi ix} - e^{(N+1)2\pi ix}}{1 - e^{2\pi ix}}$$

$$= \frac{e^{-(N+\frac{1}{2})2\pi ix} - e^{(N+\frac{1}{2})2\pi ix}}{e^{-\frac{1}{2}2\pi ix} - e^{\frac{1}{2}2\pi ix}} = \frac{\sin(2\pi(N + \frac{1}{2})x)}{\sin(2\pi\frac{x}{2})}.$$

Where we use Euler's formula for both the numerator and denominator in the last step. To conclude:

$$D_N(x) = \sum_{n=-N}^{N} e^{2\pi inx} = \begin{cases} \frac{\sin(2\pi(N+\frac{1}{2})x)}{\sin(\pi x)} & \textit{if} \quad x \in (0,1) \\ 2N+1 & \textit{if} \quad x = 0 \end{cases}.$$

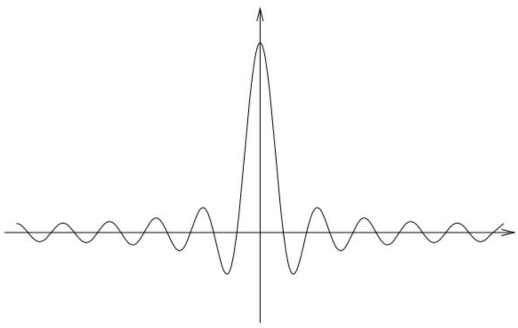

Figure 2.1: Dirichlet Kernel for large N

From the closed formula we can see that as N increases, the function will oscillate more frequently between positive and negative values (and clearly the amplitude increases as well in the neighbourhood of $x = 0$). Figure 2.1[1] gives us a perception of how $D_N(x)$ behaves for large N.

[1]Figure 2.1 taken from [12, p.50], figure 5

In the following section, we will see how the Dirichlet Kernel, and the associated closed formula for an arbitrary N, can be used to compute the partial sum of the Fourier series of a function of interest.

2.3 Convolutions

Definition 17. Let f and g be 1-periodic, continuous functions on $\mathbb{R}$. Recall that the product of two continuous functions is continuous. We define the **convolution** of f and g (denoted as $f * g$) on the interval $[0, 1]$ by

$$\begin{aligned}(f * g)(x) &= \int_0^1 f(y)g(x-y)dy \\ &= \int_0^1 f(x-y)g(y)dy.\end{aligned}$$

Our interest in the convolutions of functions originates from the study of the partial sums of Fourier series. Let $f \in C[0, 1]$, with Fourier coefficients $\widehat{f}(n) = \int_0^1 f(y)e^{-2\pi i n y}dy, n \in \mathbb{Z}$. Then the partial sum of the Fourier series of f is as before given by

$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{2\pi i n x} = \sum_{n=-N}^{N} \left(\int_0^1 f(y)e^{-2\pi i n y}dy \right) e^{2\pi i n x}.$$

Since $e^{2\pi i n x}$ is considered a constant with respect to integration to the variable y;

$$S_N(f)(x) = \sum_{n=-N}^{N} \int_0^1 f(y)e^{2\pi i n(x-y)}dy.$$

And lastly, since the sum is finite we can interchange the order between the sum and integral.

$$\begin{aligned}S_N(f)(x) &= \int_0^1 \sum_{n=-N}^{N} f(y)e^{2\pi i n(x-y)}dy = \int_0^1 f(y)\left(\sum_{n=-N}^{N} e^{2\pi i n(x-y)} \right)dy \\ &= \int_0^1 f(y)D_N(x-y)dy = (f * D_N)(x).\end{aligned}$$

Therefore the partial sum of the Fourier series of f can be expressed as the convolution of f with the Dirichlet kernel, $S_N(f)(x) = (f * D_N)(x)$.

Convolutions of two functions play a large role in Fourier analysis, and have several useful properties, such as linearity, commutativity and associativity among others, which may be found in [12, p.45][2]. The properties of convolutions are

[2] Stein and Shakarchi go further to suggest that the convolution of two functions can, in a sense, be considered as weighted averages.

often used to simplify calculations, in our case mainly for the calculation of the partial sum of a Fourier series for a given function f.

Definition 18. Let $A \subseteq X$. We define a **characteristic function** χ_A of A by:

$$\chi_A(x) := \begin{cases} 1, & x \in A \\ 0, & x \in X \setminus A \end{cases}.$$

In a sense, we can think of a characteristic function of a set as a function that indicates the membership of elements in the set. We can therefore think of the Fourier coefficients of the Dirichlet kernel as:

$$D_N(x) = \sum_{n \in \mathbb{Z}} \chi_{[-N,N]}(n) \cdot e^{2\pi i n x}$$

for $x \in [0, 1), N \in \mathbb{N}$.
Characteristic functions of sets are useful in the sense that if we have a function f with a domain X, we might be interested in the properties of the functions over a subset $A \subset X$. Then we can define a function $g = \chi_A \cdot f$. The following Lemma is an example of such a function g, which will be used in our proof of Kahane's Theorem in chapter 3:

Lemma. *Let $g_N(x) = \chi_{[0,\frac{1}{2}]}(x) \cdot \sin(2\pi(N + \frac{1}{2})x)$. Then:*

$$|S_N(g_N)(0)| \geq C \ln(N)$$

where C is a positive constant.

Proof.

$$S_N(g_N)(0) = \int_0^1 g_N(0 - y) \cdot \frac{\sin(2\pi(N + 1/2)y)}{\sin(\pi y)} dy.$$

Since $\chi_{[0,\frac{1}{2}]}(x) = 0$ for all $x > \frac{1}{2}$, we only need to integrate over $[0, \frac{1}{2}]$. Furthermore, recall that sine is an odd function.

$$S_N(g_N)(0) = \int_0^{\frac{1}{2}} \sin(2\pi(N + 1/2)(-y)) \cdot \frac{\sin(2\pi(N + 1/2)y)}{\sin(\pi y)} dy$$

$$S_N(g_N)(0) = -\int_0^{\frac{1}{2}} \frac{\sin^2(2\pi(N + 1/2)y)}{\sin(\pi y)} dy.$$

Thus

$$|S_N(g_N)(0)| = \left| -\int_0^{\frac{1}{2}} \frac{\sin^2(2\pi(N + 1/2)y)}{\sin(\pi y)} dy \right| = \int_0^{\frac{1}{2}} \frac{\sin^2(2\pi(N + 1/2)y)}{\sin(\pi y)} dy.$$

Recall, that $|\sin(\theta)| \leq |\theta|$:

$$|S_N(g_N)(0)| \geq \int_0^{\frac{1}{2}} \frac{\sin^2(2\pi(N + 1/2)y)}{\pi y} dy.$$

And after changing variables $t = 2\pi(N+1/2)y$:

$$|S_N(g_N)(0)| \geq \int_0^{\pi(N+1/2)} \frac{\sin^2(t)}{t} dt.$$

Furthermore, we can rewrite the integral as a sum of integrals over segments:

$$|S_N(g_N)(0)| \geq \sum_{k=1}^{N} \int_{(k-1)\pi}^{k\pi} \frac{\sin^2(t)}{t} dt,$$

which is certainly less than the previous line since we no longer include the last segment of length $\frac{\pi}{2}$. Next, note that $\sin^2(t)$ is bounded and periodic - and each integral in the sum will be "smaller" than the previous due to the increasing denominator as the sum progresses. Thus, we rewrite the denominator with the inequality:

$$\begin{aligned} |S_N(g_N)(0)| &\geq \sum_{k=1}^{N} \int_0^{\pi} \frac{\sin^2(t)}{k\pi} dt \\ &= \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} \int_0^{\pi} \sin^2(t) dt. \end{aligned}$$

Note that $\int_0^{\pi} \sin^2(t)dt$ is always positive. Moreover, if $t \in [\frac{\pi}{4}, \frac{\pi}{2}]$, then $\sin^2(t) \geq \sin^2(\frac{\pi}{4}) = \frac{1}{2}$. Then

$$\int_0^{\pi} \sin^2(t)dt \geq \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2(t)dt \geq \frac{1}{2}(\frac{\pi}{2} - \frac{\pi}{4}) = \frac{\pi}{8}.$$

Hence, we have

$$\begin{aligned} |S_N(g_N)(0)| &\geq \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} \int_0^{\pi} \sin^2(t)dt \geq \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2(t)dt \\ &\geq \frac{\pi}{8} \frac{1}{\pi} \sum_{k=1}^{N} \frac{1}{k} = \frac{1}{8} \sum_{k=1}^{N} \frac{1}{k}. \end{aligned}$$

Since the sum is the harmonic series, and we know $\sum_{k=1}^{N} \frac{1}{k} > \int_1^N \frac{1}{x} dx = \ln(N)$, we deduce

$$|S_N(g_N)(0)| \geq \frac{1}{8} \ln(N) = C \ln(N).$$

□

Remark. From this result, we can intuitively see how this relates to the divergence of the Fourier series of an arbitrary continuous function f. Namely, since $0 \leq |g_N(x)| \leq 1$ for all x, and the partial sum of the Fourier series of g_N is given by the convolution with the Dirichlet Kernel $S_N(g_N) = (g_N * D_N)$, we are led to believe that the Dirichlet Kernel would be the plausible cause of divergence. See Chapter 4 for more further remarks on the subject.

Chapter 3

Kahane's Theorem

3.1 Prerequisites to divergence of Fourier Series

We will go through the last definitions and useful facts we require before we state and prove Kahane's theorem. Given that we are interested in the divergence of the Fourier series of a function f, we will focus on the points x such that the Fourier series diverges.

Definition 19. Let $f \in C([0,1])$. We define the **set of divergence** of the Fourier series of f as

$$A_f := \{x \in [0,1) : \overline{\lim_{N\to\infty}} |S_N(f)(x)| = \infty\}.$$

Thus if $x \in A_f$, the Fourier series of f diverges at x. The reason we are interested in the superior limit rather than the ordinary limit of the partial sums, originates from the fact that we may not be able to the determine the limit - but if it exists, it will be equal to the superior limit.

Since our problem relates to the divergence of the Fourier series of a large subset of the set of continuous functions, we will need some explicit definitions to refer to them.

Definition 20. Let $(X \times Y, \tilde{d})$ be the metric space with $X = C([0,1))$, $Y = [0,1)$ and $\tilde{d} = d_\infty + |\cdot|$. Let $M \in \mathbb{N}$. We define $G_0(M,n)$ and $G(M,N)$ as

$$G_0(M,n) := \{(f,x) \in X \times Y : |S_n(f)(x)| > M\}$$

$$G(M,N) := \bigcup_{n \geq N} G_0(M,n).$$

We can then interpret $G_0(M,n)$ as the set where each element is the Cartesian products of a continuous function f and a specific x for which the Fourier series of f at x is greater than M. The following Lemma provides an important, but not obvious property of $G(M,N)$ and $G_0(M,n)$.

Lemma. *$G(M,N)$ is an open set.*

Proof. First we note that $G(M,N)$ is open if we can show that $G_0(M,n)$ is open for every n, since the union of open sets is open [11, p.34]. In order for $G_0(M,n)$ to be open, we need to show that for any element in $G_0(M,n)$ there is an open neighbourhood where all elements in the neighbourhood are elements of $G_0(M,n)$. So, let $(f_0,x_0) \in G_0(M,n)$ for a fixed $n \in \mathbb{N}$.

A key observation is that if f_0 and f are "close" to each other, then $S_n(f_0)$ and $S_n(f)$ will also be "close" to each other[1]. To be more precise, for any x in $[0,1)$:

$$|S_n(f)(x) - S_n(f_0)(x)| = |S_n(f - f_0)(x)| = |D_n * (f - f_0)(x)| \leq$$

$$\sup_{x' \in [0,1)} |D_n * (f - f_0)(x')|.$$

Now, by using *Young's Inequality* [1, p.157] for convolutions, we have

$$\sup_{x' \in [0,1)} |D_n * (f - f_0)(x')| \leq \sup_{x' \in [0,1)} (|f(x') - f_0(x')|) \cdot ||D_n||_{L^1([0,1))}$$

$$= d_\infty(f, f_0) \cdot ||D_n||_{L^1([0,1))}.$$

Therefore, we can conclude that

$$d_\infty(S_n(f), S_n(f_0)) \leq d_\infty(f, f_0) \cdot ||D_n||_{L^1([0,1))}$$

and thus if we let $d_\infty(f, f_0)$ be arbitrarily small, $d_\infty(S_n(f), S_n(f_0))$ will also be small.

Now, note that $S_n(f_0)$ is a continuous functions, since $S_n(f_0)$ is a finite sum where every term is a continuous function. By the definition of $G_0(M,n)$ and our assumption on f_0, x_0:

$$|S_n(f_0)(x_0)| > M.$$

Then there exists an $\epsilon > 0$ such that

$$|S_n(f_0)(x_0)| - \epsilon \cdot ||D_n||_{L^1([0,1))} > M.$$

Furthermore, since $S_n(f_0)$ is continuous there exists a $\delta > 0$ such that

$$|S_n(f_0)(x)| - \epsilon \cdot ||D_n||_{L^1([0,1))} > M$$

for $|x - x_0| < \delta$. Define $\tilde{\epsilon} = \min\{\epsilon, \delta\}$. Then for any f, x for which

$$\tilde{d}((f,x),(f_0,x_0)) = d_\infty(f, f_0) + |x - x_0| < \tilde{\epsilon},$$

[1] An explicit example is that since the calculation of Fourier coefficients require integration, any two functions that differ at finitely many points will have the same Fourier series [12, p.39].

we have

$$\begin{aligned}|S_n(f)(x)| &\geq |S_n(f_0)(x)| - |S_n(f)(x) - S_n(f_0)(x)| \\ &\geq |S_n(f_0)(x)| - d_\infty(f, f_0) \cdot ||D_n||_{L^1([0,1))} \\ &\geq \epsilon \cdot ||D_n||_{L^1([0,1))} > M.\end{aligned}$$

Hence, $(f, x) \in G_0(M, n)$.

□

The usefulness of $G_0(M, n)$ and $G(M, N)$ to the proof of Kahane's Theorem on divergence of Fourier series becomes apparent by the identity presented in the following Lemma.

Lemma.

$$\bigcap_{N,M=1}^{\infty} G(M, N) = \{(f, x) \in C([0, 1)) \times [0, 1) : x \in A_f\}$$

Proof. Let $X = C([0, 1))$ and $Y = [0, 1)$, and let

$$(f_0, x_0) \in \bigcap_{N,M=1}^{\infty} G(M, N).$$

Then $(f_0, x_0) \in G(M, N)$ for all $N, M \in \mathbb{N}$. Then

$$(f_0, x_0) \in \bigcup_{n \geq N} G_0(M, n) = \bigcup_{n \geq N} \{(f, x) \in X \times Y : |S_n(f)(x)| > M\}.$$

Therefore, for all $N, M \in \mathbb{N}$ we have $|S_n(f_0)(x_0)| > M$ for each $n \geq N$. Then

$$\overline{\lim}_{N\to\infty} |S_N(f_0)(x_0)| > M \quad \forall M \in \mathbb{N}$$

and consequently

$$\overline{\lim}_{N\to\infty} |S_N(f_0)(x_0)| = \infty.$$

And therefore, $(f_0, x_0) \in \{(f, x) \in X \times Y : x \in A_f\}$.

Conversely, if $(f_0, x_0) \in \{(f, x) \in X \times Y : x \in A_f\}$, we have

$$\overline{\lim}_{N\to\infty} |S_N(f_0)(x_0)| = \infty.$$

By the definition of the superior limit we get

$$\overline{\lim}_{N\to\infty} |S_N(f_0)(x_0)| = \inf_{N\in\mathbb{N}} \sup_{n\geq N} |S_n(f_0)(x_0)| = \infty.$$

Consequently, we have

$$\inf_{N\in\mathbb{N}} \sup_{n\geq N} |S_n(f_0)(x_0)| > M$$

for all $M \in \mathbb{N}$. Hence,

$$\sup_{n \geq N} |S_n(f_0)(x_0)| > M \quad \forall N, M \in \mathbb{N},$$

which implies that

$$(f_0, x_0) \in G_0(M, n)$$

for some $n \geq N$, for all $N, M \in \mathbb{N}$. Thus

$$(f_0, x_0) \in \bigcup_{n \geq N} G_0(M, n) \quad \forall N, M \in \mathbb{N}$$

and consequently, $(f_0, x_0) \in G(M, N)$ for all $N, M \in \mathbb{N}$. Since that is the case for all $N, M \in \mathbb{N}$, we deduce that

$$(f_0, x_0) \in \bigcap_{N,M=1}^{\infty} G(M, N).$$

□

Now that we have the necessary definitions and facts needed, we will state and prove Kahane's theorem.

3.2 Kahane's Theorem

Theorem 3 (Kahane's Theorem)**.** *Quasi all continuous functions on* $[0, 1)$ *have a Fourier series that diverges quasi everywhere.*

We will follow the presentation made by Kahane in [5, p.146].
Let $(X \times Y, \tilde{d})$ be our metric space with $X = C([0, 1))$, $Y = [0, 1)$ and $\tilde{d} = d_\infty + | \cdot |$. The theorem is proved by showing that the set of $(f, x) \in X \times Y$ with $\overline{\lim}_{n\to\infty} |S_n(f)(x)| = \infty$ is a dense G_δ in $X \times Y$. Then one can use the Kuratowski-Ulam Theorem to conclude that the set of $x \in Y$ such that $\overline{\lim}_{n\to\infty} |S_n(f)(x)| = \infty$ is a dense G_δ set in Y, for quasi all $f \in X$.

Proof. Recall, from our Lemmas in Section 3.1, that $G(M, N) = \bigcup_{n \geq N} G_0(M, n)$ and $G_0(M, n) = \{(f, x) \in X \times Y : |S_n(f)(x)| > M\}$ are open sets, and that

$$\bigcap_{N,M=1}^{\infty} G(M, N) = \big\{(f, x) \in X \times Y : \overline{\lim}_{n\to\infty} |S_n(f)(x)| = \infty\big\}.$$

We want to show that $\bigcap_{N,M=1}^{\infty} G(M, N)$ is dense, and to use Baire's Category Theorem we need to show that $G(M, N)$ is dense in $X \times Y$.

Let $(f_0, x_0) \in (X \times Y) \setminus G(M, N)$. Since

$$G(M, N) = \bigcup_{n \geq N} G_0(M, n) = \bigcup_{n \geq N} \{(f, x) \in X \times Y : |S_n(f)(x)| > M\},$$

then $|S_n(f_0)(x_0)| \leq M$ for all $n \geq N$.

Let $\epsilon > 0$, with $\epsilon > h > \frac{M}{C\ln(n)}$. Note that h can be made arbitrarily small for large enough n. Recall from the Lemma in section 2.3; if $g_n(x) = \chi_{[0,\frac{1}{2}]}(x) \cdot \sin(2\pi(n+\frac{1}{2})x)$, then $|S_n(g_n)(0)| \geq C\ln(n)$. Define

$$u_{(+)}(x) = f_0(x) + hg_n(x - x_0)$$

$$u_{(-)}(x) = f_0(x) - hg_n(x - x_0)$$

Then both pairs of $u(x)$ belong to the ϵ-neighbourhood of (f_0, x_0), since:

$$d_\infty(u_{(+)}, f_0) = \sup_{x\in[0,1)} |u_{(+)}(x) - f_0(x)| = \sup_{x\in[0,1)} |f_0(x) + hg_n(x - x_0) - f_0(x)|$$

$$= \sup_{x\in[0,1)} |hg_n(x - x_0)| = h.$$

and

$$d_\infty(u_{(-)}, f_0) = \sup_{x\in[0,1)} |u_{(-)}(x) - f_0(x)| = \sup_{x\in[0,1)} |f_0(x) - hg_n(x - x_0) - f_0(x)|$$

$$= \sup_{x\in[0,1)} | - hg_n(x - x_0)| = h.$$

Furthermore, at least one of the pairs is in $G_0(M, n)$ since:

$$|S_n(u_{(+)})(x_0)| = |S_n(f_0)(x) + hg_n(x - x_0))(x_0)|$$

$$= |S_n(f_0)(x_0) + S_n(hg_n)(0)| \geq |S_n(f_0)(x_0) + M|$$

and

$$|S_n(u_{(-)})(x_0)| = |S_n(f_0)(x) - hg_n(x - x_0))(x_0)|$$

$$= |S_n(f_0)(x_0) - S_n(hg_n)(0)| \geq |S_n(f_0)(x_0) - M|.$$

Since either $u_{(-)}(x)$ or $u_{(+)}(x)$ is in $G_0(M, n)$, it will also be in $G(M, N)$ when $n \geq N$. Therefore, $G(M, N)$ is dense in $X \times Y$.

Since $G(M, N)$ is open and dense, then by Baire's Category Theorem, $\cap_{N,M} G(M, N)$ is a dense G_δ set in $X \times Y$.

Since $[0, 1)$ is separable, and since $C([0, 1))$ and $\underline{[0, 1)}$ are complete, then by the Kuratowski-Ulam Theorem, $A_f = \{x \in [0, 1) : \overline{\lim}_{N\to\infty} |S_N(f)(x)| = \infty\}$ is a dense G_δ set in $[0, 1)$ for quasi all $f \in X$. □

Chapter 4

Further remarks

We have introduced the fundamental notions of both topology and Fourier series, and shown one of the problems encountered in Fourier analysis. Namely, that for any arbitrarily small subset $E \subset [0,1)$, there exists a continuous function f such that the Fourier series of f diverges at every point of E.

The problem of divergence of Fourier series has led to studies to investigate what conditions entail the Fourier series of a function f to converge to f. Some approaches are led by the idea that if we sum the terms in the partial sums in a different order or by different rules, it might converge. E.g. if we define

$$\sigma_N = \frac{S_0(f)(x) + ... + S_{N-1}(f)(x)}{N}$$

which is known as the N^{th} Cesàro mean of the Fourier series of f, we find that if f is continuous, then σ_N converges uniformly to f [12, p.53]. This results in that if f is continuous and $\epsilon > 0$, there exists a a trigonometric polynomial P_n such that

$$|f(x) - P_n| < \epsilon$$

for all $x \in [0,1)$ [12, p.54].
Another approach of summation is to sum the terms by their size of the Fourier coefficents - since they have the largest impact on how the Fourier series behaves. This would be a desirable approach, since e.g. in engineering with regard to signal processing and communications, as it would simplify efforts to reconstruct a continuous function. More specifically, if we define

$$(S_N(f))^*(x) = \sum_{\{k \in \mathbb{Z} : |\widehat{f}(k)| \geq \frac{1}{N}\}} \widehat{f}(k) e^{2\pi i k x}$$

then we find that

$$\overline{\lim}_{N \to \infty} |(S_N(f))^*(x)| = \infty$$

for almost all $x \in [0, 1)$ [8, p.2]. Therefore, rearranging and summing the Fourier series of f in a decreasing order is not a strong enough restriction for convergence.

Since the partial sum of the Fourier series of a function can be obtained by the convolution $(f * D_N)$, we might ask ourselves if the Dirichlet Kernel is the underlying issue of divergence. In fact, there exists a notion of good kernels - and the Dirichlet Kernel does not belong to this family of kernels. In the case above where we take the Cesàro mean of the Fourier series of f, we find that

$$\sigma_N(f)(x) = (f * F_N)(x)$$

where $F_N(x) = \frac{D_0(x)+...+D_{N-1}}{N}$ is known as the Fejér Kernel, and is what is known as a good kernel - and has the property that the limit

$$\lim_{N \to \infty} (f * F_N)(x) = f(x)$$

is uniform when f is continuous [12, p.48-53].

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