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Model Reduction of Semistable Infinite-Dimensional Control Systems

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Abstract

In this thesis, we extend parts of the framework available for model reduction of finite-dimensional stable control systems to an infinite-dimensional and semistable setting. To achieve our goals, we build upon results obtained [CKS17] where the authors find $\mathcal{H}_2$-Norm Error Estimates for the model reduction of finite-dimensional systems driven by a graph Laplacian. The difference between this and previous work is threefold: First, we consider infinite-dimensional systems as to include systems driven by Partial Differential Operators and we thus place earlier work in an appropriate Functional-Analytic setting. Second, we consider a broader class of exponentially semistable systems, not just those driven by a graph Laplacian. Third, we restrict to a class of model reductions which have a dynamic invariance with respect to their kernel and the semigroup associated to the system. For completeness, we also give a brief introduction to Semigroup Theory and provide background material from Functional Analysis. Throughout the text, the second derivative operator and heat equation on $[0, 1]$ are used as examples.

Model Reduction of Semistable Infinite-Dimensional Control Systems

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Contents

1 Introduction

The aim of this thesis is to extend parts of the framework available for model reduction of finite-dimensional control systems to an infinite-dimensional setting, to systems known as infinite-dimensional control systems and sometimes also as distributed parameter control systems. In particular, we are interested in finding a trace representation of the $\mathcal{H}_2$-norm, which essentially can be thought of as the root mean square energy of a system, that applies to semistable infinite-dimensional systems and an associated Lyupunov equation description for computing this norm seeing as these objects play a crucial part in the corresponding finite-dimensional analysis for model reduction. To achieve our goal, we analyze and extend to a Hilbert space setting a finite-dimensional result of Cheng, Kawano and Scherpen investigated in a sequence of papers: mainly [CS16] and [CKS17]. In these papers, the authors consider approximating a system driven by the negative of a Laplacian matrix[1] and among other things give a formula for the $\mathcal{H}_2$-error between the approximated system and the original system in terms of a Lyapunov equation and the trace of one of its solutions. Even though their results are quite specific to network systems - those driven by a graph Laplacian, it can be shown that their results derive from rather deep geometric notions that we exploit for greater generality, see in particular our own Theorems 4.17, 4.27 and 4.32.

Briefly, the novelty in their result lies in that they are able to give such a formula even in the case where the matrix driving the system is not stable and has a 0 eigenvalue. Typically, this degeneracy of the matrix driving the system leads to a certain integral used for the Lyapunov analysis not converging. In their work, the authors find a way around this by augmenting this integral, known originally as the controllability Gramian. They then show that this augmented integral satisfies a Lyapunov equation and is in fact the unique solution satisfying a certain constraint. Using this, they give a method for computing the $\mathcal{H}_2$-norm of the error system which arises when performing a certain model reduction technique on a system driven by a certain matrix.

Our work in this thesis consists of showing that their augmentation method holds not only in their particular setting but also in the much broader setting where the matrix driving the system is replaced by the infinitesimal generator of a C_0-semigroup and model reductions which satisfy a certain invariance criterion. We generalize their analysis concerning the augmented Gramian from the case where the kernel of the driving matrix is of dimension 1 as is the case for any Laplacian matrix, to the case where the kernel may even be infinite-dimensional. To do this, we identify their method of augmenting the Gramian which is done in a coordinate dependent fashion with a geometric procedure identifying the convergence operator with which they augment their Gramian with a projection onto the kernel of the driving operator. This allows us to show that the augmented Gramian again solves a Lyapunov equation and in the case where the driving operator is self-adjoint it is actually the unique solution invariant under the projection onto the kernel of the driving operator.

The main motivation behind wanting to extend this analysis is that many systems

[1] This is a matrix which describes the connectivity structure of a (weighted) graph, see [Chu97].

driven by partial differential equations considered both in physics and elsewhere have equilibria dependent on the initial condition and so cannot be considered exponentially stable, as this presumes a unique equilibrium, when treated on their entire domain of definition. One such example is the 1-dimensional heat equation with Neumann boundary conditions - a problem we treat extensively as an example in what follows. Moreover, model order reduction of such systems automatically becomes pertinent when one considers any numerical approach as these are restricted to treat finite-dimensional systems. In particular, this means that having a formula for the error between the actual model and the reduced order model is useful if one wishes to obtain accurate numerical results. It is for this reason that we are interested in generalizing the trace formula for the $\mathcal{H}_2$-norm to a more general class of systems which cover the partial differential equation case.

As is the case in any mathematical work, one needs to make a judgement call on assumed background and provided background. Moreover, as this thesis very much lies in the intersection between Linear Systems Theory and Analysis, distinction has to be made twice. As to linear systems and control, background knowledge will not, strictly speaking, be necessary, since we will prove key results even in the infinite-dimensional setting considered here. Nevertheless, it is useful to have background knowledge of most standard results in the finite dimensional case to provide context and understand the significance of the results obtained here, roughly corresponding to [Bro15] and [GL12], especially in regards to state space methods. Since many systems are described by ordinary or partial differential equations, knowledge corresponding to first few chapters of [Car67] (or [Tes12] for a reference in English) and [Eva98] respectively will also be useful and more or less necessary to understand what follows. With respect to analysis, our choice has been made as to reflect roughly the crossing from basic to more advanced topics in analysis. Thus, we will assume knowledge of measure-theoretic integration theory and the basics of functional analysis, such as basic Hilbert space theory and the normed linear space versions of the Uniform Boundedness Principle or Banach-Steinhaus Theorem, the Open Mapping Theorem and the Closed Graph Theorem. The main references used here for these results are [Fri70], [Fol13] and [Lue97].

This thesis is organized as follows: Section 2 deals with the theory of C_0-semigroups on Hilbert spaces and mainly states and explains theorems found in [CZ12], [Eva98] and [Kat13]. This is the main tool we use to translate the results found in [CS16] and [CKS17] to an infinite-dimensional setting. As the theory rests heavily on somewhat advanced topics in functional analysis, the reader not familiar with these is referred to the appendix for a treatment of unbounded operators, elements of their spectral theory and other technical results such as integration of operator-valued functions. This necessity is much due to the fact that a critical result in the theory of C_0-semigroups is the celebrated Hille-Yosida Theorem which in turn relies heavily on the resolvent formalism of unbounded operators but there also other reasons, including that most differential operators are unbounded on their natural domains. In Section 3, we discuss the extension of systems theory to infinite dimensions. The main reference for this section is [CZ12]. Having discussed these topics, we devote Section 4 mainly to our own work extending the results of Cheng, Kawano and Scherpen and essentially all theorems found here except for Theorem 4.4 are original. Section 5 provides a conclusion to the thesis and also discusses some potential extensions.

Before we proceed we shall make a few remarks on notation and other conventions. In what follows X will invariably be a separable Hilbert space unless otherwise stated and indeed in our own results all Hilbert spaces are assumed separable. Most often we think of X as the state space for a control system and will thus be the Hilbert space $L^2(\Omega)$ for some subset $\Omega \subseteq \mathbb{R}^n$. We have thus reserved X for the state space, whose elements we often denote x. We will want to express position in the underlying space, for instance, when writing out (partial) derivative operators. If $X = L^2(\Omega)$ or similar, then $p \in \Omega$ will denote the spatial variable. Thus, the derivative operator with respect to the spatial variable is written $\frac{d}{dp}$ (sometimes also $\frac{\partial}{\partial p}$) and the derivative of an element x in the state space is written x_p. The time derivative is often written $\dot{x} := \frac{dx}{dt}$. Further, (ϕ_n) typically denotes a sequence, possibly infinite, consisting of elements ϕ_n, often eigenvectors of a linear operator corresponding to another sequence of eigenvalues (λ_n). We should also mention before proceeding that when we write $\int$ we mean an appropriate integral, taken either in the sense of Lebesque, Bochner or Pettis, as we often find ourselves in the situation where we wish to integrate an operator. In the main text we will use these without much mention of the technical difficulties involved with more general integration theory - the key point is that many of the familiar Lebesque-integration theorems still hold for the more general class of integrals. We do however provide a brief discussion of this and further references in Appendix A.2.

2 C_0-Semigroup Theory

The fundamental notion to generalize linear systems theory to partial differential equations is that of C_0-semigroups. This also allows for generalization to delay differential equations, but here we focus entirely on the partial differential equation case.

Definition 2.1. A family $S(t)$ of bounded linear operators is called a C_0 semigroup on a Hilbert space X if

1. $S(0) = I$.
2. $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$.
3. For all $x \in X$, $\|S(t)x - x\| \to 0$ as $t \to 0$.

The first two properties above are the semigroup axioms, whereas the third is referred to as strong continuity of the semigroup. If instead of the third axiom, a semigroup $S(t)$ satisfies

$$\lim_{t \to 0^+} \|S(t) - I\| = 0$$

then it is said to be uniformly continuous, that is, the converge criterion 3. takes place in the $\mathfrak{B}(X)$ topology instead of the strong topology.

An evolution of a dynamical system is often given in incremental form rather than explicitly stating the entire trajectory (or the semigroup $S(t)$). To this idea corresponds the infinitesimal generator of a semigroup.

Definition 2.2. We say that A is the infinitesimal generator of a C_0-semigroup $S(t)$ on the space X, if for each $x \in D(A)$

$$Ax = \lim_{t \to 0^+} \frac{S(t)x - x}{t}. \tag{1}$$

We now return to the second derivative operator for an example.

Example 2.3. Consider again a simple physical model of a heated bar on $[0, 1]$

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial p^2}$$

with insulated boundary points $\frac{\partial x}{\partial p}(0,t) = \frac{\partial x}{\partial p}(1,t) = 0$ and initial distribution of heat $x(p, 0) = x_0(p)$. This can be recast as a Hilbert space ODE, $\dot{x} = Ax$ on $L^2[0, 1]$ if we define $A = \frac{d^2}{dp^2}$, and set

$$D(A) = \{x \in L^2[0,1] \mid x, x_p \in AC([0,1]), x_{pp} \in L^2[0,1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\}.$$

We will later prove that the semigroup associated to A is

$$S(t)x = \int_0^1 x(q)dq + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t} \cos(n\pi p) \int_0^1 \cos(n\pi q)x(q)dq.$$

△

Remark 2.4. Consider $\dot{x} = Ax$ and compare with the heat equation. The abstract differential equation comes in the form $\frac{dx}{dt} = Ax$ whereas the original partial differential equation often is of the form $\frac{\partial x}{\partial t} = Ax$. This is not a problem, since the operator $\frac{d}{dt}$ is the differentiation operator in the Hilbert space $L^2([0,\infty);\Omega)$ whereas $\frac{\partial}{\partial t}$ is the partial differentiation operator on $[0,\infty)\times\Omega$. Thus $\dot{x} = \frac{dx}{dt}$ is the $\|\cdot\|_X$-limit of

$$\frac{x(t+s)-x(t)}{s} = \frac{x(t+s,\cdot)-x(t,\cdot)}{s}$$

as s tends to 0. We recognize this also as the difference quotient for $\frac{\partial x}{\partial t}$ and so both differentiations are the limit of the same object and since the $\|\cdot\|_X$-topology is stronger than the pointwise topology, we have that if the Hilbert space derivative exists the abstract differential equation agrees with the original PDE.

C_0-semigroups and their infinitesimal generator are intimately connected with dynamical systems. Indeed, if one poses an equation of the form $\dot{x} = Ax$ with initial condition x_0 where A is the infinitesimal generator of a C_0-semigroup $S(t)$, then $S(t)x_0$ solves the equation. This is made precise in the following lemma.

Lemma 2.5. *Let $S(t)$ be a strongly continuous semigroup on a Hilbert space X with infinitesimal generator A. Then for all $x \in D(A)$*

$$\frac{dS(t)x}{dt} = AS(t)x = S(t)Ax.$$

Proof. Write

$$\lim_{s\to 0^+}\frac{S(t+s)x-S(t)x}{s} = S(t)\lim_{s\to 0^+}\frac{S(s)x-Ix}{s} = S(t)Ax$$

by definition of the infinitesimal generator. Since $S(t)$ commutes with $S(s)$ and I, one also obtains $S(t)Ax = AS(t)x$. □

It is often useful to have an integral formulation of the above result. To this end, take $x' \in X$ and $x \in D(A)$ as above. We then have

$$\langle x', S(t)x - x\rangle = \int_0^t \frac{d}{dt}\langle x', S(s)xds = \int_0^t \langle x', S(s)Ax\rangle ds.$$

By the arbitrariness of x', it follows that on $D(A)$ we have the identity

$$S(t)x = x + \int_0^t S(s)Axds.$$

Applying this to the kernel of A, we obtain the following corollary.

Corollary 2.6. *Suppose that $S(t)$ is a C_0-semigroup on X generated by A. Then for every $x \in \ker A$ and every $t \geq 0$, $S(t)x = x$.*

If one instead considers what this means for every possible initial condition $x \in X$, in the finite-dimensional setting, the lemma is analogous to saying that $S(t)$ is the fundamental solution of the differential equation $\dot{x} = Ax$. In fact, if A is a bounded operator, we obtain the following result familiar from linear time invariant systems.

Proposition 2.7. *Suppose that X is a Hilbert space and $A : X \to X$ is bounded. Then A generates the C_0-semigroup*

$$S(t) = e^{At} := \sum_{k=0}^{\infty} \frac{(At)^k}{k!}$$

Proof. The series converges, since its partial sums form a Cauchy sequence in $\mathfrak{B}(X)$, which is complete since X is a Hilbert space. For $n > m$ we have

$$\sum_{k=0}^{n} \frac{(At)^k}{n!} - \sum_{k=0}^{m} \frac{(At)^k}{k!} = \sum_{k=m+1}^{n} \frac{(At)^k}{k!}$$

which tends to 0 in norm as $\min(n, m) \to \infty$, by boundedness of A.

The identity property,, $S(0) = I$, holds simply by evaluating the partial sums at $t = 0$ and appealing to the norm convergence above. Next, we verify the semigroup property 2.

$$\begin{aligned} S(t)S(s) &= \sum_{k=0}^{\infty} \frac{A^k(t)^k}{k!} \sum_{l=0}^{\infty} \frac{A^l(s)^l}{l!} = \{j = k + l\} = \sum_{k=0}^{\infty}\sum_{j=k}^{\infty} \frac{A^k(t)^k}{k!} \frac{A^{j-k}(s)^{j-k}}{(j-k)!} \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{j} \frac{A^k(t)^k}{k!} \frac{A^{j-k}(s)^{j-k}}{(j-k)!} = \sum_{j=0}^{\infty} \frac{A^j}{j!} \sum_{k=0}^{j} \binom{j}{k} t^k s^{j-k} \\ &= \sum_{j=0}^{\infty} \frac{A^j(t+s)^j}{j!} = S(t+s). \end{aligned}$$

Finally, we verify the strong continuity at 0, we have for $x \in X$ that

$$\left\| \sum_{k=0}^{\infty} \frac{(At)^k}{k!} x - x \right\| = \left\| \sum_{k=1}^{\infty} \frac{(At)^k}{k!} x \right\| \leq \sum_{k=1}^{\infty} \frac{\|A\|^k t^k}{k!} \|x\| = (e^{\|A\|t} - 1)|\|x\|$$

which converges to 0 as $t \to 0$, yielding the result. □

This discussion has often alluded to that C_0-semigroups are very similar to the exponential semigroups e^{At}. In general, these concepts are not exactly the same. However, every C_0-semigroup has an exponential bound.

Proposition 2.8. *If A is the infinitesimal generator of a C_0-semigroup $S(t)$, the limit*

$$\omega_0(A) = \lim_{t\to\infty} \left(\frac{1}{t} \ln \|S(t)\| \right) = \inf_{t>0} \left(\frac{1}{t} \ln \|S(t)\| \right)$$

exists and for all $\omega > w_0$ there exists a constant M such that $\|S(t)\| \leq Me^{\omega t}$ for all $t \geq 0$.

Proof. Before giving the explicit bound on $S(t)$ only depending on t, we begin by proving that there is an M such that $\|S(t)\| \leq M$ for every $T > 0$ and $t \in [0, T]$. Suppose to arrive at a contradiction that there does not exist any such $T > 0$. Then there exists a sequence $t_n \to 0$ such that $\|S(t_n)\| \geq n$. This contradicts the conclusion of the Banach Steinhaus Theorem, i.e., $S(t_n)$ cannot be uniformly bounded, so the hypotheses of that theorem cannot hold. Therefore, there must be an $x \in X$ such that $(\|S(t_n)x\|)$ is an unbounded sequence, however, this is in contradiction to the strong continuity at 0 of $S(t)$. Thus there exists at least one such $T > 0$. For any other $t > T$ we have with $t = mT + r$, $r \in [0, T]$,

$$\|S(t)\| \leq \|S(T)\|^m \|S(r)\| \leq M^{1+m} \leq M^{1+t/T}.$$

We now characterize this bound in more detail. To this end, let $t_0 > 0$ and set $M = \sup_{t \in [0,t_0]} \|S(t)\|$ which is finite by the above argument. Consider now $t \geq nt_0$ so that $t = nt_0 + (t - nt_0)$. Then

$$\begin{aligned} \frac{1}{t} \ln \|S(t)\| = \frac{1}{t} \ln \|S(t_0)^n S(t - nt_0)\| &\leq \frac{n \log \|S(t_0)\| + \ln M}{t} \\ &= \frac{n \log \|S(t_0)\| + \ln M}{nt_0 + (t - nt_0)} \end{aligned}$$

In particular we have

$$\limsup_{t \to \infty} \frac{1}{t} \ln \|S(t)\| \leq \frac{1}{t_0} \ln \|S(t_0)\|$$

for arbitrary $t_0 > 0$. Hence

$$\limsup_{t \to \infty} \frac{1}{t} \ln \|S(t)\| \leq \inf_{t > 0} \frac{1}{t} \ln \|S(t)\| \leq \liminf_{t \to \infty} \frac{1}{t} \ln \|S(t)\|$$

so that we must have equality throughout. We denote this quantity by ω_0.

To complete the characterization of the bound, suppose that $\omega > \omega_0$ above. By above we can find t_0 such that if $t \geq t_0$ then

$$\frac{1}{t} \ln \|S(t)\| < \omega$$

wherefore $\|S(t)\| \leq e^{\omega t}$. This means that, for these $t \geq t_0$,

$$\|S(t)\| \leq e^{\omega t}.$$

However, we know that for $t \leq t_0$ we have

$$\|S(t)\| \leq M$$

for some $M > 0$. Thus on $[0, \infty)$ we have

$$\|S(t)\| \leq M e^{\omega t}.$$

□

If there exists an exponential bound which is decaying, i.e. $\omega < 0$, the semigroup is said to be exponentially stable.

Definition 2.9. A C_0-semigroup $S(t)$ is said to be exponentially stable if there exist $M, \mu > 0$ such that for all $t \geq 0$

$$\|S(t)\| \leq Me^{-\mu t}.$$

Remark 2.10. One may wonder if, as in the matrix case, exponential stability is equivalent to $\Re\lambda < 0$ for all eigenvalues λ of A. In the Hilbert space case we generally only have the inequality

$$\sup(\Re\lambda \mid \lambda \in \sigma(A)) \leq w_0(A)$$

but not equality. For a counter-example see [CZ12], Example 5.1.4.

We give another example, which will be useful in Section 3, below.

Example 2.11. Suppose that S_1, S_2 are C_0-semigroups on Hilbert spaces X_1, X_2 respectively, on which they have generators A_1, A_2. We can construct a new semigroup S on $X = X_1 \oplus X_2$, $\oplus$ being the direct sum and with inner product on X defined by $\langle x, y\rangle = \langle x_1, y_1\rangle_1 + \langle x_2, y_2\rangle_2$ where $\langle \cdot, \cdot\rangle_i, i = 1, 2$ is the inner product of X_i and where $x, y \in X$ which can be written

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

with $x_1, y_1 \in X_1$ and $x_2, y_2 \in X_2$. Now, for $x \in X$

$$S(t)x = \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} S_1(t)x_1 \\ S_2(t)x_2 \end{bmatrix}.$$

Since the matrix containing the semigroups is diagonal, there are no interaction terms, and the semigroup properties follow immediately from those of S_1, S_2 and similarly one sees that

$$\lim_{t\to 0} \frac{1}{t}(S(t)x - x) = \lim_{t\to 0} \begin{bmatrix} \frac{S_1(t)x_1 - x_1}{t} \\ \frac{S_2(t)x_2 - x_2}{t} \end{bmatrix} = \begin{bmatrix} A_1 x_1 \\ A_2 x_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

It is also interesting to note that if both S_1 and S_2 are exponentially stable then so is the semigroup S defined on the direct sum X. Indeed, if $\|S_i(t)\| \leq M_i e^{-\mu_i t}, i = 1, 2$ then

$$\|S(t)\| = \|S_1(t)\| + \|S_2(t)\| \leq M_1 e^{-\mu_1 t} + M_2 e^{-\mu_2 t} \leq 2\max(M_1, M_2) e^{-\min(\mu_1,\mu_2)t}.$$

The applied use from this example stems from the fact that we can view the new Hilbert space $X = X_1 \oplus X_2$ with semigroup $S(t)$ due to its orthogonal nature as containing all the information about the original semigroups S_1, S_2. Indeed, X, S can be viewed as the internal structure of what one in control theory would call a paralell connection of systems. This idea will be developed more later. $\triangle$

Before proceeding to study generators via their Laplace transforms, we prove a small technical lemma.

Lemma 2.12. *Suppose that A is the the infinitesimal generator of a C_0-semigroup, $S(t)$, on a Hilbert space X then A is closed and $D(A)$ is dense in X. Moreover, $\int_0^t S(s)xds \in D(A)$ for all $x \in X$.*

Proof. Consider

$$\begin{aligned}\frac{S(t)-I}{s}\int_0^t S(u)xdu &= \frac{1}{s}\int_0^t S(u+s)xdu - \frac{1}{s}\int_0^t S(u)xdu \\ &= \frac{1}{s}\int_0^s [S(t+u)-S(u)]xdu \\ &= \frac{1}{s}\int_0^s S(u)[S(t)-I]xdu.\end{aligned}$$

Sending $s \to 0$, we obtain

$$A\int_0^t S(u)xdu = \lim_{s\to 0}\frac{S(t)-I}{s}\int_0^t S(u)xdu = [S(t)-I]x$$

and in particular, for any $x \in X$ and $t > 0$

$$\int_0^t S(u)xdu \in D(A).$$

This means that any point $x \in X$ can be written

$$x = \lim_{t\to 0}\frac{1}{t}\int_0^t S(u)xdu.$$

That is, as the limit of a sequence of points entirely in the domain of A, thus verifying the density of $D(A)$ in X.

To prove that A is a closed operator, we take a sequence $(x_n) \subset D(A)$ converging to x and show that Ax_n converges to Ax. Observe

$$\frac{S(t)x-x}{t} = \lim_{n\to\infty}\frac{S(t)x_n - x_n}{t} = \lim_{n\to\infty}\frac{1}{t}\int_0^t S(s)Ax_nds = \frac{1}{t}\int_0^t S(s)Axds$$

by dominated convergence. Taking $t \to 0$ now yields closure of A. □

This implies that the spectral theory for closed densely defined operators is applicable to C_0-semigroups. We proceed along these lines in the next section.

2.1 The Resolvent and the Theorem of Hille and Yosida

The resolvent operator of the infinitesimal generator of semigroup is a very important object. Indeed, it is the Laplace Transform of the semigroup, as we shall prove below.

Lemma 2.13. *Suppose that $S(t)$ is a C_0-semigroup generated by A and growth bound ω_0. If λ, ω are such that $\Re(\lambda) > \omega > \omega_0$ then $\lambda \in \rho(A)$ and for $x \in X$:*

1. *$R(\lambda; A)x = \int_0^\infty e^{-\lambda t} S(t)x dt$ and $\|R(\lambda; A)\| \leq \frac{M}{\Re(\lambda)-\omega}$.*
2. *For $\alpha \in \mathbb{R}$, one has $\lim_{\alpha\to\infty} \alpha R(\alpha, A)x = x$.*

Proof. 1. Define for $\Re\lambda > \omega$ the family of operators

$$R_\lambda x = \int_0^\infty e^{-\lambda t} S(t)x dt.$$

By the growth bound, we get

$$\|R_\lambda\| \leq M \int_0^\infty e^{-(\Re\lambda-\omega)t} dt = \frac{M}{\Re\lambda - \omega}.$$

We will now prove that R_λ is both the left and right inverse of $(\lambda I - A)$. First, for any $x \in D(A)$, we have that

$$\begin{aligned}\frac{S(s)-I}{s} R_\lambda x &= \frac{1}{s}\int_0^\infty e^{-\lambda t}[S(t+s) - S(t)]x dt \\ &= \frac{1}{s}\left[e^{\lambda s}\int_0^\infty e^{-\lambda u} S(u)x du - e^{\lambda s}\int_0^s e^{\lambda u} S(u)x du - \int_0^\infty e^{-\lambda u} S(u)x du\right] \\ &= \frac{e^{\lambda s}-1}{s}\int_0^\infty e^{-\lambda t} S(t)x dt - \frac{e^{\lambda s}}{s}\int_0^s e^{-\lambda t} S(t)x dt.\end{aligned}$$

Taking the limit $s \to 0^+$ by using the Lebesque Differentiation Theorem we obtain

$$R_\lambda A x = A R_\lambda x = \lambda R_\lambda x - x.$$

In particular

$$R_\lambda(\lambda I - A)x = x = (\lambda I - A)R_\lambda x.$$

2. Fix $x \in X$. The domain of A is dense in X and so we can select $x' \in D(A)$ with $\|x - x'\| < \varepsilon$ and moreover by the first point we can choose, for any $\varepsilon > 0$ an α_0 such that for all $\alpha > \alpha_0$ we have $\|R_\alpha\| \leq \varepsilon$ with $\frac{\alpha}{\alpha-\omega} \leq 2$.

   Whence

$$\begin{aligned}\|\alpha R_\alpha x - x\| &= \|\alpha R_\alpha x - \alpha R_\alpha x' + \alpha R_\alpha x' - x' + x' - x\| \\ &\leq \|\alpha R_\alpha x - \alpha R_\alpha x'\| + \|(\alpha + A - A)R_\alpha x' - x'\| + \|x' - x\| \\ &\leq \frac{\alpha M}{\alpha - \omega}\|x - x'\| + \|\alpha R_\alpha A x'\| + \|x - x'\| \\ &\leq (2M + 2)\varepsilon.\end{aligned}$$

Since this holds for any $\varepsilon > 0$, we have the desired equality.

□

We are now prepared to prove the seminal Hille-Yosida Theorem.

Theorem 2.14. *A closed densely-defined operator, A, on a Hilbert space X, is the infinitesimal generator of a C_0-semigroup on X if and only there exist $M, \omega \in \mathbb{R}$ such that all real $\alpha > \omega$ are in the resolvent set of A and satisfy*

$$\|R(\alpha; A)^r\| \leq \frac{M}{(\alpha - \omega)^r}$$

for all $r \geq 1$.

Proof of the forward direction. Suppose that A is the infinitesimal generator of the C_0-semigroup $S(t)$ on X. Observe that we may write, by Lemma A.13,

$$R(\alpha; A)^r = \frac{R^{(r-1)}(\alpha; A)}{(-1)^{r-1}(r-1)!}$$

where $R^{(r-1)}$ is the derivative with respect to the parameter α of order $r-1$. Since the resolvent is available as the Laplace transform of the semigroup via Lemma 2.13, we know that α is in the resolvent set whenever $\alpha > \omega > \omega_0$ and we may write for $x \in X$

$$R(\alpha; A)x = \int_0^\infty e^{-\alpha t} S(t) x dt$$

Thus,

$$R^{(r-1)}(\alpha; A)x = \int_0^\infty (-t)^{r-1} e^{-\alpha t} S(t) x dt.$$

Since $\omega > \omega_0(A)$, the growth bound, we have that

$$\|R^{(r-1)}(\alpha; A)\| \leq M \int_0^\infty (-t)^{r-1} e^{-(\alpha-\omega)t} dt = M(r-1)!(\alpha - \omega)^{-r}.$$

Comparing this with derivative expression for the resolvent, we obtain

$$\|R(\alpha; A)^r\| \leq \frac{M}{(\alpha - \omega)^r},\ \alpha > \omega > \omega_0.$$

Proof of reverse direction. Suppose that A is a linear operator satisfying

$$\|R(\alpha; A)^r\| \leq \frac{M}{(\alpha - \omega)^r},\ \alpha > \omega > \omega_0,$$

for all $r \geq 1$ and some ω_0 and all $\alpha > \omega > \omega_0$. The idea is now to approximate A by a sequence of bounded operators. To this end, let

$$A_\alpha = \alpha^2 R(\alpha, A) - \alpha I.$$

By the bounded nature of the resolvent, this too is a bounded operator. Thus by Proposition 2.7, we have that each A_α generates a semigroup $S_\alpha(t)$ via

$$S_\alpha(t) = \sum_{k=0}^{\infty} \frac{A_\alpha^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{(\alpha^2 t)^k}{k!} R(\alpha; A)^k.$$

We now partition the proof into three parts. First, we show that the strong limit of A_α as $\alpha \to \infty$ exists and equals A. Second, we prove also that the strong limit of S_α exists. Third, we show that limit indeed constitutes a semigroup which in particular has A as its generator:

1. Note that

$$A_\alpha x = (\alpha^2 R(\alpha, A) - \alpha I)x = \alpha(\alpha R(\alpha, A) - I)x = \alpha R(\alpha; A)Ax$$

   so the convergence result is just a restatement of the second point of Lemma 2.13.

2. Observe now that by assumption on the resolvent of A and contruction of S_α that

$$\|S_\alpha(t)\| \leq e^{-\alpha t} \sum_{k=0}^{\infty} \frac{(\alpha^2 t)^k}{k!} \frac{M}{(\alpha - \omega)^k} = M e^{\frac{\alpha\omega}{\alpha-\omega} t}.$$

   In particular, for $\alpha > 2|\omega|$ sufficiently large, S_α is uniformly bounded on all intervals $[0, t], t > 0$ by $Me^{2|\omega|t}$.

   Next, since the resolvent commutes for different $\alpha, \mu \in \rho(A)$, we also see that $A_\alpha A_\mu = A_\mu A_\alpha$ and $A_\alpha T_\mu = T_\mu A\alpha$. Using this, we find that for any $x \in D(A)$, we have that

$$\begin{aligned}
\|S_\alpha(t)x - S_\mu(t)x\| &= \Big\| \int_0^t \frac{d}{ds}(S_\mu(t-s)S_\alpha(s))xds \Big\| \\
&= \Big\| \int_0^t (S_\mu(t-s)(A_\alpha - A_\mu)S_\alpha(s))xds \Big\| \\
&= \Big\| \int_0^t (S_\mu(t-s)S_\alpha(s))(A_\alpha - A_\mu)xds \Big\| \\
&\leq \Big\| \int_0^t M^2 e^{2|\omega|t}(A_\alpha - A_\mu)xds \Big\| \\
&= M^2 t e^{2|\omega|t} \|(A_\alpha - A_\mu)x\|.
\end{aligned}$$

   By the first point, this forms, separately for each t, a Cauchy sequence on $D(A)$. By the density of $D(A)$ in X and the uniformity of S_α on each compact interval we conclude that this convergence holds on all of X.

3. Now, the semigroup properties of the limit $S(t)$ of $S_\alpha(t)$ follows now immediately by point 2. As for the generator. Observe that

$$\|S_\alpha(t)A_\alpha x - S(t)Ax\| \leq \|S_\alpha(t)\|\|A_\alpha x - Ax\| + \|S_\alpha Ax - S(t)Ax\|$$

yielding strong convergence of $S_\alpha Ax \to SAx$ for all $x \in D(A)$, and this occurs uniformly on compact time intervals. Applying Lebesque's Dominated Convergence Theorem gives us

$$\lim_{\alpha\to\infty} S_\alpha x - x = \lim_{\alpha\to\infty} \int_0^t S_\alpha(s)A_\alpha x ds = \int_0^t S(s)Axds = S(t)x - x.$$

Dividing by t and taking limits as $t \to \infty$, this is enough to conclude that the generator, A' of $S(t)$ is an extension of A.

However, if $\alpha > \omega$ we have

$$(\alpha I - A)D(A) = X \text{ and } (\alpha I - A')D(A')$$

and by above we already have $AD(A) = AD(A')$ which gives

$$(\alpha I - A')D(A) = (\alpha I - A')D(A').$$

Hence $D(A) = D(A')$ by injectivity of the resolvent and A indeed is the generator of $S(t)$.

1., 2. and 3. together finish the proof. □

As a first application of the Hille-Yosida Theorem, we prove that C_0-semigroups are in a sense closed under taking adjoints. This will be very useful when dealing with inner product computations, allowing us to go back and forth between primal and dual computations.

Proposition 2.15. *If $S(t)$ is a C_0-semigroup with infinitesimal generator A on a Hilbert space X, then $S^*(t) = [S(t)]^*$ is also a C_0-semigroup on X but with infinitesimal generator A^*.*

Proof. By Lemma A.10 it is clear that $R(\alpha, A^*) = R(\alpha, A)^*$ for real α and since these have the same norm as $R(\alpha, A)$, we may conclude by Hille-Yosida that A^* generates a C_0-semigroup, say $T(t)$. To see that $T(t) = [S(t)]^*$, write using the Laplace characterization of the resolvent

$$\begin{aligned}\langle x', \int_0^\infty e^{-\lambda t}T(t)xdt\rangle &= \langle x', R(\lambda, A^*)x\rangle \\ &= \langle R(\lambda, A)x', x\rangle \\ &= \langle \int_0^\infty e^{-\lambda t}S(t)x'dt, x\rangle \\ &= \langle x', \int_0^\infty e^{-\lambda t}[S(t)]^*xdt\rangle.\end{aligned}$$

This holds for all $x, x' \in X$ and for all λ with $\Re\lambda > \omega$. Thus by uniqueness of the Laplace transform we conclude that $T(t) = S^*(t)$. □

2.2 Riesz Spectral Operators

In the theory of finite dimensional control, the singular value decomposition is a tool of great importance and is so in particular in model reduction where it, for instance, is used to obtain reduction by balanced truncation, see [GL12] Chapter 9. Here we will consider a class of operators which admit a decomposition which roughly speaking mirrors the SVD in finite dimensions. That is, we want to consider operators A which satisfy for $x \in D(A)$

$$Ax = \sum_{i=1}^{\infty} \lambda_n \langle x, \psi_n \rangle \phi_n$$

where λ_n are the eigenvalues of A, ϕ_n its eigenvectors and ψ_n the eigenvectors of A^*. This will provide a rich trove of examples for our own work in Section 4, and includes the second derivative operator as an example.

To make the theory precise, we first need the notion of a Riesz Basis.

Definition 2.16. A sequence of vectors (ϕ_n) in a Hilbert space X forms a Riesz Basis for X if $\overline{\text{span}}(\phi_n) = X$ and there exist constants m, M such that for any $N \in \mathbb{N}$ scalars $\alpha_n, n = 1, .., N$ the followings holds

$$m \sum_{n=1}^{N} |\alpha_n|^2 \leq \left\| \sum_{n=1}^{N} \alpha_n \phi_n \right\|^2 \leq M \sum_{n=1}^{N} |\alpha_n|^2. \tag{2}$$

As one might expect from the discussion above, the key property is that the eigenvectors of an operator form a Riesz Basis. Of course, any orthogonal basis (ϕ_n) is a Riesz basis since we then have equality in (2) for $m = M = 1$ using the orthogonality of the ϕ_n.

Lemma 2.17. *Suppose that A is a closed linear operator on a Hilbert space X and that A has simple eigenvalues (λ_n) with eigenvectors (ϕ_n) forming a Riesz Basis. Then*

1. *The eigenvectors (ψ_n) corresponding to eigenvalues $(\bar{\lambda}_n)$ of A^* can be chosen such that $\langle \phi_n, \psi_m \rangle = \delta_{n,m}$. That is, (ϕ_n, ψ_n) are biorthogonal.*

2. *Every $x \in X$ can be represented uniquely as*

$$x = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n$$

and there exist $m, M > 0$ such that

$$m \sum_{n=1}^{\infty} |\langle x, \psi_n \rangle|^2 \leq \|z\|^2 \leq M \sum_{n=1}^{\infty} |\langle x, \psi_n \rangle|^2.$$

Proof. 1. Write

$$\lambda_n\langle \phi_n, \psi_m\rangle = \langle A\phi_n, \psi_m\rangle = \langle \phi_n, A^*\psi_m\rangle = \langle \phi_n, \bar{\lambda}\psi_m\rangle = \lambda_m\langle \phi_n, \psi_m\rangle$$

and since the eigenvalues are nonrepeated, this implies $\langle \phi_n, \psi_m\rangle = \alpha_m\delta_{n,m}$ for some $\alpha_m \in \mathbb{C}$ and we obtain the result by scaling ψ_m accordingly, i.e., dividing by $\bar{\alpha}_m$.

2. Since $\overline{\operatorname{span}}\,\phi_n = X$, we may write an $x \in X$ we may choose a sequence $x^p \to x$ of the form

$$x^p = \sum_{k=1}^{p} \alpha_k^p \phi_k.$$

Moreover, biorthogonality gives that

$$\alpha_j^p = \langle x^p, \psi_j\rangle \to \langle x, \psi_j\rangle$$

as $p \to \infty$.

Next, by the fact that ϕ_n constitutes as Riesz basis, we may write

$$m\sum_{j=1}^{p} |\langle x^p, \psi_j\rangle|^2 = m\sum_{j=1}^{p} |\alpha_j^p|^2 \le \|x^p\|^2 \le M\sum_{j=1}^{p} |\langle x^p, \psi_j\rangle|^2.$$

To obtain the result, we shall need that $(\langle z, \psi_j\rangle) \in l^2$. Write

$$\begin{aligned}\sqrt{\sum_{j=1}^{q} |\langle x, \psi_j\rangle|^2} &\le \sqrt{\sum_{j=1}^{q} |\langle x, \psi_j\rangle - \langle x^p, \psi_j\rangle|^2} + \sqrt{\sum_{j=1}^{q} |\langle x^p, \psi_j\rangle|^2} \\ &\le \sqrt{\sum_{j=1}^{q} |\langle x, \psi_j\rangle - \langle x^p, \psi_j\rangle|^2} + \frac{1}{\sqrt{m}}\|x^p\|.\end{aligned}$$

By convergence of $x^p \to x$ the first term can be made arbitrarily small for each q, and for the same reason, the second term is uniformly bounded, which gives $(\langle z, \psi_j\rangle) \in l^2$. Therefore,

$$x = \lim_{p\to\infty} x^p = \lim_{p\to\infty} \sum_{k=1}^{\infty} \langle x^p, \psi_k\rangle\phi_k = \sum_{k=1}^{\infty} \langle x, \psi_k\rangle\phi_k$$

and the norm estimate for x follows by taking limits of the corresponding estimate for x^p.

□

If we were just concerned with finite-dimensional operators, this lemma would be enough for the SVD-like form in the beginning of the section since the representation in the lemma for $x \in X$ could then just be applied to Ax. In the functional-analytic setting one needs to worry about convergence. Nevertheless, this motivates the following definition.

Definition 2.18. Suppose that A is a closed linear operator on a Hilbert space, X and let λ_n, ϕ_n denote its eigenvalues and eigenvectors. If the λ_n are simple with $(\lambda_n) \subset \mathbb{C}$ totally disconnected and (ϕ_n) forms a Riesz Basis, one calls A a Riesz Spectral Operator.

Observe that these are essentially the hypotheses of Lemma 2.17, with the addition that (λ_n) is totally disconnected. This is a technical condition used in the control literature and is used, for instance, to derive an approximate controllability test for distributed parameter control, see [CZ12]. We include it here only because we do not wish to stray from convention, however, for our considerations it is of no importance.

The theorem below shows that the SVD-like form does hold for Riesz Spectral Operators, and moreover, a similar form holds for the associated semigroup.

Theorem 2.19. *Suppose that A is a Riesz Spectral Operator on a Hilbert space X. If the $(\lambda_n), (\phi_n)$ are the eigenvalues and eigenvectors of A and if (ψ_n) are the biorthogonal eigenvectors of A^*. Then*

1. $\rho(A) = \{\lambda \in \mathbb{C} \mid \inf_n |\lambda - \lambda_n| > 0\}, \sigma(A) = \overline{\{\lambda_n\}}$ *and for $\lambda \in \rho(A)$, $R(\lambda, A)$ has the form*

$$R(\lambda; A) = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle \cdot, \psi_n \rangle \phi_n;$$

2. *The operator A can be written*

$$Ax = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n;$$

for $x \in D(A)$ where $D(A)$ is given explicitly by $D(A) = \{x \in X | \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \psi_n \rangle|^2 < \infty\}$.

3. *A generates a C_0-semigroup if and only if $\sup_n \lambda_n < \infty$ and then the associated semigroup is given by*

$$S(t) = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle \cdot, \psi_n \rangle \phi_n.$$

Proof. 1. Take λ such that $\inf_{\lambda_n \in \sigma_p(A)} |\lambda - \lambda_n| \geq \alpha > 0$. Observe that

$$\Big\| \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle x, \psi_n \rangle \phi_n \Big\|^2 \leq M \sum_{n=1}^{\infty} \frac{1}{|\lambda - \lambda_n|^2} |\langle x, \psi_n \rangle|^2 \leq \frac{M}{m\alpha^2} \|x\|^2$$

using the bounds on the Riesz form for x, showing that the form for the resolvent is bounded. Denote now

$$f_N(\lambda)x = \sum_{n=1}^{N} \frac{1}{\lambda - \lambda_n} \langle x, \psi_n \rangle \phi_n$$

and note in particular that as $N \to \infty$ this is the form we want to show that $R(\lambda; A)$ has. Furthermore, it is easy to see, since f_N acts orthogonally on x that

$$(\lambda I - A) f_N(\lambda)x = \sum_{n=1}^{N} \langle x, \psi_n \rangle \phi_n \to x \text{ as } N \to \infty.$$

Now, since A is closed so that $f_N(\lambda)x$ and $(\lambda I - A) f_N(\lambda)x$ converge in the X-topology. Denoting the desired form of the resolvent by $f_\infty(\lambda)$, we obtain for any $x \in X$

$$(\lambda I - A) f_\infty(\lambda)x = x$$

so it is a right inverse. Let now instead $x \in D(A)$. We may write

$$(\lambda I - A)x = (\lambda I - A) f_\infty(\lambda)(\lambda I - A)x.$$

Wherefore

$$0 = (\lambda I - A)x - (\lambda I - A)x = (\lambda I - A)[x - f_\infty(\lambda)(\lambda I - A)x].$$

This means that $f_\infty(\lambda)$ is both a right and a left inverse (on $D(A)$) proving that indeed $f_\infty(\lambda) = R(\lambda; A)$ and $\lambda \in \rho(A)$. Now, the resolvent set of A is open, so the spectrum is closed, and therefore we also have that any member of the resolvent set satisfies $\inf_{\lambda_n \in \sigma_p(A)} |\lambda - \lambda_n| > 0$ (the reverse inclusion of the characterization of $\rho(A)$).

2. Let $S = \{x \in X | \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \psi_n \rangle|^2 < \infty\}$. We will first show that $S \subseteq D(A)$ and that the expansion for Ax holds on S, the usefulness of this characterization of $D(A)$ being square-summability. Take $x \in S$ and define $x_N = \sum_{n=1}^{N} \langle x, \psi_n \rangle \phi_n$. Then, as $N \to \infty$, we have

$$x_N \to x \text{ and } Ax_N \to \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n$$

in the X-topology. By closedness of A, it follows that $x \in D(A)$ and that indeed

$$Ax = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n.$$

As for the reverse inclusion, take $x \in D(A)$ and write $y = (\lambda I - A)x$ with $\lambda \in \rho(A)$. Thus, by the first bulletin,

$$x = (\lambda I - A)^{-1} y = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle y, \psi_n \rangle \phi_n = \sum_{n=1}^{\infty} \langle x, \psi_n \rangle \phi_n.$$

Therefore $\frac{1}{\lambda-\lambda_n} \langle y, \psi_n \rangle = \langle x, \psi_n \rangle$ and we may compute, using $\mu = \inf |\lambda - \lambda_n|$ that

$$\begin{aligned} \sum_{n=1}^{\infty} |\lambda_n|^2 |\langle x, \psi_n \rangle|^2 &= \sum_{n=1}^{\infty} \Big| \frac{\lambda_n}{\lambda - \lambda_n} \Big|^2 |\langle y, \psi_n \rangle|^2 = \sum_{n=1}^{\infty} \Big| \frac{\lambda}{\lambda - \lambda_n} - 1 \Big|^2 |\langle y, \psi_n \rangle|^2 \\ &\leq \sum_{n=1}^{\infty} \Big| \frac{|\lambda|}{\mu} + 1 \Big|^2 |\langle y, \psi_n \rangle|^2 \leq \sum_{n=1}^{\infty} \Big| \frac{|\lambda|}{\mu} + 1 \Big|^2 \|y\|^2. \end{aligned}$$

That is, $x \in S$ and so $D(A) = S$.

The necessity of $\sup_{n \geq 1} \Re \lambda_n < \infty$ is a consequence of the Hille-Yosida Theorem. Taking such a $\lambda > \omega = \sup_{n \geq 1} \Re \lambda_n < \infty$, we may write

$$(\lambda I - A)^{-1} x = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle x, \psi_n \rangle \phi_n, \text{ and so } (\lambda I - A)^{-r} x = \sum_{n=1}^{\infty} \frac{1}{(\lambda - \lambda_n)^r} \langle x, \psi_n \rangle \phi_n.$$

This means that we may estimate the resolvent as

$$\|R(\lambda; A)^r x\|^2 \leq M \sum_{n=1}^{\infty} \frac{1}{|\lambda - \lambda_n|^{2r}} |\langle x, \psi_n \rangle|^2 \leq \frac{M}{m} \frac{\|x\|^2}{(\Re \lambda - \omega)^{2r}}$$

and so the Theorem of Hille and Yosida gives us that A generates a C_0-semigroup $S(t)$ with $\|S(t)\| \leq \sqrt{M/m} e^{\omega t}$.

As for the characterization of $S(t)$, let (us ever so slightly abuse notation and) write

$$e^{At} x = \sum_{n=1}^{\infty} e^{\lambda_n t} \langle x, \psi_n \rangle \phi_n$$

which is bounded for all $t > 0$. Whenever $\Re \lambda > \omega$ we can take the Laplace transform

$$\int_0^{\infty} e^{-\lambda t} e^{At} x dt = \sum_{n=1}^{\infty} \int_0^{\infty} e^{-(\lambda - \lambda_n)t} \langle x, \psi_n \rangle \phi_n dt = \sum_{n=1}^{\infty} \frac{1}{\lambda - \lambda_n} \langle x, \psi_n \rangle \phi_n = R(\lambda; A)x.$$

We conclude by noting that the Laplace transform is injective and since the Resolvent is the Laplace tranform of the associated semigroup, we actually have $S(t) = e^{At}$. □

To illustrate the strength of this theorem, we show how it easily characterizes the semigroup structure of the heat equation on $[0, 1]$.

Example 2.20. Let us revisit $X = L^2[0,1]$ with $A = \frac{d^2}{dp^2}$ with

$$D(A) = \Big\{x \in L^2[0,1] \Big| x, \frac{dx}{dp} \in AC[0,1], \frac{d^2x}{dp^2} \in L^2[0,1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\Big\}.$$

It was previously shown that the eigenvectors are $v_n(p) = \cos n\pi p, n \geq 0$ and from elementary Fourier analysis, it is well known that this actually constitutes an orthogonal basis for $L^2[0,1]$. It is known from elementary Fourier Analysis (see [Rud06], Chapter 4) that $(1, \sqrt{2}\cos(n\pi p), n \geq 1)$ forms an orthogonal basis and thus in particular it is a Riesz basis and so by Theorem 2.19 it follows that A and its associated semigroup $S(t)$ have representation

$$Ax(\cdot) = \langle x(\cdot), 1\rangle + 2\sum_{n=1}^{\infty} -n^2\pi^2\langle x(\cdot), \cos(n\pi\cdot)\rangle \cos(n\pi\cdot)$$
$$S(t)x = \langle x(\cdot), 1\rangle + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x(\cdot), \cos(n\pi\cdot)\rangle \cos(n\pi\cdot)$$

which confirms the claim in Example 2.3. △

This last example shows how the analysis of a C_0-semigroup is substantially simplified if we may decompose it along its eigenvectors. This decomposition actually has at least two very useful properties, the obvious one being orthogonality. A second property of this decomposition, more subtle and perhaps even more useful, is that the eigenvectors are invariants under both the generator and the semigroup.

Definition 2.21. Let V be a subspace of a Hilbert space X with C_0-semigroup $S(t)$ defined thereon. We say that V is $S(t)$-invariant if for all $t \geq 0$

$$S(t)V \subseteq V.$$

If A is the generator of a C_0-semigroup, we say that a subspace $V \subseteq X$ is A-invariant if

$$A(V \cap D(A)) \subseteq V.$$

If A is allowed to be unbounded, as is typically the case, one can show that A-invariance does not necessarily imply $S(t)$-invariance as we would expect for matrices (or even bounded operators). Since the concepts of A and $S(t)$-invariance are central to our model reduction technique in Section 4 we need to intuitively understand why this is not the case. The idea in terms of the heat equation is roughly speaking that temperature does not change locally from 0 in small time, but globally, we expect the distribution of heat to eventually flatten out and so if there is a mass of heat anywhere, there will eventually be a mass of heat everywhere. We make this explicit by an example below.

Example 2.22. Let us continue with our heat equation example. So, set $X = L^2[0,1]$ and $A = \frac{d^2}{dp^2}$ with

$$D(A) = \Big\{x \in L^2[0,1] \,\Big|\, x, \frac{dx}{dp} \in AC[0,1], \frac{d^2x}{dp^2} \in L^2[0,1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\Big\}.$$

Take now the subspace

$$V = \{x \in C^\infty([0,1]) \mid x = 0 \text{ on } [0,1/4) \cup (3/4,1]\}.$$

Simply differentiating any such $x \in V$ twice shows that $A(V \cap D(A)) \subseteq V$. It is known from elementary calculus that there exists a function which is C^∞ on $[1/4,3/4]$, taking the value 0 on both endpoints and having Lebesque mass 1. Thus, let x be any such function glued together with the 0 function on $[0,1/4)$ and $(3/4,1]$. Observe that this function still has Lebesque mass 1 and lies in $V \cap D(A)$. However, for any $p \in [0,1]$ and any $\varepsilon > 0$,

$$\begin{aligned} \|S(t)x(p) - 1\| &= \left\| \langle x(\cdot), 1\rangle + 2\sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x(\cdot), \cos(n\pi\cdot)\rangle \cos(n\pi p) - 1 \right\| \\ &\leq 2 \left\| \sum_{n=1}^{\infty} e^{-n^2\pi^2 t}\langle x(\cdot), \cos(n\pi\cdot)\rangle \cos(n\pi p) \right\| \leq \varepsilon \end{aligned}$$

if t is made sufficiently large. In particular, $S(t)x \notin V$ and so we have constructed a counterexample for the claim that A-invariance should imply $S(t)$-invariance. △

If we combine Theorem 2.19 with Definition 2.21, we obtain the aformentioned mentioned invariance.

Proposition 2.23. *Let A be a Riesz spectral operator on X with eigenvectors (ϕ_n) and which generates a C_0-semigroup $S(t)$. Then any subset V of X given by*

$$V = \overline{\text{span}}_{n\in\mathcal{I}}(\phi_n), \text{ for } \mathcal{I} \subseteq \mathbb{N}$$

is $S(t)$-invariant and A-invariant.

One can actually prove that these are the only closed $S(t)$-invariant subspaces of X, but this is considerably more difficult. Seeing, as we only need this direction in the sequel, we stop ourselves here.

2.3 Infinite-Dimensional Differential Equations

Lemma 2.5 can be interpreted as saying that a semigroup $S(t)$ is the fundamental solution to the homogenous differential equation

$$\dot{x} = Ax$$

where A is the corresponding infinitesimal generator. The analogy can be extended to inhomogeneous equations of the following form, often referred to as equations of evolution:

$$\begin{cases} \dot{x}(t) & = Ax(t) + f(t), \\ x(0) & = x_0. \end{cases} \tag{3}$$

As is typical in the study of more complicated differential equations, one needs to take care when deciding on what solution concept to use.

Definition 2.24. A continuously differentiable function $t \mapsto x(t)$ is said to be a classical solution of (3) if for every t, $x(t) \in D(A)$ and $x(t)$ satisfies (3).

For classical solutions, the variation of parameters formula holds.

Proposition 2.25. *Assume that $f \in C([0,T];X)$ and that x is a classical solution of (3). Then also $Ax \in C([0,T];X)$ and the solution is given by*

$$x(t) = S(t)x_0 + \int_0^t S(T-s)f(s)ds. \tag{4}$$

Proof. By assumption $\dot{x}$ and f are elements of $C([0,T];X)$, and therefore so is also $Ax = \dot{x} + f$.

To prove the variation of parameters formula, let $t \in [0,T]$ and consider the quantity $S(t-s)x(s)$ on $[0,t)$. Then consider the difference quotient

$$\begin{aligned} \frac{S(t-s-h)x(s+h) - S(t-s)x(s)}{h} &= \frac{S(t-s-h)x(s+h) - S(t-s-h)x(s)}{h} \\ &+ \frac{S(t-s-h)x(s) - S(t-s)x(s)}{h}. \end{aligned}$$

For the first term, observe that since $S(t)$ is uniformly bounded on any compact interval and using the strong continuity of the semigroup gives us that it converges to $S(t-s)\dot{x}$. The last term converges to $-AS(t-s)x(s)$ since $x \in D(A)$. Therefore

$$\begin{aligned} \frac{d}{ds}\Big(S(t-s)x(s)\Big) &= S(t-s)\dot{x}(s) - A(S(t-s)x(s) \\ &= S(t-s)[Ax(s) + f(s)] - S(t-s)Ax(s) \\ &= S(t-s)f(s). \end{aligned}$$

The variation of parameters formula then follows by integrating, since t was fixed. □

We can show that these solutions, albeit under quite restrictive regularity assumptions on f, are unique. Thus, when f is sufficiently nice, the variation of parameters formula constructively gives the solution.

Theorem 2.26. *Let X be a Hilbert space and assume that A is the infinitesimal generator of a C_0-semigroup, $S(t)$, thereon. If $f \in C^1([0,T];X)$ and $x_0 \in D(A)$ then the solution $x(t)$ given by (4) is continuously differentiable on $[0,T]$ and furthermore it is unique in the class of classical solutions.*

Proof. As for uniqueness, if there are two different solutions $x_1(t), x_2(t)$, we consider their difference $\Delta(t) = x_1(t) - x_2(t)$. Clearly, this satisfies $\dot{\Delta}(t) = A\Delta, \Delta(0) = 0$. Now we use the semigroup $S(t)$ and remark that $y(s) = S(t-s)\Delta(s)$ is constant since

$$\dot{y}(t) = \frac{d}{dt}S(t-s)\Delta(s) = 0.$$

Therefore $\Delta(t) = y(t) = 0$.

With regards to existence, we need to show that (4) is an element of $C^1([0,T];X) \cap D(A)$ and that this actually satisfies the differential equation. Now $x(t) = S(t)x_0 + y(t)$ where

$$\begin{aligned} y(t) &= \int_0^t S(t-s)f(s)ds = \int_0^t S(t-s)\left(f(0) + \int_0^s \dot{f}(\tau)d\tau\right)ds \\ &= \int_0^t S(t-s)f(0)ds + \int_0^t \int_\tau^t S(t-s)\dot{f}(\tau)dsd\tau \end{aligned}$$

by Fubini. Since $y(t)$ is representable as integral over the semigroup, it follows by Lemma 2.12 that $y(t)$ is an element of $D(A)$.

To prove that $y(t)$ solves the zero-initial condition problem, write

$$\begin{aligned} Ay(t) &= [S(t) - I]f(0) + \int_0^t [S(t-\tau) - I]\dot{f}(\tau)d\tau \\ &= S(t)f(0) + \int_0^t S(t-\tau)\dot{f}(\tau)d\tau - f(t) \end{aligned}$$

which is allowed since A is closed and since

$$\int_0^t \|A\int_\tau^t S(t-s)\dot{f}(\tau)ds\|d\tau = \int_0^t \|S(t-\tau)\dot{f}(\tau) - f(\tau)\|d\tau < \infty.$$

Therefore

$$\begin{aligned} \frac{dy}{dt}(t) &= S(t)f(0) + \int_0^t S(s)\dot{f}(t-s)d \\ &= S(t)f(0) + \int_0^t S(t-s)\dot{f}(s)ds \\ &= Ay(t) + f(t) \end{aligned}$$

as required, where we have used that for any g, $S * g = g * S$; that is, convolution is commutative.

□

Note that we generally consider A to be an unbounded operator, so requiring f to be smooth is a comparatively strong regularity assumption, and will often not hold in applications. One still wishes to have a solution concept under these circumstances. If the solution is only available in integral form, as motivated by the variation of parameters formula, it instead is called mild.

Definition 2.27. If $f \in L^2([0,T];X)$ and x satisfies

$$x(t) = S(t)x_0 + \int_0^t S(T-s)f(s)ds$$

for all t then x is a mild solution of (3).

Remark 2.28. In fact, one can show that these solutions are equivalent to the weak solutions known from PDE theory, see Chapter 3.1 of [CZ12].

We should also note that if $f(t)$ is of the form $Gx(t)$, where G is a bounded linear operator, one can shown that $A+G$ actually generates a new semigroup which solves the system. This fact is used extensively in infinite-dimensional feedback control, but we shall not need it in the sequel and so shall not take the detour. A detailed treatment of this can be found in chapters 3 and 5 of [CZ12].

3 Distributed Parameter Systems Theory

Just as an ordinary differential equation describes the motion of a single point in space, a partial differential equation describes the motion of an entire manifold in space. For instance, an ODE may describe the evolution of temperature at perhaps a single or several points and the corresponding situation for a PDE is to describe the temperature evolution of the entire space in which these points lie. Another way to look at this situation is to say that the solution of an ordinary differential equation produces a point for each time $t \mapsto f(t)$ and a partial differential equation produces an entire function $t \mapsto f(t, \cdot)$ for each point in time; the state space corresponding to an ordinary differential equation is some manifold Ω, whereas the state space of a partial differential equation corresponds to some set of functions from a manifold. The first situation is typically finite-dimensional, whereas the second is inherently infinite-dimensional.

3.1 Infinite-Dimensional Dynamical Systems

We shall here consider abstractly what is meant by a dynamical system given by a C_0-semigroup.

Definition 3.1. By a dynamical system determined by a C_0-semigroup, $S(t)$, defined on a Hilbert space X we shall mean the set

$$\mathcal{S} = \{x \in X : x = S(t)x_0, x_0 \in X\}.$$

The map $x \mapsto S(t)x$ is called the flow of the dynamical system.

In the sequel, we shall often be concerned with the asymptotic behavior of dynamical systems. If the system settles at a point as time progresses and does not move, we say that such a point is an equilibrium point. One of the weaker notions of an equilibrium is given below.

Definition 3.2. A point $x_e \in X$ is said to be a Lyapunov equilibrium of a dynamical system $\mathcal{S}$ if for every open U of X containing x_e there exist an open subset O of X containing $S(t)x_e$ we have $S(t)O \subseteq U$.

That is, a point x_e is a Lyapunov equilibrium if we cannot distinguish it over time via the topology of X. Intuitively then, as the convergence requirement for exponential stability of a semigroup $S(t)$ occurs in the uniform topology, which is one of the strongest one usually works in, we expect such equilibria to also be equilibria in the sense of Lyapunov.

Proposition 3.3. *If a dynamical system $\mathcal{S}$ is given by an exponentially stable semigroup $S(t)$, then the point $0 \in X$ is a Lyapunov equilibrium.*

Proof. Observe that $S(t)X \subseteq X$ trivially. Let U be any open set of X containing the point $0 \in X$. We need to show that there exists a subset O of X, also containing the origin, with $S(t)O \subseteq U$ for all $t \geq 0$.

To see this, note that since $0 \in U$, and the ε-balls form a basis for the norm topology, U contains at least one of the sets $B(0, \varepsilon), \varepsilon > 0$ and we can simpy select $O = B(0, \varepsilon/M)$ where M is chosen such that $\|S(t)\| \leq Me^{-\mu t}$ for some $\mu > 0$. □

The theory of dynamical systems is extremely rich, and we have here just provided preliminary notions necessary to understand the main point of this thesis, which is of course semistability and in particular its relation to control theory, which is the topic of the next section. Further reading on this topic may be found in [Rob01] and also [Tes12] for the finite-dimensional case.

3.2 Distributed Parameter Control Theory

In this section we describe the Systems Theory necessary for our purposes. The exposition is mainly based on [CZ12] but [BDPDM07] is used as an auxilliary reference. A control system is essentially a dynamical system, as discussed in the last section, with the possibility for steering, or control, of the main variable of interest, the state x, via an input, u. The systems treated here and in the remainder of this thesis are linear and of the form below.

$$\begin{cases} \dot{x} &= Ax + Bu, \ x(0) = x_0 \\ y &= Cx \end{cases} \tag{Σ}$$

Here, the state variable, x, is a member of a separable Hilbert space X and we assume that A is the infinitesimal generator of a C_0-semigroup, $S(t)$, on X. The control, or input, u, is a member of another function space, U, and similarly for the output, y, which a member of a third function space, Y. Further these spaces are connected via the following operators: The input operator, $B \in \mathfrak{B}(U, X)$, and the output operator, $C \in \mathfrak{B}(X, Y)$ act on the control (input) $u \in L^2([0,T]; U)$ and observation (output) $y \in L^2([0,T]; Y)$ to steer the state and produce an output respectively. Typically, X, Y, U are themselves Lebesque-type spaces such as L^2 or the Sobolev space H^2. Systems satisfying the above hypotheses will be denoted $\Sigma(A, B, C)$, $\Sigma(A, B, -)$ or $\Sigma(A, -, B)$ if either the output or input operator is irrelevant or just Σ for short.

Remark 3.4. The convention that Pritchard and Salamon usually apply is to consider a set of Hilbert spaces

$$W \subseteq X \subseteq V$$

with continuous dense injections. Moreover, the more general system allows for outputs of the form $y = Cx + Du$. The reason to introduce these auxiliary spaces is to allow for potentially unbounded operators, $B \in \mathfrak{L}(U, V), D \in \mathfrak{L}(U, Y), C \in \mathfrak{L}(W, Y)$. This adds an additional layer of technical difficulty to the problem which we do not wish to treat here, so we make the simpler assumptions above. The more general situation is treated extensively in [Sal87]. We should note that allowing for unbounded input and output operators is not merely a technical curiosity but has applied interest for instance when modeling point actuators and sensors as Dirac measures. The reason for not including Du is that this results in an infinite $\mathcal{H}_2$-norm.

Example 3.5. A simple example of a control system can be given in terms of the heat equation. Consider a simple physical model of a heated bar on $[0,1]$

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial p^2} + u(t).$$

with insulated boundary points $\frac{\partial x}{\partial p}(0,t) = \frac{\partial x}{\partial p}(1,t) = 0$, initial distribution of heat $x(p,0) = x_0(p)$ and a source term u. This can be recast in terms of a system $A = \frac{d^2}{dp^2}$, $B = I$ so that $\Sigma = (A, I, -)$ on $L^2[0,1]$. Here with $X = U = L^2[0,1]$ and

$$\begin{aligned} D(A) =&\{x \in L^2[0,1] \mid x, x_p \text{ are absolutely continuous}, \\ & x_{pp} \in L^2[0,1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\}. \end{aligned}$$

△

3.3 Controllability

The notion of controllability essentially asks whether we in some time period (potentially infinite) can attain any desired state (potentially asymptotically). In contrast to finite-dimensional systems theory, where there are several equivalent and intuitive characterizations, for distributed parameter systems, the notion of controllability is more subtle. Indeed, there are at least three definitions which all reduce to the standard notion in the finite dimensional setting. Here we will consider only *approximate controllability*, mainly as this concept is both intuitively reasonable and allows for generalization of Reachability and the Gramian-Lyupunov equation theory and so should be recognizable to those familiar with the finite-dimensional theory.

Definition 3.6.

1. the system $\Sigma(A, B, -)$ is *approximately controllable in time t* if for any $\bar{x}$ in the state space and any $\varepsilon > 0$ there exists an input such that $d(x(t), \bar{x}) < \varepsilon$ with u as input. Equivalently, $\overline{\operatorname{ran} \mathcal{B}^t} = X$ where $\mathcal{B}^t$ is the controllability map

$$\mathcal{B}^t u = \int_0^t S(t-s)Bu(s)ds.$$

   $\mathcal{B}^\infty = \mathcal{B}$ is called the extended controllability map.

2. The *reachability space* of $\Sigma(A, B, -)$ is given by set of all states that can be attained by some control from the origin.

$$\begin{aligned} \mathcal{R} :&= \{x \in X \mid \exists t > 0, u \in L^2([0,t];U) \text{ such that } x = \mathcal{B}^t u\} \\ &= \bigcup_{t>0} \operatorname{ran} \mathcal{B}^t. \end{aligned}$$

3. If $\overline{\mathcal{R}} = X$, then $\Sigma(A, B, -)$ is *approximately controllable* (without reference to any particular time). Equivalently, for any $\bar{x} \in X$ there exists $t > 0$, $\varepsilon > 0$ and input u such that $d(x(t), \bar{x}) < \varepsilon$ with u as input.

4. The Controllability Gramian is defined as $P = \mathcal{B}\mathcal{B}^*$.

Remark 3.7. The attentive reader will notice that the controllability map corresponds to the mild solution of a system with zero initial condition as defined via the variation of parameters formula. Thus, approximate controllability really asks whether any state can be asymptotically and approximately (in the topology of X) attained as a mild solution for the control system (Σ).

It is possible to describe controllability in an algebraic manner as the following theorem illustrates.

Theorem 3.8. *If $\mathcal{B}$ and P are bounded the following are equivalent:*

1. $\Sigma(A, B, -)$ *is approximately controllable,*
2. $\ker \mathcal{B}^*$ *is trivial,*
3. $P > 0$.

Proof. To show equivalence of 1. and 2., since $\mathcal{B}$ is bounded, we may use range-nullspace to note that

$$\overline{\mathcal{R}} = \overline{\operatorname{ran} \mathcal{B}} = \overline{\operatorname{ran} \mathcal{B}^{**}} = (\ker \mathcal{B}^*)^{\perp}$$

Hence the left-hand side equals X if and only if $\ker B^*$ is void. To see that 2. and 3. are equivalent. Note that for any $u \in U$

$$\langle Px, x\rangle_X = \langle \mathcal{B}\mathcal{B}^*, x\rangle_X = \langle \mathcal{B}^* x, \mathcal{B}^* x\rangle_X = \|\mathcal{B}^* x\|_X^2$$

and that $\ker \mathcal{B}^*$ is void iff $\mathcal{B}^* u \neq 0$ for all u iff $\|\mathcal{B}^* u\|_X > 0$ for all u. □

This extends classic results but with a small caveat: The description is not entirely algebraic. Even though P is the solution of a particular Lyupunov equation, this equation is functional analytic in nature due to the non-finiteness of the dimension. The above result motivates to some extent the very introduction of the controllability Gramian. We give necessary conditions for its hypotheses below.

Lemma 3.9. *Suppose A generates an exponentially stable semigroup. Then*

$$P \in \mathfrak{B}(X), \text{ and } \mathcal{B} \in \mathfrak{B}\left(\left(L^2[0,\infty); U\right), X\right).$$

Proof. $\mathcal{B}$ is well-defined and bounded since for each t

$$\begin{aligned} \left\| \int_0^t S(s)Bu(s)ds \right\| &\leq \int_0^t \|S(s)Bu(s)\| ds \leq M \|B\|_{\mathfrak{B}(U,X)} \int_0^t e^{-\alpha t} \|u(s)\| ds \\ &\leq \frac{M}{\sqrt{2\alpha}} \|B\|_{\mathfrak{B}(U,X)} \|u\|_{L^2([0,t),U)} \end{aligned}$$

by Cauchy-Schwartz and where α is the growth index of the semigroup S. Obviously P is then also bounded since it is a composition of bounded operators. □

We now give the Lyapunov result for the controllability Gramian.

Theorem 3.10. *Consider $\Sigma(A, B, -)$ and suppose A generates an exponentially stable C_0-semigroup. Then P is the unique self-adjoint solution to the Lyupunov equation*

$$\langle Px, A^*x'\rangle + \langle A^*x, Px'\rangle = -\langle B^*x, B^*x'\rangle$$

with $x, x' \in D(A^*)$ and the inner product taken in X.

Proof. We prove uniqueness, since a generalized form of the Lyapunov equation actually holds even for semistable systems, and follows from Theorem 4.25 proved later.

Suppose now P' is another self-adjoint operator satisfying the Lyupunov equation and consider $\delta = P - P'$:

$$\langle \delta x, A^*x'\rangle + \langle A^*x, \delta x'\rangle = 0.$$

Thus, if we let $x = S^*(t)x_0$ and $x' = S^*(t)x_0'$ for $x_0, x_0' \in D(A^*)$. Whence we obtain

$$\frac{d}{dt}\langle S^*(t)x_0, \delta S(t)x_0'\rangle = 0$$

which after integration becomes

$$\langle S^*(T)x_0, \delta S^*(T)x_0'\rangle = \langle x_0', \delta x_0'\rangle$$

and since $S^*(T)x_0 \to 0$ as $T \to \infty$ by exponential stability so that

$$\langle x_0, \delta x_0'\rangle = 0$$

for $x_0, x_0' \in D(A^*)$. Since $D(A^*)$ is dense in X^* it follows that $\delta = 0$ or $P = P'$. □

Remark 3.11. There is also the dual notion of approximate observability. Briefly, a system $\Sigma(A, -, C)$ is approximately observable if and only if $\Sigma(A^*, C^*, -)$ is approximately controllable. This is a deep duality which is found throughout Systems Theory and in various forms.

3.4 Input-Output Behavior

There is a special operator which characterizes the input-output behavior of a system in the time-domain.

Definition 3.12. The impulse response of Σ is given by $h(t) = CS(t)B$ for all $t \geq 0$.

Intuitively and informally, the impulse response at t is just what happens (in the long run) to a system with zero initial condition if one uses a Dirac delta mass as input $u = \delta$ localized in time at t. A purely motivational and nonrigorous computation based on variation of parameters then gives

$$C\int_0^\infty S(s)B\delta(s-t)ds = CS(t)B.$$

A natural question to ask is how to assign a size to the system Σ. Ideally, such a size should reflect the input-output behavior of the system. We make the following definition.

Definition 3.13. The $\mathcal{H}_2$-norm of a system $\Sigma = (A, B, C)$ with impulse response $h(t)$ is given by

$$\|\Sigma\|_{\mathcal{H}_2} = \sqrt{\int_0^\infty \operatorname{tr}\big(h(t)h^*(t)\big)dt}.$$

Another way of writing this, using the definition of the trace, is

$$\|\Sigma\|_{\mathcal{H}_2} = \sqrt{\int_0^\infty \sum_{i=1}^\infty \|h(t)e_i\|^2 dt}$$

for some orthonormal basis e_i of U so that the $\mathcal{H}_2$-norm is the sum of the energies of h in each basis direction.

We now prove that $\|\cdot\|_{\mathcal{H}_2}$ is actually a norm on a suitable vector space.

Proposition 3.14. *The space of all impulse responses deriving from systems of the form $\Sigma(A, B, C)$ with input-output spaces U and Y forms a vector space, $\mathcal{V}$. On the subspace $\mathcal{W}$ of $\mathcal{V}$, consisting of all Σ with finite $\|\Sigma\|_{\mathcal{H}_2}$, $\|\cdot\|_{\mathcal{H}_2}$ is a norm.*

Proof. First, note that $\mathcal{V}$ is a vector space over $\mathbb{C}$ since first $\alpha \in \mathbb{C}$ we have that αh is the impulse reponse of $\Sigma(A, B, \alpha C)$, so scalar multiplication $(\alpha, h) \mapsto \alpha h$ is well-defined. Second, for $h_1 = C_1S_1B_1, h_2 = C_2S_2B_2 \in \mathcal{V}$, we have pointwise addition

$$h_1(t) + h_2(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} S_1(t) & 0 \\ 0 & S_2(t) \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \tag{5}$$

where the semigroup, defined on $X = X_1 \oplus X_2$ above is generated by

$$\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

with A_1, A_2 defined on the domain $D(A_1) \oplus D(A_2)$, making addition well-defined and $\mathcal{V}$-closed. See also Example 2.11 for details on the direct sum semigroup. The associative and distributive laws are inherited from matrix multiplication and addition can be seen to be commutative by changing places of the indices in (5).

To see that $\|\cdot\|_{\mathcal{H}_2}$ is a norm on this space, take again $\alpha \in \mathbb{C}$ and $h_1, h_2 \in \mathcal{V}$. Then if $\alpha\Sigma_1$ is the system corresponding to αh_1 we have

$$\begin{aligned} \|\alpha\Sigma_1\|_{\mathcal{H}_2} &= \sqrt{\int_0^\infty \sum_{i=1}^\infty \|\alpha h(t)e_i\|^2 dt} = \sqrt{\int_0^\infty \sum_{i=1}^\infty |\alpha|^2 \|h(t)e_i\|^2 dt} \\ &= |\alpha| \sqrt{\int_0^\infty \sum_{i=1}^\infty \|h(t)e_i\|^2 dt} = |\alpha| \|\Sigma_1\|_{\mathcal{H}_2}. \end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|\Sigma_1+\Sigma_2\|_{\mathcal{H}_2} &= \sqrt{\int_0^\infty \sum_{i=1}^\infty \|(h_1(t)+h_2)e_i\|^2 dt} \leq \sqrt{\int_0^\infty \sum_{i=1}^\infty \|h_1(t)e_i\|^2 + \|h_2 e_i\|^2 dt} \\
&= \sqrt{\int_0^\infty \sum_{i=1}^\infty \|h_1(t)e_i\|^2 dt + \int_0^\infty \sum_{i=1}^\infty \|h_2 e_i\|^2 dt} \\
&\leq \sqrt{\int_0^\infty \sum_{i=1}^\infty \|h_1(t)e_i\|^2 dt} + \sqrt{\int_0^\infty \sum_{i=1}^\infty \|h_2 e_i\|^2 dt} = \|\Sigma_1\|_{\mathcal{H}_2} + \|\Sigma_2\|_{\mathcal{H}_2}
\end{aligned}$$

As for the nonnegativity, we have $\|\cdot\|_{\mathcal{H}_2} \geq 0$ by construction. Finally, if $\|\Sigma_1\|_{\mathcal{H}_2} = 0$ it is immediate that all the functions $\|h_1(t)e_i\|^2 = \langle h_1(t)e_i, h_1(t)e_i\rangle$ are almost everywhere 0, but since e_i is an orthonormal basis this is sufficient to conclude that $h_1(t)$ is almost everywhere 0. □

The last proposition might seem only a technical detail, allowing us to actually call $\|\cdot\|_{\mathcal{H}_2}$ a norm. There are, however, system-theoretic reasons for wanting this as well, as this confirms that parallell connection (addition of impulse responses) is a well-defined operation, and produces a new system from two component systems.

Remark 3.15. We have purposefully avoided the frequency domain description here and will not say much about it. However, if for nothing but etymological interest, it should be stated that the name $\mathcal{H}_2$-norm actually derives from a corresponding frequency domain description of h. One can show that h is the Laplace transform of the so-called transfer function of Σ, which outputs according to $\hat{y}(s) = G(s)\hat{u}(s)$, where $\hat{y}$ and $\hat{u}$ are the Laplace transforms of the input and output signals. Via the Paley-Wiener Theorem one can then identify the time domain definition with a corresponding definition based on the actual Hardy space norm, and not the Lebesque-norm construction we have used. Details in the finite-dimensional case can be found in [GL12], Chapter 3. Information about the Hardy spaces in the operator-valued case can be found in [RR97]. More about the frequency domain description of distributed parameter systems can be found in Chapters 4.3, 7 and 8 of [CZ12].

We have thus concluded the background necessary for our own work. In the next section we study the main topics of this thesis; model reduction and semistability.

4 Model Reduction of Semistable Systems in Infinite Dimensions

In this section we consider the main problem of this thesis, namely, given a system

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx, x(0) = x_0 \end{cases} \tag{Σ}$$

to find a reduced system

$$\begin{cases} \dot{v} &= \hat{A}v + \hat{B}u \\ \hat{y} &= \hat{C}v, v(0) = v_0 \end{cases} \tag{$\hat{\Sigma}$}$$

which approximates the initial system well in both a qualitative system-theoretic manner and quantitatively in the sense of the $\mathcal{H}_2$-norm. As usual in this thesis, (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are such that $A, \hat{A}$ generate C_0-semigroups on separable Hilbert spaces X, $V \subset X$ and $B, \hat{B}$, $C, \hat{C}$ are bounded linear operators.

In the finite-dimensional case, (Σ) is typically a system of ordinary differential equations in $X = \mathbb{R}^n$ for some integer n and the goal would then be to be find a reduced system of ordinary differential equations $(\hat{\Sigma})$ on some lower-dimensional space $V = \mathbb{R}^k, k < n$ which approximates the original system well in some metric. The fundamental idea behind model reduction here is essentially the same: we want to find dynamics $(\hat{\Sigma})$ defined on a subspace V of X which approximate (Σ) in the particular metric induced by the $\mathcal{H}_2$-norm. An example of original and reduced spaces in the infinite-dimensional case could be $X = L^2(M)$ for some$m$-dimensional manifold and then V could be for instance $V = L^2(N)$ where N is an n-dimensional manifold $n < m$. However, there are many more interesting subspaces V of this X, including subspaces which don't necessarily reduce the dimension of the manifold on which the PDE exists. Such examples include the eigenspaces of the operator A. This last choice of V is illustrated in Example 4.5 later.

A large issue with model reduction in infinite dimensions is that the map $A \mapsto \sigma A \pi$ which is typically used to induce model reductions in the finite-dimensional case does not necessarily imply a nice relation between semigroups $S, \hat{S}$, making coordinate-free computations difficult. One way to deal with this is to assume a commutativity property between A and $\hat{A}$ in relation to the reducing map σ. Such a procedure is actually well known in mathematical biology and chemistry when there is no input nor output operator and is then known as a lumping. Since we are in need of a concept to fit the entire system Σ and not just the driving operator A, we will give it a different name to avoid overload of terminology[2].

Definition 4.1. An invariant model reduction of Σ onto V is a triple $(\pi, \sigma, \hat{A})$ where $\pi : X \to V$ is a bounded surjective operator, $\sigma : V \to X$ is a bounded operator and

[2]In the definition below one could say that an invariant model reduction consists of a lumping for A and a description of the reduced input and output operators consistent with this lumping.

$\hat{A}$ satisfies $\hat{A}\pi x = \pi A x$ for all $x \in D(A)$. The reduced input and output operators are given by $\hat{B} = \pi B$ and $\hat{C} = C\sigma$.

Put differently, π and σ together produce new dynamics on the reduced space V via surjection and embedding. The following diagram is perhaps useful to understand the input and output behaviors of the original and reduced systems with respect to π and V:

$$\begin{array}{ccc} D(A) \subset X & \xrightarrow{A} & X \\ \pi\downarrow & & \downarrow\pi \\ \pi D(A) \subset V & \xrightarrow{\hat{A}} & V \end{array}$$

Remark 4.2. $\pi = 0$ gives trivially an invariant model reduction for any system and so does $\pi = I, \hat{A} = A$.

As a trivial consequence of the Open Mapping Theorem, we remark that V can essentially be viewed as a subspace of X since π is a bounded surjection and so its restriction is a bounded bijection. We obtain:

Lemma 4.3. *The map π restricts to a bounded linear operator with bounded inverse $\pi_| := \pi_{|(\ker \pi)^\perp} : (\ker \pi)^\perp \to V$.*

The commutativity means that the model reduction is faithful in the sense that one can work either in the original model directly and then reduce or work in the reduced state space V and then apply the reduced model. To illustrate the use of the commutativity assumption, we prove the theorem below, found originally in [AR$^+$13] for the infinite-dimensional case.

Theorem 4.4. *Suppose that $\pi : X \to V$ is a bounded linear map and that A is the infinitesimal generator of a C_0-semigroup $S(t)$. Then the following are equivalent:*

1. $\ker \pi$ *is $S(t)$-invariant for each $t \geq 0$.*
2. *There exists $\hat{A} : \pi(D(A)) \to V$ generating a C_0-semigroup $\hat{S}(t)$ on V with $\pi A = \hat{A}\pi$ on $D(A)$ and in this case $\pi S(t) = \hat{S}(t)\pi$ for each $t \geq 0$ on X.*

Proof. Assume first that $\ker \pi$ is invariant under $S(t)$ for $t \geq 0$ and define

$$\hat{S}(t)v = \pi S(t)x,$$

where x is chosen such that $v = \pi x$ and this is well-defined precisely since the kernel of π is invariant under $S(t)$. The fact that $\hat{S}(t)$ now constitutes a strongly continuous semigroup follows almost immediately from that $S(t)$ is such a semigroup. One has

$$\hat{S}(0)v = \hat{S}(0)\pi x = \pi S(0)x = \pi x = v$$

and for $s, t \geq 0$ the semigroup property

$$\hat{S}(t+s)v = \pi S(t+s)x = \pi S(t)S(s)x = \hat{S}(t)S(s)x = \hat{S}(t)\hat{S}(s)\pi x = \hat{S}(t)\hat{S}(s)v.$$

The strong continuity now follows from the estimate

$$\|\hat{S}(t)v - v\| = \|\pi S(t)x - \pi x\| \leq \|\pi\|\|S(t)x - x\|$$

and the strong continuity of $S(t)$.

We now check that the generator $\hat{A}$ of this new semigroup verifies the desired properties. Now for $v \in V$

$$\hat{A}v = \lim_{t\to 0} \frac{\hat{S}(t)v - v}{t} = \lim_{t\to 0} \frac{\pi S(t)x - \pi x}{t} = \pi \lim_{t\to 0} \frac{S(t)x - x}{t} = \pi A x$$

so that $\hat{A}$ is indeed defined on all of $\pi D(A)$ and $\hat{A}\pi x = \pi A x$ on $D(A)$. Furthermore this implies $\pi D(A) \subseteq D(\hat{A})$.

For the reverse inclusion, choose a $\lambda \in \mathbb{C}$ which lies in $\rho(A) \cap \rho(\hat{A})$ which is possible by Lemma 2.13. Choose now any element $v \in D(\hat{A})$ so that we may write $v = (\lambda I - \hat{A})^{-1}w$ for some $w = \pi x \in V$ with $x \in X$. Thus by the Laplace characterization of the resolvent, we obtain

$$\begin{aligned} v &= \int_0^\infty e^{-\lambda t}\hat{S}(t)w dt = \int_0^\infty e^{-\lambda t}\hat{S}(t)\pi x dt \\ &= \int_0^\infty e^{-\lambda t}\pi\hat{S}(t)x dt = \pi\int_0^\infty e^{-\lambda t}\hat{S}(t)x dt \\ &= \pi(\lambda I - A)^{-1}x \in \pi(D(A)), \end{aligned}$$

using that $x \in X$ and therefore $R(A;\lambda)x \in D(A)$. Thus we have equality between $D(\hat{A})$ and $\pi D(A)$.

To see that 2. implies 1. assume that 2. holds, suppose that $\hat{A}$ generates $\hat{S}(t)$, and consider the maps

$$\begin{aligned} &t \mapsto \hat{S}(t)v_0, \\ &t \mapsto \pi S(t)x_0, \end{aligned}$$

with $v_0 = \pi x_0, x_0 \in D(A)$. By the assumption that $\pi A = \hat{\pi}$ one notes that either of these maps is a solution to the problem

$$\begin{cases} \dot{v}(t) &= \hat{A}v(t), \\ v(0) &= v_0. \end{cases}$$

However, by Theorem 2.26 this problem has a unique solution from which we conclude that the maps are equal. In particuar this means that for any $x \in X$

$$\hat{S}(t)\pi x = \pi S(t)x.$$

Thus if $x \in \ker\pi$ so that $\pi x = 0$ then $\pi S(t)x = \hat{S}(t)(\pi x) = 0$ so that also $S(t)x \in \ker\pi$ for all $t \geq 0$. That is, $S(t)\ker\pi \subseteq \ker\pi$, as required. □

This theorem is roughly speaking the reason for considering model reductions of this form. The formula $\pi S(t) = \hat{S}(t)\pi$ is immensely useful in computations since for operators in infinite dimensions a representation $S(t) = e^{At}, \hat{S}(t) = e^{\hat{A}t}$ is not available in general. An alternative would be to use a variation of parameters type formula for $\hat{S}(t)$ by considering something looking formally like $\pi A = (\pi - I + I)A = A + (\pi - I)A$. However, this would involve significantly more computational work if possible at all.

Fortunately, the theory is sufficiently rich to include many partial differential operators and commonly used model reductions of these. We revisit the heat equation example below and prove that truncation of the series representation of its solution qualifies as an invariant model reduction.

Example 4.5. Consider a simple physical model of a heated bar on $[0, 1]$

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial p^2} + u(t)$$

with insulated boundary points $\frac{\partial x}{\partial p}(0, t) = \frac{\partial x}{\partial p}(1, t) = 0$, initial distribution of heat $x(p, 0) = x_0(p)$ and a source term u. This can be recast in terms of a system $A = \Delta$, $\Sigma = (A, I, -)$ on $L^2[0, 1]$ with $D(A) = \{x \in L^2[0, 1] \mid x, x_p \text{ are absolutely continuous}, x_{pp} \in L^2[0, 1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\}$.

Moreover, the semigroup associated with A is

$$S(t)x = \int_0^1 x(q)dq + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\cos(n\pi p)\int_0^1 \cos(n\pi q)x(q)dq$$

Now, in previous examples it was verified that A is self-adjoint and that $(\cos(\pi np))$ are eigenvectors of A. Indeed, according to Proposition 2.23 the invariant subspaces are closures of the spans of subcollections of eigenvectors of A and therefore we may choose π as the projection onto the closure of any such eigenspaces. This means that one can define a new semigroup

$$\hat{S}(t)v = \chi_{[0\in\mathcal{I}]}\int_0^1 v(q)dq + \sum_k 2e^{-n_k^2\pi^2 t}\cos(n_k\pi p)\int_0^1 \cos(n_k\pi q)v(q)dq$$

on the reduced state space $V = \overline{\text{span}}\{\cos(n_k\pi p)\}_{k\in\mathcal{I}}$. That is, truncating the eigenvalues of the heat equation is an invariant model reduction. Note also that σ is just the embedding of the eigenspace V into X. △

The theory is, however, not without restriction. The clustering projection considered in [CKS17] is not an invariant model reduction as given in Definition 4.1. We give a counterexample based on this below.

Example 4.6. We use an example from [CKS17]. Let

$$A = \begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 3 & -1 & -1 \\ -2 & -1 & 5 & -2 \\ -1 & -1 & -2 & 4 \end{bmatrix}, \pi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

with $\hat{A} = \pi A \pi'$ it is then not hard to verify that

$$\hat{A}\pi = \begin{bmatrix} 4 & -1 & -3 & -3 \\ -1 & 3 & -2 & -2 \\ -3 & -2 & 5 & 5 \end{bmatrix} \neq \begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 3 & -1 & -1 \\ -3 & -2 & 3 & 2 \end{bmatrix} = \pi A.$$

This shows that the so-called clustering projections of [CKS17] are not in general invariant model reductions. △

4.1 Semistability

The presence of the non-decaying term in Example 4.5 means that the semigroup $S(t)$ is not exponentially stable. However, the eigenvalues do at least satisfy $\Re\lambda \leq 0$ similar as is the case for the system considered by [CKS17]. To deal with this, we introduce a weaker notion of stability applicable to infinite-dimensional systems, found also in [HB13].

Definition 4.7. Suppose A generates a C_0-semigroup $S(t)$ on X. A, $S(t)$ are said to be exponentially semistable if for every $x \in X$ there exists $x_e \in \ker A$ and scalars $M, \mu > 0$ such that $\|S(t)x - x_e\| \leq Me^{-\mu t}\|x - x_e\|$. The point x_e is called the equilibrium point corresponding to x.

This means that the kernel of the infinitesimal generator, A, corresponds to the set of equilibrium points of the dynamical system induced by $S(t)$. This is perhaps not so surprising, since nonrigorously, one may imagine that an $x \in X$ under the semigroup $S(t)$ is shifted approximately to $x + (\Delta t)Ax$ in an infinitesimal time-step Δt and if x lies in the kernel then $x + (\Delta t)Ax = x$. For a more formal motivation, see the discussion following Lemma 2.5.

Remark 4.8. Observe that exponential semistability is stronger than requiring that the eigenvalues of a matrix, A, satisfy $\Re\lambda_i \leq 0$. To see this, consider $A = iI$ which satisfies $\Re\lambda_i(A) \leq 0$ but is not exponentially semistable since the kernel of A is empty and $S(t) = Ie^{it}$ for which $S(t)x$ does not converge for any $x \neq 0$ in the strong topology. We will give a precise characterization for matrices in terms of their eigenvalues later.

Two issues with Definition 4.7 need to be pointed out before proceeding: x_0, μ and M may all depend on x, so there is a distinct lack of uniformity in the problem. We will mostly be able to deal with this by applying Banach-Steinhaus' principle of uniform boundedness to S or some variation thereof. Moreover, if A is semistable it is stable iff $\ker A = \{0\}$ since $S(t) = I$ on $\ker A$. The second point is foundational in the geometric understanding of the problem and a large part of the theory is a result of careful consideration of the interaction of the subspaces V and $\ker A$.

Now, before going any further with this definition and to better our understanding of the concept, let us consider the finite-dimensional case and an example.

Proposition 4.9. *Suppose that $A \in \mathfrak{B}(\mathbb{C}^n)$. Then A is exponentially semistable if and only if $\Re\lambda \leq 0$ for all eigenvalues λ of A and there are no purely imaginary eigenvalues.*

Proof. Suppose that A is exponentially semistable. It is immediate by definition that for every $x \in \mathbb{C}^n$

$$\lim_{t\to\infty} S(t)x = \lim_{t\to\infty} e^{At}x = x_e$$

for some $x_e \in \ker A \subseteq \mathbb{C}^n$. Clearly this requires $\Re\lambda \leq 0$, so suppose to arrive at a contradiction that one eigenvalue satisfies $\lambda = \alpha i, \alpha \in \mathbb{R}$. Choose x as an associated eigenvector. Then we have

$$\lim_{t\to\infty} e^{At}x = \lim_{t\to\infty} \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} x = \lim_{t\to\infty} \sum_{k=0}^{\infty} \frac{(\alpha i)^k t^k}{k!} x = \lim_{t\to\infty} e^{\alpha i t} x$$

which does not even converge, so the requirements on the eigenvalues of A are necessary.

Suppose now that $\Re\lambda \leq 0$ for all eigenvalues λ of A and that there are no purely imaginary eigenvalues. Let J be the Jordan form for A and we first show that J is exponentially semistable. Now if K is the number of distinct nonzero eigenvalues of A, we may write

$$e^{Jt} = \begin{bmatrix} e^{J_1 t} & 0 & 0 & \dots & 0 \\ 0 & e^{J_2 t} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & 0 \\ \vdots & \vdots & & e^{J_K t} & 0 \\ 0 & 0 & 0 & 0 & I_{\dim\ker A} \end{bmatrix} \text{ with } J = \begin{bmatrix} J_1 & 0 & 0 & \dots & 0 \\ 0 & J_2 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & 0 \\ \vdots & \vdots & & J_K & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and Jordan blocks $J_i, i = 1, \dots k$. From this we conclude that for any x

$$\left\| e^{Jt}x - \begin{bmatrix} 0 & 0 \\ 0 & I_{\dim\ker A} \end{bmatrix} x \right\| \leq \max_{i=1,\dots K} M_i e^{-\min_{i=1,\dots K} \mu_i t}$$

where we use that each Jordan block satisfies an exponential stability condition so that they each have rates $-\mu_i$ and constant bounds M_i. Since $\begin{bmatrix} 0 & 0 \\ 0 & I_{\dim\ker A} \end{bmatrix}$ maps to the kernel of J this proves exponential semistability of J.

To finish the proof, we show that exponential semistability does not depend on the choice of basis. Let $A = PJP^{-1}$. Then

$$e^{Jt} = \sum_{k=1}^{\infty} \frac{J^k t^k}{k!} = \sum_{k=1}^{\infty} \frac{P^{-1} A^k P t^k}{k!} = P^{-1} e^{At} P$$

Therefore, if J is exponentially semistable with bounds M, μ, we have that, if x_e is the equilbrium point of x under J

$$\|e^{At}Px - Px_e\| = \|Pe^{Jt}P^{-1}Px - Px_e\| \leq \|P\| M e^{-\mu t} \|x - x_e\|$$

so that the point Px converges to the equilibrium Px_e under A, and since P is invertible we conclude the proof by nothing that this means that every point has an equilibrium point under A as well. □

Example 4.10. Suppose that A is the negative of a Laplacian matrix, L, so that $A = -L$. It is well known that the Laplacian matrix is symmetric positive semidefinite. In particular, this implies that A is symmetric, with $\Re\lambda_i \leq 0$ and no modes on the imaginary axis. Proposition 4.9 now shows that the semigroup generated by A is exponentially semistable. △

Note that the example above implies that our definition of semistability encompasses the network model in [CS16] where a first-order system driven by a Laplacian matrix is considered. A further motivation for studying semistability is the fact that the second derivative operator in general is not stable, however, as we shall later see, it is semistable.

Example 4.11. We show that $A = \frac{d^2}{dp^2}$ is not exponentially stable. By Example 2.20 presented earlier, 0 is an eigenvalue of A, so that $S(t) = I$ on the nontrivial subset $\ker A$. In particular this implies $\|S(t)\| \geq \|I\| = 1$ for all t, thus barring any possibility of exponential stability. △

Before delving deeper into semistability itself, we explore its relation to other concepts. It is quite clear from the definition that exponential semistability is a weaker requirement than stability. However, one may still wonder if it is stronger than Lyapunov stability. The answer is in the affirmative and the proof is essentially the same as in the case for exponential stability.

Proposition 4.12. *Suppose that A is the infinitesimal generator of an exponentially semistable semigroup on a Hilbert space X. Then every point $x_e \in \ker A$ is a Lyapunov equilibrium.*

Proof. Let U be any open set of X containing the point $x_e \in \ker A$. We need to show that there exists a subset O of X , with $S(t)O \subseteq U$ for all $t \geq 0$.

To see this, note that since $x_e \in U$, and the ε-balls form a basis for the norm topology, U contains at least one of the sets $B(x_e, \varepsilon), \varepsilon > 0$ and set $O = B(x_e, \varepsilon/(L+\varepsilon))$ where M is chosen such that $\|S(t)x - x_e\| \leq Me^{-\mu t}\|x_e - x\|$ for some $\mu > 0$. Now, if $x \in B(x_e, \varepsilon/M)$ we have that

$$\|S(t)x - x_e\| \leq Me^{-\mu t}\|x_e - x\| < \frac{M}{\varepsilon}e^{-\mu t}.$$

Thus for every $x \in O, S(t)x \in U$, which concludes the proof. □

The next lemma emphasizes the importance of the generator kernel, the proof of which shows us that $S(t) - S_\infty$ has nice stability properties.

Lemma 4.13. *If $S(t)$ is an exponentially semistable semigroup the limiting operator $S_\infty : X \to \ker A \subset X$ of $S(t), t \to \infty$ exists, is bounded and idempotent.*

Proof. Consider

$$\begin{aligned}\|S(t)-S(s)\| &= \sup_{\|x\|=1, x\in X} \|S(t)x - S(s)x\| = \sup_{\|x\|=1, x\in X} \|S(t)x - x_e - (S(s)x - x_e)\| \\ &\leq \sup_{\|x\|=1, x\in X} 2Me^{-\mu \min(s,t)}\|x - x_e\| \\ &\leq \sup_{\|x\|=1, x\in X} 2Me^{-\mu \min(s,t)}(1 + \|x_e\|)\end{aligned}$$

Note that this still depends on the distance from of the origin of the equilibrium point $\|x_e\|$. To alleviate this, I will establish a uniform bound on the family $S(t)$. Observe that by assumption of semistability, for each $x \in X$

$$\|S(t)x\| \leq \|x_0\| + \|S(t)x - x_0\| \leq \|x_0\| + M\|x - x_0\|$$

so that $\sup_t \|S(t)x\| < \infty$ for each $x \in X$. By the Banach-Steinhaus theorem this means that $\|S(t)\|$ is uniformly bounded, by say M'. Suppose now that there exists x with $\|x_e\| > M'$. Write to arrive at a contradiction

$$M' > \|x_e\| = \lim_{t\to\infty} \|S(t)x\| \leq \lim_{t\to\infty} \|S(t)\|\|x\| = \lim_{t\to\infty} \|S(t)\| \leq M'.$$

Hence

$$\|S(t) - S(s)\| \leq 2Me^{-\mu \min(s,t)}(1 + M')$$

and so since $S(t) \in \mathfrak{B}(X)$ is Cauchy in t, there exists a limiting operator S_∞ which is bounded by completeness of $\mathfrak{B}(X)$. Moreover, $S_\infty x = x_e \in \ker A$ and indeed

$$0 = \|x_e - x_e\| = \|S(t)x_e - x_e\| = \lim_{t\to\infty} \|S(t)x_e - x_e\| = \|S_\infty x_e - x_e\|$$

so that $S_\infty x_e = x_e$. That is, for all $x \in X$, $S_\infty^2 x = S_\infty x$. □

Note that the results are anticipated by the proof of the lemma characterizing the finite-dimensional case, where we had all Jordan blocks converging to 0 except that corresponding to the kernel. The operator S_∞ has a particularly nice form when A is self-adjoint.

Corollary 4.14. *Suppose that A is self-adjoint. Then S_∞ is the orthogonal projection onto the kernel of A.*

Proof. A is self-adjoint if and only if each operator $S(t)$ is self-adjoint. Since S_∞ is the limit of a sequence of self-adjoint operators it is also self-adjoint. To sum up, S_∞ is a self-adjoint operator satisfying $S_\infty^2 = S_\infty$ which precisely means that S_∞ is an orthogonal projection and by Lemma 4.13 its image is $\ker A$. □

The methods used to prove the above results, such as the Banach-Steinhaus Theorem, are only necessary due to the infinite-dimensional nature of the state space. We can perhaps better understand the results by again reverting to the finite-dimensional case.

Example 4.15. Suppose that $A \in \mathfrak{B}(\mathbb{C}^n)$ is exponentially semistable and self-adjoint. Since A is self-adjoint we may write the spectral decomposition of A as

$$A = UDU^*$$

where U is unitary and with $D = \operatorname{diag}(\lambda_1, \dots, \lambda_K, 0, \dots, 0)$. Further, we know by Proposition 4.9 that the eigenvalues $\lambda_1, \dots, \lambda_K$ all have negative real part. Therefore

$$\lim_{t\to\infty} e^{At} = \lim_{t\to\infty} \sum_{k=0}^{\infty} \frac{(UDU^*)^k t^k}{k!} = \lim_{t\to\infty} Ue^{Dt}U^* = U\pi_{\ker D}U^* = \pi_{\ker A}$$

where $\pi_{\ker A}$ is the projection onto the kernel of A, and $\pi_{\ker D}$ the matrix with an identity block in the bottom right of size $\dim \ker A$ and zeroes everywhere else.

△

Remark 4.16. As tempted as one may be to define a new, stable, semigroup $T(t) = S(t) - S_\infty$, this does not work directly since for $0 \neq x \in \ker A$

$$T(0)x = S(0)x - S_\infty x = x - S_\infty x = 0 \neq Ix.$$

The lemma implies that the equilibrium points cannot be too far away from the initial condition in norm. Indeed, we showed above that $\|x_e\| \leq \|S_\infty x\|$, thus allowing for a uniform bound of $\|x - x_e\|$ in terms of $\|x\|$. Indeed, not only do we show that the equilibrium point depends continuously on x, it depends linearly on x, precisely via the mapping S_∞. This removes one component of the lack of uniformity problem in semistability and inspires the following results which gives alternate characterizations of exponential semistability, useful both in practice and to better understand the concept. We state this more delicately in the theorem below.

Theorem 4.17. *If $S(t)$ is a C_0-semigroup with generator A the following are equivalent:*

1. *$S(t)$ is exponentially semistable.*
2. *There exists a bounded operator $S_\infty : X \to \ker A$ which is idempotent on $\ker A$ and constants $\mu, L > 0$ such that such that for every $x \in X$ $\|(S(t) - S_\infty)x\| \leq Le^{-\mu t}\|x\|$.*

Proof. 1. implies 2. by Lemma 4.13 and since

$$\begin{aligned}\|(S(t) - S_\infty)x\| &= \|S(t)x - x_e\| \leq Me^{-\mu t}\|x - x_e\| \\ &= Me^{-\mu t}\|x - S_\infty x\| \leq \|I - S_\infty\|Me^{-\mu t}\|x\|.\end{aligned}$$

so S_∞ is the desired operator. Conversely, it is easy to see that 2. implies 1. since one may write

$$\|(S(t)x - S_\infty x)\| = \|(S(t) - S_\infty)(x - S_\infty x)\| \leq Le^{-\mu t}\|x - S_\infty x\|$$

so $S_\infty x$ is the equilibrium point corresponding to x. □

There actually exists at least one more characterization of exponential semistability via a semistable version of the Datko Theorem proved in [HB13]. However, its proof is rather long and for our purposes the characterization above suffices. Using the lemma, we can also finish the previous example and show that the second derivative operator is actually semistable.

Example 4.18. Recall that the associated semigroup may be represented as a Fourier sum. I.e. the formula

$$S(t)x = \int_0^1 x(q)dq + \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\cos(n\pi p)\int_0^1 \cos(n\pi q)x(q)dq$$

holds. Thus, observing that

$$S_\infty x = \lim_{t\to\infty} S(t)x = \int_0^1 x(q)dq$$

is a constant function, so certainly lies in $\ker A = \{x \in D(A) \mid \frac{d^2}{dx^2} = 0\}$. We take this as our equilibrium point x_e. Then

$$\begin{aligned}
\|S(t)x - x_e\| &= \left\|\sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\cos(n\pi p)\int_0^1 \cos(n\pi q)x(q)dq\right\| \\
&\le \sum_{n=1}^{\infty} 2e^{-n^2\pi^2 t}\left\|\cos(n\pi p)\int_0^1 \cos(n\pi q)x(q)dq\right\| \\
&\le 2e^{-\pi^2 t}\sum_{n=1}^{\infty}\left\|\cos(n\pi p)\int_0^1 \cos(n\pi q)x(q)dq\right\| \le 2e^{-\pi^2 t}\|x\|
\end{aligned}$$

where the last step follows from Bessel's inequality is applied, realizing that we have the Fourier cosine expansion of x (save the constant term, which was eliminated by the equilibrium point). The proof of semistability is completed by observing that $x_e = S_\infty x$ so that the above constitutes the required bound of the second characterization of semistability in Theorem 4.17. △

Using the preceding arguments we can now show that the invariant model reductions preserve semistability.

Proposition 4.19. *If A is semistable on the Hilbert space X and $(\pi, \hat{A})$ is an invariant linear model reduction onto V then $\hat{A}$ is semistable on V.*

Proof. Let $v \in V$. First, observe that any $v \in V$ can, by Lemma 4.3, be written $v = \pi x = \pi_| x$ for $x = \pi_|^{-1} v \in X$. Denote the equilibrium point of x by x_e, which exists by semistability of S. Then using the second characterization of semistability in Theorem 4.17

$$\begin{aligned}
\|\hat{S}(t)v - \pi x_e\| &= \|\hat{S}(t)\pi x - \pi x_e\| = \|\pi S(t)x - \pi x_e\| \le \|\pi\|\|S(t)x - \pi x_e\| \\
&\le |\pi\| L e^{-\mu t}\|x\| = |\pi\| L e^{-\mu t}\|\pi_|^{-1} v\| \le \frac{\|\pi\|}{\|\pi_|^{-1}\|} L e^{-\mu t}\|v\|.
\end{aligned}$$

Hence πx_e is the desired equilibrium point and the result follows. □

Remark 4.20. The proof above may seem quite trivial but the result actually rests heavily on Theorem 4.17. To be precise, there is nothing that a priori guarantees that x_e is in the image of $\pi_|^{-1}$. The theorem allows us to do away with the norm bound on the V-equilibrium point πx_e and solely work with x which by assumption lies in the image of $\pi_|^{-1}$.

Observe further that $\frac{\|\pi\|}{\|\pi_|^{-1}\|} \leq \frac{\|\pi_|\|}{\|\pi_|^{-1}\|} \leq 1$ so the growth bound is actually the same for both models.

The reduced model actually preserves more than just semistability. If the original model is approximately controllable, then so is the reduced model on the reduced state space V.

Proposition 4.21. *Let $\Sigma(A, B, -)$ be an approximately controllable control system on the Hilbert space X and $(\pi, \hat{A})$ be an invariant linear model reduction onto V. Then the reduced model $\Sigma(\hat{A}, \hat{B}, -)$ is approximately controllable on the reduced space V.*

Proof. Suppose that the reachability subspace of $\Sigma(A, B, -)$ is dense in X. Then any $x \in X$ can be written as the limit of a sequence of elements in the reachability space

$$x = \lim_{n\to\infty} \int_0^{\tau_n} S(\tau - s)Bu_n ds$$

for $\tau_n > 0, u_n \in U$. But for any $v \in V$, of the model reduction satisfies for some $x \in X$

$$\begin{aligned} v = \pi x = \pi \lim_{n\to\infty} \int_0^{\tau_n} S(\tau - s)Bu_n ds &= \lim_{n\to\infty} \int_0^{\tau_n} \pi S(\tau - s)Bu_n ds \\ = \lim_{n\to\infty} \int_0^{\tau_n} \hat{S}(\tau - s)\pi Bu_n ds &= \lim_{n\to\infty} \int_0^{\tau_n} \hat{S}(\tau - s)\hat{B}u_n ds. \end{aligned}$$

We conclude: for every $v \in V$ there is a sequence of elements in the reachability subspace of $\Sigma(\hat{A}, \hat{B}, -)$ that converge to v, i.e., the reachability subspace for the reduced model is also dense. The interchanges of the limit and integral with π are justified by that first, π is bounded, and second by that the integrands are bounded operators.

□

Having established that this model reduction preserves several desirable structures of the control system, Σ, we proceed now first to generalize the notion of a Gramian for semistable systems. These will later be used to give a formula for the $\mathcal{H}_2$-norm difference between the original and reduced systems.

4.2 The Gramian Revisited

Remark 4.20 above also means that we should not hope to carry out the analysis for a stable perturbation of A but actually have to deal with the instability directly. One such way to do this is found in [CKS17] where the authors introduce a network Gramian, augmented to suit the semistability due to the presence of a driving Laplacian matrix. However, their method is actually much further reaching, and we define a variation of it below.

Definition 4.22. The semistability Gramian of an exponentially semistable system $\Sigma(A, B, -)$ is given by

$$P_\infty = \int_0^\infty (S(t) - S_\infty)BB^*(S(t) - S_\infty)^* ds$$

where the integral is taken in the sense of Pettis.

Remark 4.23. If A is stable then $\ker A = \{0\}$ then $S_\infty = 0$ since 0 is the only equilibrium point and thus P_∞ reduces to the ordinary controllability Gramian as defined earlier.

Lemma 4.24. *The semistability Gramian of an exponentially semistable system $\Sigma(A, B, -)$ exists and is bounded; $P_\infty \in \mathfrak{B}(X)$.*

Proof. Define a family

$$P_t = \int_0^t (S(s) - S_\infty)BB^*(S(s) - S_\infty)^* ds$$

Using Theorem 4.17 to bound $(S(t)-S_\infty)$ by an exponential growth condition, we obtain pointwise

$$\begin{aligned}
\|P_t x\| &= \left\| \int_0^t (S(s) - S_\infty)BB^*(S(s) - S_\infty)^* x ds \right\| \\
&\leq \int_0^t \|(S(s) - S_\infty)BB^*(S(s) - S_\infty)^* x\| ds \\
&\leq \|B\|^2 L^2 \int_0^t e^{-2\mu s} ds \|x\| = \|B\|^2 L^2 \frac{1 - e^{-2\mu t}}{2\mu} \|x\| \\
&\leq \frac{\|B\|^2 L^2}{2\mu} \|x\|.
\end{aligned}$$

Observe that the constants L, μ a priori depend on x. More precisely there exists a pointwise norm bound for $P_t x, x \in X$ which however is independent of t. Hence by Banach-Steinhaus there exists a uniform bound $K \in \mathbb{R}$ for which $\|P_t\| \leq K, \forall t$ yielding

$$\|P_\infty\| = \|\lim_{t\to\infty} P_t\| = \lim_{t\to\infty} \|P_t\| \leq \lim_{t\to\infty} K = K.$$

□

Theorem 4.25. *For every $x \in D(A^*)$, P_∞ satisfies the semistability Lyapunov equation*

$$AP_\infty x + P_\infty A^* x = -(I - S_\infty)BB^*(I - S_\infty)^* x.$$

Proof. Let $x, x' \in D(A^*)$ and observe that, if integrable, we have formally

$$\int_0^\infty \frac{d}{dt} \langle B^*[S(t) - S_\infty]^* x, B^*[S(t) - S_\infty]^* x' \rangle dt = -\langle B^*(I - S_\infty)^* x, B^*(I - S_\infty)^* x' \rangle$$

Moreover, using the fact that $\frac{dS(t)}{dt} = AS(t) = S(t)A$,

$$\begin{aligned}\frac{d}{dt}\langle B^*[S(t)-S_\infty]^*x, B^*[S(t)-S_\infty]^*x'\rangle &= \langle B^*A^*[S(t)]^*x, B^*[S(t)-S_\infty]^*x'\rangle \\ &\quad + \langle B^*[S(t)-S_\infty]^*x, B^*A^*[S(t)]^*x'\rangle.\end{aligned}$$

Now

$$\begin{aligned}\int_0^\infty \langle B^*A^*[S(t)]^*x, B^*[S(t)-S_\infty]^*x'\rangle dt &= \int_0^\infty \langle [S(t)A]^*x, BB^*[S(t)-S_\infty]^*x'\rangle dt \\ &= \int_0^\infty \langle [S(t)-S_\infty]A]^*x, BB^*[S(t)-S_\infty]^*x'\rangle dt \\ &= \int_0^\infty \langle A^*x, [S(t)-S_\infty]BB^*[S(t)-S_\infty]^*x'\rangle dt \\ &= \left\langle A^*x, \int_0^\infty [S(t)-S_\infty]BB^*[S(t)-S_\infty]^*x' dt\right\rangle \\ &= \langle A^*x, P_\infty x'\rangle\end{aligned}$$

where we used that Lemma 4.13 implies that $S_\infty A = 0$. Similar computations show

$$\langle B^*[S(t)-S_\infty]^*x, B^*A^*[S(t)]^*x'\rangle = \langle P_\infty x, A^*x'\rangle.$$

Therefore

$$\langle P_\infty x, A^*x'\rangle + \langle A^*x, P_\infty x'\rangle = -\langle B^*(I-S_\infty)^*x, B^*(I-S_\infty)^*x'\rangle.$$

Since $D(A^*)$ is dense in X this implies

$$AP_\infty x + P_\infty A^*x = -(I-S_\infty)BB^*(I-S_\infty)^*x$$

for every $x \in D(A^*)$. To finish the proof, note that the required integrability to justify our formal computations follows from

$$\begin{aligned}\left|\frac{d}{dt}\langle B^*[S(t)-S_\infty]^*x, [S(t)-S_\infty]^*x'\rangle\right| &\le \left|\langle B^*A^*[S(t)]^*x, B^*[S(t)-S_\infty]^*x'\rangle\right| \\ &\quad + \left|\langle B^*[S(t)-S_\infty]^*x, B^*A^*[S(t)]^*x'\rangle\right| \\ &\le \|A^*x\|\|x'\|\|B^*\|^2L^2e^{-2\mu t} \\ &\quad + \|A^*x'\|\|x\|\|B^*\|^2L^2e^{-2\mu t}\end{aligned}$$

where we used the second characterization of semistability to obtain a pointwise bound on $S(t) - S_\infty$. □

4.3 $\mathcal{H}_2$-Error Estimates

The aim is to obtain error estimates between the reduced and original model under the $\mathcal{H}_2$-norm. Recall that the $\mathcal{H}_2$-norm of a system $\Sigma = (A, B, C)$ with impulse response $h(t) = CS(t)B$ is given by

$$\|\Sigma\|_{\mathcal{H}_2} = \sqrt{\int_0^\infty \operatorname{tr}(hh^*)dt}. \tag{6}$$

Observe that there are two obvious ways in which the integral (6) can fail to be finite. Either $h(t)$ does not decay at infinity or the trace does not exist, i.e., $h(t)$ is not Hilbert-Schmidt (the reader is referred to the appendix for a brief introduction of this class). The problem that $h(t)$ might not decay at infinity mainly is due to the semistability of $S(t)$. To understand this, note that the fact that the kernel of A is nontrivial means that $S(t)$ has fixed points for all t and in particular its limit is non-zero. It should also be said that this is not an infinite-dimensional problem but similar issues may arise in a finite-dimensional setting. On the other hand, when A is Riesz Spectral, the non-existence of the trace corresponds to the sequence of singular values of A not decaying fast enough and this issue is purely infinite-dimensional.

Since we are mainly concerned with estimating errors between two system we will not have to demand stability of the system, but simply that the original and reduced systems synchronize at infinity. One might say that the model reduction respects the equilibria of the original system, so that both models agree asymptotically. We give sufficient conditions for this below.

Proposition 4.26. *Let $\Sigma(A, -, -)$ be an exponentially semistable system and suppose that $(\pi, \sigma, \hat{A})$ is an invariant model reduction of this system with $\sigma\pi$ restricting to the identity on* $\ker A$. *Then for all initial conditions $\|S(t)x - \sigma\hat{S}(t)\pi x\| \to 0$.*

Proof. By commutativity $\hat{S}(t)\pi x = \pi S(t)x$. So we may write for any $x \in X$ with equilibrium point $x_e \in \ker A$

$$\begin{aligned}\|S(t)x - \sigma\pi S(t)x\| &= \|(S(t)x - x_e) - (\sigma\pi S(t)x - x_e)\| \\ &= \|(S(t)x - x_e) - (\sigma\pi S(t)x - \sigma\pi x_e)\| \\ &\leq \|I - \sigma\pi\| M e^{-\mu t}\|x - x_e\|\end{aligned}$$

proving the result. □

The synchronization result above guides our intuition for the hypotheses necessary for the main result, which we state immediately below.

Theorem 4.27. *Suppose that $\Sigma(A, B, C)$ is a distributed parameter system on a separable Hilbert space X where A generates a semistable C_0-semigroup $S(t)$ that B and C are bounded and that $(\sigma, \pi, \hat{A})$ is an invariant model reduction thereof where $\sigma\pi$ restricts* *to the identity on* $\ker A$*. Then if* $(I-\sigma\pi)S(t)$ *is Hilbert-Schmidt the model error is given by*

$$\|\Sigma - \hat{\Sigma}\| = \sqrt{\operatorname{tr}\Big(C(I-\sigma\pi)P_0(I-\sigma\pi)^*C^*\Big)}$$

where P_0 *is the semistability Gramian of* Σ *which for* $x \in D(A^*)$ *satisfies*

$$APx + PA^*x = -(I-S_\infty)BB^*(I-S_\infty)^*x.$$

Proof. Write

$$\begin{aligned} h_I(t) - \hat{h}_I(t) &= CS(t)B - C\sigma\hat{S}(t)\hat{B} = CS(t)B - C\sigma\hat{S}(t)\pi B \\ &= CS(t)B - C\sigma\pi S(t)B = C(I-\sigma\pi)S(t)B. \end{aligned}$$

Thus, since $(I-\sigma\pi)S(t)$, is Hilbert-Schmidt and since B and C are bounded it follows that also $h_I(t) - \hat{h}_I(t)$ is Hilbert-Schmidt. Now

$$\begin{aligned} \|\Sigma - \hat{\Sigma}\|^2_{\mathcal{H}_2} &= \int_0^\infty \operatorname{tr}\Big([h_I(t)-\hat{h}_I(t)][h_I(t)-\hat{h}_I(t)]^*\Big)dt \\ &= \int_0^\infty \sum_{i=1}^\infty \Big\langle (h_I(t)-\hat{h}_I(t))^*e_i, h_I(t)-\hat{h}_I(t))^*e_i \Big\rangle\, dt \\ &= \sum_{i=1}^\infty \int_0^\infty \Big\langle (h_I(t)-\hat{h}_I(t))^*e_i, h_I(t)-\hat{h}_I(t))^*e_i \Big\rangle\, dt, \\ &\quad \text{by the Monotone Convergence Theorem} \\ &= \sum_{i=1}^\infty \int_0^\infty \langle (C(I-\sigma\pi)S(t)B)^*\, e_i, ((I-\sigma\pi)S(t)BC)\, e_i\rangle\, dt \\ &= \sum_{i=1}^\infty \int_0^\infty \langle e_i, C(I-\sigma\pi)[S(t)-S_\infty]BB^*[S(t)-S_\infty]^*(I-\sigma\pi)^*C^*e_i\rangle\, dt \end{aligned}$$

since by the lemma above $(I-\sigma\pi)S_\infty = 0$. Observe also that the use of the Monotone Convergence Theorem is justified by applying it to the real valued sequence of functions

$$\sum_{i=1}^n \Big\langle (h_I(t)-\hat{h}_I(t))^*e_i, h_I(t)-\hat{h}_I(t))^*e_i \Big\rangle : [0,\infty) \to \mathbb{R}.$$

The limit of this sequence exists by the Hilbert-Schmidt assumption, and so is a well-defined function. Next, we want to move the integral inside of the inner product. To do this, we interpret the expression as an integral in the sense of Pettis and then in the next step use that this integral commutes with bounded operators, so as to find that this equals (see appendix A.2 for the definition of the Pettis integral and further references),

$$\begin{aligned} &\sum_{i=1}^\infty \Big\langle e_i, \int_0^\infty (I-\sigma\pi)C[S(t)-S_\infty]BB^*[S(t)-S_\infty]^*dt(I-\sigma\pi)^*C^*e_i \Big\rangle \\ =&\sum_{i=1}^\infty \Big\langle e_i, (I-\sigma\pi)C\int_0^\infty [S(t)-S_\infty]BB^*[S(t)-S_\infty]^*dt(I-\sigma\pi)^*C^*e_i \Big\rangle. \end{aligned}$$

Therefore,

$$\|\Sigma - \hat{\Sigma}\|^2_{\mathcal{H}_2} = \operatorname{tr}\left(C(I-\sigma\pi) \int_0^\infty [S(t) - S_\infty]BB^*[S(t)-S_\infty]^* dt (I-\sigma\pi)^* C^* \right)$$
$$= \operatorname{tr}\left(C(I-\sigma\pi) P_\infty (I-\sigma\pi)^* C^* \right)$$

where P_∞ is the semistability Gramian defined in Definition 4.22. The operator Lyupunov equation for P_∞ was shown to hold in Theorem 4.25. □

Sufficient conditions for the impulse responses to be Hilbert-Schmidt can be based on constraining the input operator B and the output operator C to be of finite rank, see [CS01]. It is also interesting to note that one recovers a familiar identity if one considers a trivial model reduction $\pi = 0$ one obtains the $\mathcal{H}_2$-norm of Σ itself whenever A is exponentially stable.

Corollary 4.28. *Suppose that $\Sigma(A, B, C)$ is a distributed parameter system on a separable Hilbert space X where A generates an exponentially stable C_0-semigroup $S(t)$ and that B and C are bounded. Then if $S(t)$ is Hilbert-Schmidt the model error is given by*

$$\|\Sigma\|^2_{\mathcal{H}_2} = \sqrt{\operatorname{tr} CPC^*}$$

where P is the controllability Gramian of Σ which for $x \in D(A^*)$ satisfies

$$APx + PA^*x = -BB^*x.$$

We conclude this section by showing that the heated bar example with eigenvalue truncation satisfies the hypotheses of Theorem 4.27 and then use that Theorem to compute the error between a truncated series and the original model.

Example 4.29. Consider again the example with the heated bar on $[0, 1]$ with semigroup

$$S(t)x = \int_0^1 x(q)dq + \sum_{n=1}^\infty 2e^{-n^2\pi^2 t}\cos(n\pi p) \int_0^1 \cos(n\pi q)x(q)dq.$$

which can equivalently be written

$$S(t)x = \langle x, 1\rangle + \sum_{n=1}^\infty e^{-n^2\pi^2 t} \langle x(\cdot), \cos(n\pi \cdot)\rangle \cos(n\pi \cdot).$$

The Hilbert-Schmidt norm can be estimated as

$$\operatorname{tr}(S^*S) = \sum_{n=0}^\infty \langle S\cos(n\pi x), S\cos(n\pi x)\rangle + \sum_{n=1}^\infty \langle S\sin(n\pi x), S\sin(n\pi x)\rangle$$
$$\leq 1 + \sum_{n=1}^\infty e^{-2n^2\pi^2 t} < \infty$$

for each $t \geq 0$, and so $S(t)$ is Hilbert-Schmidt. Moreover, it was previously shown that mode truncation is an invariant model reduction. Since the eigenvalue $\lambda = 0$ corresponds to $\ker A$, if the 0-eigenvalue is included in the projection set $\mathcal{I}$ then one can use π equal to the projection on $\overline{\operatorname{span}}(\cos(n_k \pi p))_{k\in\mathcal{I}} \subset X$ and σ the inclusion into X.

Since we know the solution of the heat equation and if we further suppose that the input operator is $B = I$, we can explicitly compute the semistability Gramian. Using that the second derivative operator and its associated semigroup are self-adjoint operators, we find

$$\begin{aligned} P_\infty &= \int_0^\infty [S(t) - S_\infty][S(t) - S_\infty]^* dt = \int_0^\infty [S(t) - S_\infty][S(t) - S_\infty] dt \\ &= \int_0^\infty S(t)S(t) - 2S_\infty + S_\infty dt = \int_0^\infty S(t)S(t) - S_\infty dt. \end{aligned}$$

Let us examine closer the choice of the model reduction $\pi = \pi_N$ which projects onto the closure of the span of the eigenvectors $\{0, \phi_1, \ldots, \phi_N\}$ and σ the asscoiated inclusion. Then

$$\begin{aligned} (I - \sigma\pi)P_\infty(I - \sigma\pi) &= (I - \sigma\pi)\int_0^\infty S(t)S(t) - S_\infty dt(I - \sigma\pi) \\ &= \int_0^\infty [(I - \sigma\pi)S(t)S(t)(I - \sigma\pi)] - S_\infty dt = \int_0^\infty \hat{S}(t)\hat{S}(t) - S_\infty dt \end{aligned}$$

using first that the Pettis integral commutes with bounded operators, second that $\sigma\pi$ is an identity on $\ker A$ and finally the definition of an invariant model reduction. Observe now that

$$(I - \sigma\pi)S(t)x = \sum_{n=N+1}^\infty e^{-n^2\pi^2 t}\langle x(\cdot), \cos(n\pi\cdot)\rangle \cos(n\pi\cdot)$$

Since $\hat{S}(t) = \hat{S}^*(t)$ also $\bar{S}(t) = \bar{S}^*(t)$, and so the trace is computed as

$$\begin{aligned} \operatorname{tr}((I - \sigma\pi)P_\infty(I - \sigma\pi)) &= \operatorname{tr}\int_0^\infty \bar{S}^*(t)\bar{S}(t)dt - S_\infty dt \\ &= \int_0^\infty \operatorname{tr}(\bar{S}^*(t)\bar{S}(t)dt - S_\infty)dt \\ &= \int_0^\infty \sum_{n=N+1}^\infty \|S(t)e_n\|^2 dt = 2\int_0^\infty \sum_{n=N+1}^\infty \|e^{-n^2\pi^2 t}\cos(n\pi\cdot)\|^2. \end{aligned}$$

If we compute

$$\begin{aligned} \|e^{-n^2\pi^2 t}\cos(n\pi\cdot)\|^2 &= \int_0^1 \left|e^{-n^2\pi^2 t}\cos(n\pi q)\right|^2 dq \\ &= e^{-2n^2\pi^2 t}\int_0^1 |\cos n\pi q|^2 dq = \frac{e^{-2n^2\pi^2 t}}{2} \end{aligned}$$

we can insert this into the earlier expression to obtain, via Theorem 4.27,

$$\begin{aligned}\|\Sigma - \hat{\Sigma}\|_{\mathcal{H}_2} &= \int_0^\infty \sum_{n=N+1}^{\infty} e^{-2n^2\pi^2 t} = \sum_{n=N+1}^{\infty} \int_0^\infty e^{-2n^2\pi^2 t} \\ &= \sum_{n=N+1}^{\infty} \frac{1}{2\pi^2 n^2}.\end{aligned}$$

We have thus obtained an exact expression for $\mathcal{H}_2$-error of our main example. △

In fact, if we analyze the computations in Example 4.29 we notice that the procedure is rather general, as we have really only used Riesz spectral properties save for the final explicit expression. Thus, the reasoning applies at least to the class of self-adjoint Riesz spectral operators which generate exponentially semistable semigroups. It can be imagined that this therefore is a quite useful procedure also more generally.

Observe also that the finiteness of the error in the Riesz case depends crucially on the eigenvalues λ_n and the convergence of a series of the form

$$\sum_{n\in J} \frac{1}{\lambda_n}.$$

Heuristically then, in order for the $\mathcal{H}_2$-error to be finite, we need that the real parts of the eigenvalues of the generator tend to $-\infty$ sufficiently fast.

Remark 4.30. Presented differently, the procedure in Example 4.29 gives a norm bound of the integrated Hilbert-Schmidt norm of a semigroup and an approximating sequence. If we phrase this in numerical PDE language, we have above constructed an error bound for an eigenvalue based discretization scheme and are automatically guaranteed its convergence in Hilbert-Schmidt norm by the semistability of the associated infinitesimal generator. Given that the Hilbert-Schmidt norm is stronger than, for instance, the ordinary operator norm, this is a rather strong statement about the discussed convergence (and we have even considered an integrated version thereof).

In the next section we give further results which may hopefully be useful in applied computations.

4.4 Computational Considerations

Theorem 4.27 gives the model error in terms of the trace of the semistability Gramian introduced in definition 4.22. However, as the solution to the associated Lyapunov equation is not necessarily unique, the only computational tool we have provided thus far requires the explicit computation of the Gramian which in turn necessitates the computation of the Semigroup $S(t)$. If the objective is to apply model reduction to a partial differential equation, this amounts to actually solving the partial differential equation, which may well be the very task one intended to avoid. Here, we further explore the Lyapunov approach for computation of the Gramian.

Lemma 4.31. *Assume that P_1 is a self-adjoint solution of the semistability Lyapunov equation*

$$\langle P_1 x, A^* x' \rangle + \langle A^* x, P_1 x' \rangle = -\langle B^*(I - S_\infty)^* x, B^*(I - S_\infty)^* x' \rangle$$

where A is the infinitesimal generator for an exponentially semistable C_0-semigroup on a separable Hilbert space, X, and suppose $x, x' \in D(A^*)$. If P_2 is another self-adjoint operator, then each of the statements below implies the next. If A in addition is self-adjoint, all the statements are equivalent.

1. *P_2 also satisfies the semistability Lyapunov equation.*
2. *$\Delta = P_2 - P_1$ satisfies for each $x, x' \in D(A^*)$*
$$\langle S_\infty x, \Delta S_\infty x' \rangle = \langle x, \Delta x' \rangle.$$
3. *There exists an operator $\Pi : X \to X$ that Π maps onto a subspace W of $\ker A^*$ such that the solutions satisfy the relation $P_2 = P_1 + \Pi$.*

Proof. We first show that $1 \Rightarrow 2$. Let P_2 be another self-adjoint solution of the Lyapunov equation and consider $\Delta = P_1 - P_2$. For $x, x' \in D(A^*)$, it follows by direct computation that

$$\langle x, \Delta A x' \rangle + \langle A x, \Delta x' \rangle = 0.$$

If we let $x = S(t)x_0, x' = S(t)x'$, this can be rewritten as

$$\begin{aligned} 0 &= \langle S(t)x_0, \Delta A S(t) x_0' \rangle + \langle A S(t) x_0, \Delta S(t) x_0' \rangle \\ &= \langle S(t)x_0, \Delta \frac{d}{dt} S(t) x_0' \rangle + \langle \frac{d}{dt} S(t) x_0, \Delta S(t) x_0' \rangle \\ &= \frac{d}{dt} \langle S(t) x_0, \Delta S(t) x_0' \rangle. \end{aligned}$$

Integrating this equation from 0 to ∞ we obtain

$$\langle S_\infty x_0, \Delta S_\infty x_0' \rangle = \langle x_0, \Delta x_0' \rangle.$$

Now $2 \Rightarrow 3$. To see this, we may simply take $\Pi = \Delta$, since

$$\begin{aligned} &\langle S_\infty x, \Delta S_\infty x' \rangle = \langle x, \Delta x' \rangle \\ \Leftrightarrow &\langle S_\infty^* \Delta S_\infty x, x' \rangle = \langle \Delta x, x' \rangle \end{aligned}$$

and since $x, x' \in D(A^*)$ where $D(A^*)$ is dense in X, we indeed have for any $\bar{x} \in X$

$$\Delta \bar{x} = S_\infty^* \Delta S_\infty \bar{x} = S_\infty^* (\Delta S_\infty \bar{x}) \in \ker A^*.$$

Finally, $3 \Rightarrow 1$ in the case A is self-adjoint. This follows since by construction of Π we have geometrically that Π maps to the kernel of A^*, so $A\Pi = A^*\Pi = 0$ since A is self-adjoint. But then also $0 = (A\Pi)^* = \Pi^* A^*$. Direct computation now shows that

$$\begin{aligned}\langle P_2 x, A^* x' \rangle + \langle A^* x, P_2 x' \rangle &= \langle (P_1 + \Pi)x, A^* x' \rangle + \langle A^* x, (P_1 + \Pi)x' \rangle \\ &= \langle Ix, (P_1 + \Pi)^* A^* x' \rangle + \langle (P_1 + \Pi)^* A^* x, Ix' \rangle \\ &= \langle Ix, (P_1)^* A^* x' \rangle + \langle (P_1)^* A^* x, Ix' \rangle \\ &= \langle P_1 x, A^* x' \rangle + \langle A^* x, P_1 x' \rangle \\ &= -\langle B^*(I - S_\infty)^* x, B^*(I - S_\infty)^* x' \rangle.\end{aligned}$$

□

Using this lemma, we can give an explicit method for computation of the semistability Gramian without explicit reference to the semigroup whenever the generator is self-adjoint.

Theorem 4.32. *Let A be the self-adjoint exponentially semistable generator of a C_0-semigroup $S(t)$ on a separable Hilbert space, X, and let B be bounded. Suppose further that P is an arbitrary solution to the semistability Lyapunov equation*

$$\langle Px, A^* x' \rangle + \langle A^* x, Px' \rangle = -\langle B^*(I - S_\infty)^* x, B^*(I - S_\infty)^* x' \rangle$$

then the semistability Gramian can be computed as

$$P_\infty = P - S_\infty P.$$

In particular, P_∞ is the unique solution to the semistability Lyapunov equation satisfying the constraint

$$P_\infty = (I - S_\infty) P_\infty.$$

Proof. Observe that

$$S_\infty S(t) = \lim_{s \to \infty} S(s) S(t) = \lim_{s \to \infty} S(t + s) = S_\infty$$

and by Lemma 4.13 we already have $S_\infty^2 = S_\infty$. This implies that

$$S_\infty [S(t) - S_\infty] = 0.$$

which in turn implies that

$$S_\infty P_\infty = 0.$$

If P is any other solution to the semistability Lyapunov equation, substituting the third characterization of Lemma 4.31 yields

$$S_\infty (P + \Pi) = 0$$

or

$$S_\infty \Pi = -S_\infty P$$

and since S_∞ acts identically and is idempotent on $\operatorname{im} \Pi \subseteq \ker A^* = \ker A$ we obtain

$$\Pi = -S_\infty P.$$

□

Remark 4.33. Since A is self-adjoint it follows that so is S_∞ which means that S_∞ actually is the projection onto the kernel of A. Therefore $I - S_\infty$ is in turn the projection onto the complement of the kernel of A so that the semistability Gramian actually is a minimal solution in norm; $\|P_\infty\| \leq \|P\|$ for all solutions P of the semistability Lyapunov equation since $\|S_\infty\| \leq 1$.

Theorem 4.32 is actually quite strong as it does not at all depend on the restrictive assumptions made about the structure of the model reduction in Theorem 4.27. Its significance here is that it allows us to compute the model error in Theorem 4.27 without explicit mention of the semigroup $S(t)$. Instead one may apply the following program:

1. Compute the kernel of A.
2. Find an arbitrary solution, P, of the semistability Lyapunov equation.
3. Apply the projection onto the orthogonal complement of the kernel of A according to $P_\infty = (I - S_\infty)P$.
4. Compute the trace as in Theorem 4.27.

Observe that S_∞ is known without computing $S(t)$, since, if A is self-adjoint, it is just the projection onto the kernel of A. We also wish to add a disclaimer; by no means we suggest that the solution of the operator Lyapunov equation to be particularly easy. What we are saying is that the method proposed above circumvents the computation of the semigroup which typically involves solving a partial differential equation for arbitrary initial conditions.

Remark 4.34. The lemma and theorem in this subsection are an adaption of corresponding results in [CS16] which only applies to finite-dimensional systems where A is the negative of a Laplacian matrix of a graph which is semistable with $\dim \ker A = 1$. By reinterpreting their proofs in geometric language, we allow for arbitrarily large instability, possibly $\dim \ker A = \infty$ as long as A generates a semistable C_0-semigroup.

5 Discussion and Conclusion

In this thesis we extended the classical trace-$\mathcal{H}_2$ formula to infinite-dimensional systems and have applied this to a particular class of model reductions of exponentially semistable systems. We have also extended the characterization of the Gramian, P_∞, appearing in this formula to an operator-theoretic setting as the solution of a Lyupunov equation and managed to uniquely identify this solution even in the case where the system under consideration is only exponentially semistable. Moreover, this class of model reductions seems suitable for engineering applications as these reductions preserve both our notion of controllability and stability. Throughout the thesis, we have used the second derivative operator as an example. This has allowed us to also uncover an intresting connection between model reduction of infinite-dimensional control systems and numerical schemes for partial differential equations, see Example 4.29 and the remark following it.

The main drawback of the theory presented above is that the class of invariant model reductions is not as general as one could hope. For instance, the clustering projection [CKS17] was seen to not be included in this class. The main reason for this is that both here and in the article by Cheng, Kawano and Scherpen, the derivation of the trace formula involved specific properties of the structure of the Laplacian matrix and its relation to their model reduction technique in terms of the nodes of the graph that particular Laplacian represented. Since we do not have a particular structure for our operators, other than that they are semistable, this structure has to be obtained elsewhere, and in this case the choice was made to impose it through the model reduction and its interaction with the kernel of the driving operator. This does, however, beg the question if more general results could be obtained in terms of the class of model reductions used. Given the existence of formulas for additive and multiplicative perturbations of semigroups we deem this likely and thus think that our results should be seen merely as a first step, or an indication, toward these hypothetical, more general results. For such perturbation results, see [EN99].

5.1 Ideas for Further Research

Alternately, one may be interested in considering operators which have some structure, similar to that of a Laplacian matrix, such as the class of elliptic partial differential operators generated by covariance matrices as $A = -\nabla \cdot C \cdot \nabla$ or similar, where $C \geq 0$ is a covariance matrix for some n-dimensional process. If one identifies each basis vector in the ambient Euclidean space for the covariance matrix with a physical, or financial property, such as the value of a stock, one could perhaps again apply a clustering-like model reduction where stocks that are highly correlated with each other are bundled into an index (or cluster). Processes driven by such generators are not without interest, since they occur frequently in the stochastic differential equation literature as the generators of certain Itô diffusions, see in particular the discussion of the Feynman-Kac Theorem in [Øks03]. Such processes are abundant in Financial Mathematics, which may serve as further motivation to endeavour in this direction.

If one instead considers the optimal control of such processes, Itô diffusions, one obtains a partial differential equation known as the Hamilton-Jacobi-Bellman equation which is somewhat reminiscent in structure of those considered here but is, however, nonlinear. This in itself precludes many of the arguments used in our work to be translated directly as they rely much on linear structures, such as the Lyapunov equation. Nevertheless, when phrased appropriately, it can be shown that the associated optimal control problem has a nonlinear as semigroup infinitesimal generator, known as the Hamiltonian of that problem, so perhaps there is some hope for generalization after all. A nonlinear extension of the theory presented here would thus allow for treatment of the Hamilton-Jacobi-Bellman equation, which notoriously suffers from the curse of dimensionality (see [Bel57]) and so reduction of this class of partial differential equations would be of applied interest. The semigroup approach to optimal control of diffusions is discussed in detail in [FS06].

We also think that Example 4.29 illustrates interesting connections to the numerical analysis of partial differential equations, where Hilbert-Schmidt norms of approximations to PDEs have already been considered in for instance [KLS15] and [Ros91]. This connection may be interesting to investigate further.

For no other reason than that the author's interests currently are geared toward partial differential equations, we have focused on these as the main application of distributed parameter control. Another class of problems which we have largely ignored in this thesis are those concerned with delay differential systems of the form $\dot{x}(t) = Ax(t-\tau) + Bu$ for some $\tau > 0$. One may imagine that the lag τ means that one has to expand the state space to include a value for each t in an interval of length τ, making it infinite-dimensional. It is shown in [CZ12] that these also fall under the class of C_0-semigroup distributed parameter control systems so that our results apply. Results much reminiscent of ours for this subclass, but avoiding the semigroup approach, can be found [JVM11] and [JDM13] where the authors also give a numerical scheme to solve the associated Lyapunov equation. It would certainly be interesting to investigate in more detail how our methods could apply to their problems or perhaps how their methods could be extended to the more general situation presented here.

A Background in Functional Analysis

As this thesis deals extensively with operators on normed linear spaces, we will want to make precise the terminology used with respect to these. Our brief overview of the functional analytic topics treated here is mostly inspired by [Rud91] and [RS72]. [Kat13] is also used as a more advanced reference. For introductory material we mainly refer to [Fri70] and [Lue97].

A normed linear space $(X, \|\cdot\|)$ is a vector space X equipped with a norm $\|\cdot\|$. Recall that a normed linear space is complete if every Cauchy Sequence is indeed a convergent sequence. If the normed linear space is complete with respect to the norm topology it is known as a Banach space, and if it in addition to being complete furthermore is equipped with an inner product $\langle\cdot,\cdot\rangle$ which satisfies $\langle\cdot,\cdot\rangle = \|\cdot\|^2$ it is a Hilbert space. Sometimes we will have to deal with several different normed linear spaces simultaneously which are somehow related and if the norm is not clear from context, we shall equip it with the ambient space as a subscript, $\|\cdot\|_X$ and similarly for Hilbert spaces.

Definition A.1. By an operator between normed linear spaces, X, Y, we shall mean a linear map, defined on a subspace $D(A) \subseteq X$, $A : D(A) \to Y$.

Thus in its most general form presented here, an operator will not be defined on the entire space, nor shall it be bounded, which entails:

Definition A.2. An operator $A : X \to Y$ is bounded if its $\mathfrak{B}(X,Y)-$norm is finite. This norm is defined by

$$\|A\|_{\mathfrak{B}(X,Y)} = \sup_{x\in X, \|x\|=1} \|Ax\|_Y.$$

It can be shown that $\mathfrak{B}(X,Y)$, the space of all bounded operators defined on the entire space X and codomain Y, constitutes a Banach space in its own right under the norm above, see Chapter 4.4 in [Fri70]. Note that if at least one of two operators A, B is unbounded, then operator addition and multiplication have to be done with great care. The natural domains are

$$\begin{aligned} D(A+B) &= D(A) \cap D(B), \\ D(AB) &= \{x \in D(B) : Bx \in D(A)\}. \end{aligned}$$

Under these hypotheses the associative laws of addition and multiplication hold. However, for operators A, B, C, as for the distributive laws, one only has

$$\begin{aligned} (A+B)C &= AC + BC, \\ C(A+B) &\subset CA + CB, \end{aligned}$$

where the abuse of notation $A \subset B$ means that $A = B$ on $D(A) \subset D(B)$. One also says that B is an extension of A. A more detailed discussion is found in Chapter 13 of [Rud91]. Unfortunately, boundedness is a condition far too strong for most differential operators so these considerations cannot be forgone. It turns out however, that a weaker, yet highly useful condition is more often satisfied.

Definition A.3. Let X, Y be normed linear spaces and $A : D(A) \subseteq X \to Y$ a linear operator. The graph of A, $G(A)$ is the set $\{(x, Ax) \mid x \in D(A)\}$. A is then said to be closed if $G(A)$ is a closed subset of $X \times Y$ in the topology generated by $\|(x,y)\|_{X\times Y} = \|x\|_X + \|y\|_Y$.

Similarly, one can define $\langle \cdot, \cdot \rangle_{X\times Y} = \langle \cdot, \cdot \rangle_X + \langle \cdot, \cdot \rangle_Y$. We also observe that it immediately follows by the Closed Graph Theorem (Chapter 4.6 in [Fri70]) that an operator A is bounded if and only if $D(A) = X$ and A is closed. This again illustrates the importance of the domain when considering operators in general.

Definition A.4. Suppose that $A : X \to X$ is an operator and that $D(A)$ is dense in X. Then one defines the adjoint of A, as the operator A^* defined on

$$D(A^*) = \{y \in X \mid \exists y^* \in X \text{ for which } \langle Ax, y\rangle = \langle x, y^*\rangle, \forall x \in X\}.$$

The adjoint is then defined by $A^*y = y^*$ where y^* satisfies the above. To be precise, A^* satisfies $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $y \in D(A^*)$.

If $A = A^*$ and $D(A) = D(A^*)$ then one says that A is self-adjoint. The adjoint and graph of A are intimitely related.

Lemma A.5. *Suppose that A is a densely defined operator on A. Then $G(A^*) = [JG(A)]^\perp$ where $J : X \times X \to X \times X$ is defined by $J(x,y) = (-y, x)$. In particular, if A is closed then $X \times X = [JG(A)] \oplus G(A^*)$.*

Proof. Observe that the following statements are equivalent:

- $(y, y^*) \in G(A^*)$.
- $(Ax, y) = (x, y^*)$ for every $x \in D(A)$.
- $(-Ax, x) \perp (y, y^*)$ for every $x \in D(A)$.
- $(y, y^*) \in [J(G(A)]^\perp$.

The second statement follows from the first since $J^2 = -I$ so that J is unitary and since A is closed $G(A)$ is a closed subset of $X \times X$, $X \times X$ can be written as an orthogonal direct sum. □

The above lemma is extremely important, and in one way or another underpins each of the properties of the adjoint as established in the next lemma.

Lemma A.6. *Suppose that A is a densely defined operator on X and suppose that $B \in \mathfrak{B}(X)$. Then:*

1. *If A is closed then so is A^* and furthermore $D(A^*)$ is dense in X.*
2. *If A is bounded so is A^*.*
3. *$(A + B)^* = A^* + B^*$ with domain $D((A + B)^*) = D(A^*)$.*

4. $(BA)^* = A^*B^*$ *with domain* $D((BA)^*) = D(A^*)$.

5. *If* $A^{-1} \in \mathfrak{B}(X)$ *then* $(A^*)^{-1} \in \mathfrak{B}(X)$ *and* $(A^*)^{-1} = (A^{-1})^*$.

Proof. 1. Let $z \perp D(A^*)$. We want to show that this implies $z = 0$. To see this, for $y \in D(A)$ we have $\langle z, y\rangle = 0$ and so

$$\langle (0, z), (-T^*y, y)\rangle = 0.$$

But then $(0, z) \in [JG(A^*)]^\perp = G(A)$ which means that $z = A(0) = 0$ as required.

2. Recall

$$D(A^*) = \{y \in X \mid \exists y^* \in X \text{ for which } \langle Ax, y\rangle = \langle x, y^*\rangle, \forall x \in X\}.$$

Now, since A is bounded, $\langle A\cdot, y\rangle$ is a bounded linear functional and the existence of such a y^* follows by the Riesz Representation Theorem (see [Fri70]) for every $y \in X$. Thus $D(A^*) = X$ and again since A is bounded it is certainly closed, and thus A^* is also closed by the first point. We have shown that A^* is closed with domain X so it is bounded (by the Closed Graph Theorem).

3. Suppose $\exists y, y^* \in X$ with $\langle (A + T)x, y\rangle = \langle x, y^*\rangle$ for all $x \in X$. In this case

$$\langle Ax, y\rangle = \langle x, y^* - B^*y\rangle$$

and so $D((A + B)^*) \subset D(A^*)$ and $(A + B)^* = A^* + B^*$ on $D(A^*)$. The argument can be reversed for the other inclusion.

4. Without reference to boundedness of B we can obtain $A^*B^* \subset (BA)^*$. Namely if $x \in D(BA)$ and $y \in D(A^*B^*)$ we get

$$\langle Ax, B^*y\rangle = \langle x, A^*B^*y\rangle.$$

As $x \in D(A)$ and $B^*y \in D(B^*)$ we find

$$\langle BAx, y\rangle = \langle Ax, B^*y\rangle$$

and so, since $Ax \in D(B)$ and $y \in D(B^*)$, we obtain

$$\langle BAx, y\rangle = \langle x, A^*B^*y\rangle$$

proving the first inclusion.

If B is bounded, one obtains $D(B^*) = X$ from which

$$\langle Ax, B^*y\rangle = \langle BAx, y\rangle = \langle x, (BA)^*y\rangle$$

proving that also $(BA)^* \subset A^*B^*$.

5. If $A^{-1} \in \mathfrak{B}(X)$ by the second statement also $(A^{-1})^* \in \mathfrak{B}(X)$. Now for $x, y \in X$

$$\langle((A^{-1})^* A^* x, y\rangle = \langle A^* x, A^{-1} y\rangle = \langle x, AA^{-1} y\rangle = \langle x, y\rangle.$$

Since this holds for every $x, y \in X$, we have that $(A^*)^{-1} = (A^{-1})^* \in \mathfrak{B}(X)$.

□

The lemma makes computation with unbounded operators easier. Using it, we will now give an example of a closed and self-adjoint operator which is neither bounded nor has an inverse.

Example A.7. Let $X = L^2[0,1]$ and define $A = \frac{d^2}{dp^2}$ with

$$D(A) = \left\{x \in L^2[0,1] \middle| x, \frac{dx}{dp} \in AC[0,1], \frac{d^2x}{dp^2} \in L^2[0,1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\right\}$$

where $AC[0,1]$ is the space of absolutely continuous functions on $[0,1]$. A is not bounded, since for the sequence

$$x_n = \frac{\cos(nx)}{\|\cos(nx)\|}$$

we have $\|x_n\| = 1$ but $\|Ax_n\| = n^2$ for all n by construction. It is also not too hard to see that A does not have an inverse, since $\{0\} \neq \operatorname{span}(1) = \ker A$.

Nevertheless, we can construct an inverse of $A + I$ as follows. Define on X,

$$Bx(p) = \int_0^p g(p,q)h(q)dq + \int_p^1 g(q,p)h(q)dq \text{ where}$$
$$g(p,q) = \cot(1)\cos(p)\cos(q) + \sin(q)\cos(p).$$

It is easy to see that this integral opeartor is in $\mathfrak{B}(X)$. Now, defining the function $y(p) = Bx(p)$, we obtain

$$\begin{aligned}\frac{dy}{dp}(p) = &\int_0^p [\cos q \cos p - \cot(1)\cos(q)\sin(p)]x(q)dq \\ &- \int_p^1 [\sin(q)\sin(p) + \cot(1)\cos(q)\sin(q)]x(q)dq.\end{aligned}$$

After a second round of differentiation, we are left with

$$\frac{d^2y}{dp^2}(p) = x(p) - y(p).$$

In particular $(A + I)B = I$ and so B is the inverse of $A + I$. Since B is bounded, it is also closed. Moreover, the graph of B is topologically equivalent to that of its inverse, so the graph of $A + I$ is also closed. Finally, the class of closed operators is stable under continuous perturbations and so $A = A + I - I$ is also closed.

Moreover, we can compute the adjoint of $(I+A)^{-1}$. Namely

$$\begin{aligned}
&\langle (I+A)^{-1}x, y\rangle \\
&\quad = \int_0^1 \int_0^p g(p,q)x(q)dq\overline{y(p)}dp + \int_0^1 \int_x^1 g(q,p)x(q)dq\overline{y(p)}dp \\
&\quad = \int_0^1 \int_q^1 g(p,q)x(q)\overline{y(p)}dqdp + \int_0^1 \int_0^q g(q,p)x(q)\overline{y(p)}dqdp \\
&\quad = \int_0^1 \overline{y(p)} \int_q^1 g(p,q)x(q)dqdp + \int_0^1 \overline{y(p)} \int_0^q g(q,p)x(q)dqdp \\
&\quad = \int_0^1 y(p)\overline{\int_q^1 g(p,q)x(q)dqdp} + \int_0^1 y(p)\overline{\int_0^q g(q,p)x(q)dqdp} \\
&\quad = \langle x, (I+A)^{-1}y\rangle.
\end{aligned}$$

Since $(I+A)^{-1}$ is bounded, this is sufficient to conclude that $(I+A)^{-1}$ is self-adjoint. It follows that also $(I+A)^* = I+A$ and thus also $A = A^*$. △

Remark A.8. Throughout this thesis, this example for an operator will be a reoccuring theme. It should also be noted that there is essentially nothing stopping us from using its higher dimensional analogue, the Laplace operator, except that this would force us to consider Sobolev spaces explicitly. Indeed, the natural domain for both $\frac{d^2}{dt^2}$ and the Laplace operator is a Sobolev space called H^2. However, when the ambient Euclidean space has dimension 1, the characterization of $D(A)$ is also possible since absolute continuity is sufficient for almost everywhere differentiability and allows us to bypass Sobolev space theory which we do not wish to treat in this thesis. More about this can be found in Chapter 5 of [Eva98] and Chapter 7 of [GT15].

A.1 Elements of Spectral Theory

Definition A.9. If X is a normed linear space, and A an operator on X, one makes the following definitions.

- The *resolvent set* of A, denoted $\rho(A)$ is the set of $\lambda \in \mathbb{C}$ such that $\lambda I - A$ has a bounded inverse. In this case $R(\lambda; A) = (\lambda I - A)^{-1}$ is known as the resolvent operator of A.
- The *spectrum* of A, denoted $\sigma(A)$ is the complement of the resolvent set. That is, $\lambda \in \sigma(A)$ iff $\lambda I - A$ does not have a bounded inverse.

The following properties of the resolvent are useful in computations.

Lemma A.10. *The resolvent operator has the following properties:*

1. *For $\mu, \lambda \in \rho(A)$, one has $R(\mu; A) - R(\lambda; A) = (\lambda - \mu)R(\lambda; A)R(\mu; A)$. In particular $R(\lambda; A), R(\mu; A)$ commute.*

2. $R(\lambda; A)^* = R(\bar{\lambda}; A^*)$.

Proof. 1. Write

$$\begin{aligned} R(\mu; A) - R(\lambda; A) &= R(\mu; A)[A - \lambda I]R(\lambda; A) - R(\lambda; A)[A - \mu I]R(\mu; A) \\ &= (\lambda - \mu)R(\lambda; A)R(\mu; A) \end{aligned}$$

since $R(\lambda; A)$ commutes with $A - \lambda I$ for any λ in the resolvent set by definition of the resolvent set.

2. This is an immediate consequence of Lemma A.6, points 3 and 5.

□

In contrast to the resolvent, the spectrum of an operator is in general anything but nice. Unlike the finite dimensional situation, there are several ways in which an element of $\mathbb{C}$ can fail to be in the resolvent set and not all such λ in the spectrum are eigenvalues.

Definition A.11. If $\lambda \in \sigma(A)$, one makes the following distinctions.

1. If $\lambda I - A$ has dense range and $(\lambda I - A)^{-1}$ exists but is unbounded then λ is said to lie in the continuous spectrum of A, $\sigma_c(A)$.
2. If $(\lambda I - A)^{-1}$ exists but its domain is not dense in Y, then λ is said to lie in the residual spectrum of A, $\sigma_r(A)$.
3. If $\lambda I - A$ fails to be injective, λ lies in the point spectrum of $A, \sigma_p(A)$.

The point spectrum of A is often referred to as the set of eigenvalues of A and it is clear that $\lambda \in \sigma_p$ if and only there exists $x \neq 0$, such that $\lambda Ix - Ax = 0$. In this case, x is known as the eigenvector associated to λ. When X is a function space and A a differential operator, the eigenvectors are also sometimes known as eigenfuntions. Returning to our standard example, we characterize the eigenvalues and eigenvectors of the second derivative operator on $[0, 1]$ below.

Example A.12. Consider again $X = L^2[0, 1]$ and $A = \frac{d^2}{dp^2}$ with

$$D(A) = \Big\{x \in L^2[0,1] \Big| x, \frac{dx}{dp} \in AC[0,1], \frac{d^2x}{dp^2} \in L^2[0,1], \frac{dx}{dp}(0) = \frac{dx}{dp}(1) = 0\Big\}.$$

Now, v is an eigenvector iff $\frac{d^2v}{dp^2} = \lambda v$. This is just an ordinary differential equation for v and its solutions satisfying $v \in D(A)$ are $\lambda_n = -n^2\pi^2, v(p) = \cos(n\pi p)$ which thus constitute the eigenvalues and eigenvectors of the second derivative operator. △

In the sequel, we will need further properties of the resolvent, in particular, as we will later show, that the Laplace transform of a strongly continuous semigroup coincides with the resolvent of its generator and so statements concerning the resolvent translate directly to the frequency domain analysis of strongly continuous semigroups. Above all, the resolvent operator is highly regular, and enjoys an analyticity property similar to the function $\frac{1}{1-x}$ known from complex analysis.

Lemma A.13. *For any closed operator A, the resolvent set $\rho(A)$ is open and on $\rho(A)$ the resolvent $R(\lambda; A)$ has a power series expansion. Furthermore, the resolvent is holomorphic thereon in the sense that its weak derivative exists and satisfies*

$$\frac{d}{d\lambda} R(\lambda; A) = -R(\lambda; A)^2.$$

Proof. Repeated application of the resolvent equation yields for $\lambda, \mu \in \rho(A)$ and arbitrary natural number n that

$$(\lambda I - A)^{-1} = (\mu - \lambda)^{n+1} (\mu I - A)^{-n-1} (\lambda I - A)^{-1} + \sum_{k=0}^{n} (\mu - \lambda)^k (\mu I - A)^{-k-1}.$$

Since the resolvent is bounded for any μ, one may choose λ sufficiently close to μ such that $\|(\mu - \lambda)(\mu I - A)^{-1}\| < 1$ in which case actually the series converges uniformly and one has for such λ, μ

$$(\lambda I - A)^{-1} = \sum_{k=0}^{\infty} (\mu - \lambda)^k (\mu I - A)^{-k-1}.$$

The key observation is that, first, there actually is a power series expansion of the resolvent, and second, that this holds in a neighborhood of μ, which thus proves the first two statements. The final statement follows by examing, for $x, x' \in X$, that the (weak) difference satisfies

$$\langle x, R(\lambda + h; A)x' \rangle - \langle x, R(\lambda; A)x' \rangle = \langle x, -hR(\lambda; A)R(\lambda + h; A)x' \rangle.$$

by the resolvent equation. Thus

$$\frac{\langle x, R(\lambda + h; A)x' \rangle - \langle x, R(\lambda; A)x' \rangle - \langle x, -hR(\lambda; A)R(\lambda + h; A)x' \rangle}{|h|}$$

which has the same limit when $h \to 0$ as the quotient

$$\frac{\langle x, R(\lambda + h; A)x' \rangle - \langle x, R(\lambda; A)x' \rangle - \langle x, -hR(\lambda; A)R(\lambda; A)x' \rangle}{|h|}$$

so that we may conclude the weak derivative of the resolvent is $-R(\lambda; A)^2$. □

A more detailed discussion of the relationship between spectral theory and complex analysis is found in Chapter 7 of [Kre78] and in Chapter 3 of [HP96]. In the second reference it is also shown that weak differentiability of a complex variable implies uniform differentiability, thus justifying us writing $\frac{d}{d\lambda} R(\lambda; A) = -R(\lambda; A)^2$ for the complex derivative. Essentially, the power series representation for the resolvent also implies holomorphicity via an infinite-dimensional analogue of the Cauchy-Hadamard Theorem, but we do not wish to get into complex analytic details so we proved these facts separately.

A.2 Integration and Traces of Operators

The primary notion of integrability in this thesis is that of the Bochner Integral. As is in the the theory for the Lebesque integral, the Bochner integral is well-defined on a suitable class of measurable functions.

Definition A.14. Let $f : \Omega \to X$ be a function from a set Ω to a separable Hilbert space, X. If there exists a sequence (f_n) of simple functions convering to f in the norm-topology of X_2 we say that f is (strongly Bochner) measurable. Moreover, the integral of f is defined as the strong limit

$$\int_\Omega f dt = \lim_{n \to \infty} \int_\Omega f_n dt \tag{7}$$

If the limit (7) exists for a function f, it is said to be (Bochner) integrable.

The key point is that both Fubini's Theorem as well as Lebesque's Dominated Convergence Theorem and Lebesque's Differentiation Theorem continue to hold for Bochner integrals, see [DUJ77] and [HP96]. Unfortunately, this notion is in general too strong for us to able to integrate operators in full generality. We thus introduce an even weaker notion of integral, known as the Pettis integral.

Definition A.15. Let X, X' be separable Hilbert spaces and let $F : \Omega \to \mathfrak{B}(X, Y)$. If the complex-valued function $\langle x', F(t)x\rangle, t \in \Omega$ is integrable for all $x \in X, x' \in X'$ we say that $F(t)$ is Pettis integrable. Moreover, this integral is defined by

$$\langle x', \int_\Omega F(t)dtx\rangle = \int_\Omega \langle x', F(t)x\rangle dt$$

An overview of these integrals is given [CZ12] For details, we refer to the highly readible text [DUJ77] and also [DS58]. A useful property of these integrals is that they commute with bounded linear operators.

Another notion that needs to be extended to operators is that of a trace. Nuclear operators are roughly speaking operators for which this extension occurs naturally. As the trace of the controllability map and Gramian allows one to characterize the norm of a certain systems, the extension of this notion is relevant to our study.

Definition A.16. Let X be a separable Hilbert space, $(e_i)_{i=1}^{\infty}$ an orthonormal basis and $B \in \mathfrak{B}(X)$. Then

$$\mathrm{tr} B = \sum_{i=1}^{\infty} \langle Be_i, e_i\rangle$$

is called the trace of B.

If $\mathrm{tr}|B| < \infty$ then B is said to be of trace class or 1-nuclear. Similarly, if $\mathrm{tr} B^*B < \infty$ then B is said to Hilbert-Schmidt or 2-nuclear.

The separability assumption is necessary for the concept in the above form to be well-defined since a Hilbert space X admits a countable orthonormal basis if and only if it is separable, see [Fri70]. Luckily, the L^p spaces are separable so this poses no immediate restriction to us.

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