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Real Homotopy Theory of the Framed Little N-disks Operad

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Erik Lindell

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Erik Lindell

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REAL HOMOTOPY THEORY OF THE FRAMED LITTLE n-DISKS OPERAD

ERIK LINDELL

Abstract. The subject of this thesis is the real homotopy theory of the framed little n-disks operad. In particular, we study *graph complexes* that serve as algebraic models for this operad. Such a model was found by Anton Khoroshkin and Thomas Willwacher in their paper [KW17] and in this paper we prove that a similar, but arguably more useful, complex is also such a model. The article is written to be comprehensible for students from the master's level and upwards, so a background on operads is given, as well as a quite thorough background on the algebraic tools that are used in the proofs.

Contents

1. Introduction

A graph complex is a differential graded vector space where the elements are formal linear combinations, or possibly series, of graphs. Many problems from topology and geometry can be reformulated as problems concerning the (co)homology of some graph complex. For example, graph complexes can be used as algebraic models for other mathematical structures, such as *operads*, which are the objects of interest in this paper. Specifically, we shall use graph complexes to study one of the currently most imporant operads in algebraic topology: *the framed little n-disks operad*.

There exist several reasons for the significance of the framed little n-disks operad. For example, it plays an imporant role in the current study of the real and rational homotopy theory of embeddings spaces of manifolds. It can be proven (see for example [BW13, Proposition 6.1]) that if M and N are manifolds such that $\dim N - \dim M \geqslant 3$, then there there exists a weak equivalence

$$\mathrm{Emb}(M,N) \xrightarrow{\sim} \mathrm{Hom}^h_{D_n^{fr}}(\mathrm{Emb}_M, \mathrm{Emb}_N), \tag{1}$$

where $\mathcal{D}_n^{fr}$ is the framed little n-disks operad, Emb_M and Emb_N are certain right modules $\mathcal{D}_n^{fr}$ and $\mathrm{Hom}^h_{D_n^{fr}}(\mathrm{Emb}_M, \mathrm{Emb}_N)$ is the derived mapping space (see for example [Hirschhorn04, Chapter 8] for a definition) of right $\mathcal{D}_n^{fr}$-module maps between them.

The subject of this paper is the real homotopy theory of the framed little n-disks operad. In particular, we study *algebraic models* of this operad. By a model of a topological operad $\mathcal{T}$, such as the framed little n-disks operad, we mean an operad in the category of differential graded vector spaces that is weakly equivalent to the operad of chains on $\mathcal{T}$. Up to homotopy, such a model thus completely characterizes the structure of $\mathcal{T}$. This means that in situations where we are only interested in a topological operad $\mathcal{T}$ up to homotopy, such as when studying the homotopy theory of $\mathrm{Emb}(M,N)$ using (1), we may instead work with a model of $\mathcal{T}$, which may be more practical. The models we are going to work with are all graph complexes.

We define algebraic models of operads properly in Section 3.3. We then introduce the necessary theory to construct a model of the framed little n-disks operad, which was introduced in the paper [KW17] by Anton Khoroshkin and Thomas Willwacher, and is based on Kontsevich's graph complex graphs_n, introduced in [Kontsevich99]. In the last section of the paper, we construct a new graph complex, which we prove is also a model for the framed little n-disks operad. This is a new and original result. The benefit of this new model is mainly that it comes with an action by a large dg Lie algebra, which makes it possible to use this model (or actually a slight extension of this model) to compute the so called homotopy derivations of the framed little n-disks operad. This is work that is currently being done by Simon Brun and Thomas Willwacher.

1.1. **Plan of the paper.** This paper is intended for mathematicians from the master's student level, such as the author himself, and upwards. The ambition is therefore that the paper should be possible to follow for anyone who has not studied more than some basic homological algebra and algebraic topology, with maybe only a few glances at the sources if necessary. For this reason we start in **Section 2** by giving a brief overview of the basics and conventions that will be used, with sources for the reader unfamiliar with these topics. This section can thus safely be skipped or skimmed by the more experienced reader, or referred back to when necessary.

We start the exposition in **Section 3**, with an introduction to operads. A number of supplementary examples is given for the reader unfamiliar with operads, while the rest of the section is quite narrowly focused on properties and constructions that we will use later. We end the section by properly defining the framed little n-disks operad. For a thorough introduction to operads, see [LV12], which is the standard introductory work to the subject.

We will see that the framed little n-disks operad can be constructed as an operadic semidirect product between the original little n-disks operad and the special orthogonal group. This group will play an important role for the rest of the paper for this reason, which is why it is the subject of **Section 4**. Here we recall the real homology of $SO(n)$ and of its classifying space $BSO(n)$ and then construct the Koszul complex $H_\bullet(BSO(n)) \otimes H_\bullet(SO(n))$, which we prove is an acyclic chain complex.

In **Section 5**, we move on to discuss the *Maurer-Cartan* equation in dg Lie algebras. Elements satisfying this equation will be of crucial importance in our later constructions. We prove some of their basic properties, demonstrate their connection to Quillen's functors $\mathcal{C}_*$ and $\mathcal{L}$ and introduce the resolution $\mathcal{LC}_*(L)$ of a dg Lie algebra L. We also show how this resolution can be used to resolve $H_\bullet(SO(n))$, a result that we will need to use again in the last section of the paper.

After this, in **Section 6**, we prove some lemmas from homological algebra that will be used repeatedly in our later proofs. The first lemmas we prove come from the theory of spectral sequences. Since we have no need for the full machinery of spectral sequences, however, we will not give a proper introduction to the subject in this section and never actually define what a spectral sequence is. The reader familiar with the subject can safely skip this section, after taking note of Propositions 6.6 and 6.9 and Example 6.8, which will be referred to at several later points. A reader who wants a brief introduction to spectral sequences can see for example [FHT01, Chapter 18].

In **Section 7**, we finally move on to introduce graph complexes and define Kontsevich's complexes graphs_n and GC_n, which lay the basis for the rest of the paper, since it is from these two that all remaining complexes we introduce will be constructed. The most important section is 6.4, where we prove that GC_n acts on graphs_n as a dg Lie algebra.

With this, we have the necessary background to demonstrate the construction of Khoroshkin and Willwacher's model for the framed little n-disks operad, which is what we do in **Section 8**. Here we use most of the results from Section 5. The key finding of Khoroshkin and Willwacher is a Maurer-Cartan element in the dg Lie algebra $H^\bullet(BSO(n))\widehat{\otimes}\mathsf{GC}_n$. In the case where n is even, this enables us to construct an action by $H_\bullet(SO(n))$ on graphs_n, in a way such that the resulting semi-direct product $\mathsf{graphs}_n \circ H_\bullet(G)$ is a model for the framed little n-disks operad. For n even, they also prove that the this complex is a model for the homology of the framed little n-disks operad, which proves that the operad is *formal.*

When n is odd, the Maurer-Cartan element has a more complicated form, which forces us to replace $H_\bullet(G)$ with a resolution. By the results that we will prove in Section 5, the Maurer-Cartan element gives us a map of dg Lie algebras from the resolution $\mathcal{LC}_*(\pi^{\mathbb{R}}(SO(n)))$ to GC_n (where we view $\pi^{\mathbb{R}}(SO(n))$ as an abelian dg Lie algebra with the zero differential). By composing with the action by GC_n on graphs_n and then applying the universal enveloping algebra functor, we get a Hopf algebra action

$$\widehat{H}_\bullet(SO(n)) := \mathcal{U}(\mathcal{LC}_*(\pi(G))) \circlearrowright \mathsf{graphs}_n.$$

Since the universal enveloping algebra functor preserves quasi-isomorphisms, this is a resolution of $H_\bullet(G) = \mathcal{U}\pi^{\mathbb{R}}(SO(n))$. This allows us to construct the semi-direct product

$$\mathsf{graphs}_n \circ \widehat{H}_\bullet(SO(n)),$$

which Khoroshkin and Willwacher prove is a model for the framed little n-disks operad, in their paper. In this case, they also prove that the operad is *not* formal.

This model allows us to find new models for the operad by constructing complexes that are weakly equivalent to this complex, instead of having to prove that they model the framed little n-disks operad directly. This is the subject of **Section 9**, where the original results of the paper are contained. In the complex defined by Khoroshkin and Willwacher the operadic composition is "twisted" by the aformentioned Maurer-Cartan element. Instead, we define a complex which has a natural action by the Lie algebra of this element, which allows us to use it to twist the differential of the complex. This complex, which we will denote by graphs_n^{dec} thus has a more complicated differential, but a less complicated operadic structure. The goal of this section is to prove that this complex is weakly equivalent to the original one, by constructing an explicit zigzag connecting them. Even though we know that the operad is formal in the even case, we start by constructing the zigzag in this case, as an illustration of the idea, and then extend this idea to the odd case. Since the operad is non-formal in the case of odd n, this is the case we are mainly interested in.

1.2. **Acknowledgments.** My deepest gratitude goes to my supervisor, Thomas Willwacher, for agreeing to supervise this thesis and for all the guidance and wisdom he has given me throughout the project. The credit for the ideas behind the new results in this thesis should all be given to him.

Next, I want to thank Alexander Berglund for his invaluable help in preparing me for this project and for his support and feedback during its duration.

Lastly, I would like to thank Najib Idrissi-Kaïtouni, Julien Ducoulombier, Stefano D'Alesio, Simon Brun and Matteo Felder for taking time to answer many of my questions, as well as for making me feel welcome and part of their team at the ETH.

2. Notation, conventions and basics

In this section we shall fix some notation and also describe what basics are necessary for a reader, as well as give references for those basics.

First note that we are consistently working over the reals in this paper, so for example homology will always be implicitly taken with $\mathbb{R}$-coefficients. We will also only consider real homotopy, i.e. $\pi^{\mathbb{R}}(G) := \pi(G) \otimes \mathbb{R}$.

Differential graded algebra.

The setting throughout the paper will be that of differential graded vector spaces, algebras, coalgebras and Lie algebras. For an introduction, see for example [FHT01, Chapters 3 and 21].

Throughout the paper we shall use homological conventions unless otherwise stated, so differentials always have degree -1, for example. We use the notation $V[n]$ for the nth suspension of a graded vector space, i.e. the space where we raise the degree of every element by n. This is explicitly constructed as $V[n] = \mathbb{R}s \otimes V$, where s is some generator of degree n. If we have some element x in a graded vector space V, we will therefore write sx for the corresponding element in $V[n]$, when it is important to distinguish the two. In the same setting, if x denotes an element of $V[n]$, we denote the corresponding element of V by $s^{-1}x$. Otherwise, we will often abuse notation a bit and simply write x for both elements.

The coalgebras we consider are coaugmented and counitary, which means that we can write $C = \mathbb{Q} \oplus \bar{C}$, where $\bar{C}$ is the kernel of the counit map $\varepsilon : C \to \mathbb{R}$.

At several points, we shall use the *free graded commutative algebra* functor, which goes from vector spaces to graded commutative algebras. Specifically, the free graded commutative algebra on a vector space V, which we shall denote by ΛV, is the quotient of the tensor algebra $T(V)$, by the ideal generated by all commutators $v \otimes w - (-1)^{|v||w|} w \otimes v$.

Using the free graded algebra ΛV, we define the *Koszul sign* related to a permutation σ in S_r, and the elements $x_1, \dots, x_r$ in V, to be the sign $\varepsilon(\sigma; x_1, \dots, x_r)$ such that

$$x_1 x_2 \cdots x_r = \varepsilon(\sigma) x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(r)}.$$

If for example σ is the transposition $(k\ l)$ in S_r, for $1 \leqslant k < l \leqslant r$, then $\varepsilon(\sigma; x_1, \dots, x_r) = (-1)^{|x_k||x_l|}$, so any Koszul sign can be determined by decomposing the given permutation as a composition of transpositions and using this. We will use Koszul signs at several points in certain maps of graded objects, even in cases where the object is not a free graded commutative algebra.

Related to this is also the so called *Koszul sign convention* that we shall use for maps in the graded setting. This is the convention that given maps $f : V \to V'$ and $g : W \to W'$ of graded vector spaces, there is a implicit sign factor in the tensor product $f \otimes g$:

$$(f \otimes g)(v \otimes w) = (-1)^{|g||v|} f(v) \otimes g(w),$$

for $v \in V$ and $w \in W$. One way of viewing this is that the map g needs to "jump over" the element v in the expression, so we introduce the Koszul sign related to this transposition.

Sweedler notation. When using a coproduct in any of the coalgebras that will appear, we will use so called *sumless Sweedler notation.* Recall that in a coalgebra C the coproduct $\Delta : C \to C \otimes C$ evaluated on $x \in C$ can be written as a sum

$$\Delta(x) = \sum_{i=1}^{k} (x_i', x_i''),$$

Since we will often take the coproduct in a tensor product of coalgebras, we will write (x_i', x_i'') instead of $x_i' \otimes x_i''$ in a coproduct, to avoid confusion. In sumless Sweedler notation we let the sum be implied and simply write

$$\Delta(x) = (x', x''),$$

or for an iterated coproduct $\Delta^{N-1}(x) = (x', x'', \cdots, x^{(N)})$. Whenever such an expression appears in the text, recall that a summation is implicit. Note also that when an expression like $(x', x'')(y', y'')$ appears, there are two implicit summations:

$$(x', x'')(y', y'') = \sum_i \sum_j (x_i', x_i'')(y_j', y_j'').$$

Algebraic models. In the category of dg vector spaces, we say that a map $V \to V'$ is a *weak equivalence* if it is a quasi-isomorphism, i.e. if it induces an isomorphism on homology. Then we write $V \xrightarrow{\sim} V'$. We say that V and V' are *weakly equivalent* if there exist spaces $V_0, \dots, V_k$, where $V_0 = V$ and $V_k = V'$, and if for each $0 \leqslant l \leqslant k-1$ there exists a either a weak equivalence $V_l \to V_{l+1}$ or a weak equivalence $V_{l+1} \to V_l$:

$$V = V_0 \xrightarrow{\sim} V_1 \xleftarrow{\sim} \cdots \xrightarrow{\sim} V_{k-1} \xleftarrow{\sim} V_k = V'.$$

We call the set of maps a *zigzag* between V and V'. A zigzag is thus, informally, a sequence of weak quasi-isomorphisms between V and V. Note that V and V' are weakly equivalent as dg vector spaces if and only if they have isomorphic homology. If V is weakly equivalent to V' we also say that V is a *model* for V', and vice versa. If X is a topological space, then the chain complex $C_\bullet(X)$ is a dg vector space, and we say that V is a model for X if it is weakly equivalent to $C_\bullet(X)$.

Classifying spaces. Throughout the paper we will work with the *classifying space* of the group $SO(n)$. The classifying space BG of a topological group G is the base space of the *universal G-bundle*

$$EG \to BG,$$

where a (principal) G-bundle is a fiber bundle $P \to B$ where P comes with an action $G \times P \to P$, which preserves the fibers of the bundle, and acts freely and transitively on these. This implies that the fibers are homeomorphic to G itself, so in particular $P/G \cong B$. The universal bundle $EG \to BG$ can be explicitly constructed using the *bar construction* (see for example [May99, Chapter 16.5]).

3. Operads

In this section we first review the basic definitions and constructions from operad theory that we will use. Then we give some elementary examples of operads and finally we define our main operad of interest: the framed little n-disks operad. We shall only consider *symmetric* operads in this paper, so whenever we use the term operad this is the type of operad we are referring to.

3.1. **Operads and morphisms of operads.**

Operads can be defined in any symmetric monoidal category (for a definition of symmetric monoidal category, see for example [MacLane71, Chapter XI.1]). This general definition, which can be found in [LV12, Chapter 5.2], is more abstract than what is necessary for our purposes, however, so for the sake of clarity we shall make a definition, from [LV12, Chapter 5.3], that makes sense in the categories of vector spaces, topological spaces and differential graded vector spaces. Let $\mathcal{C}$ be one of these categories. We start by defining something called a *symmetric sequence.*

Definition 3.1. A symmetric sequence $\mathcal{P}$ in the category $\mathcal{C}$ is a sequence of objects

$$(\mathcal{P}(0), \mathcal{P}(1), \ldots),$$

indexed by the natural numbers, together with a right action by the symmetric group S_r on each object $\mathcal{P}(r)$, which we for $\mu \in \mathcal{P}(r)$ and $\sigma \in S_r$ denote by μ^σ.

A morphism of symmetric sequences $\mathcal{P}$ and $\mathcal{Q}$, in the category $\mathcal{C}$, is a sequence of $\mathcal{C}$-morphisms $\mathcal{P}(r) \to \mathcal{Q}(r)$, for $r \geqslant 0$, that are also S_r-equivariant.

By adding some more structure to a symmetric sequence, we can define an operad in $\mathcal{C}$. Note that we in this definition for brevity use $\otimes$ to denote the tensor product in $\mathcal{C}$, independently of which category we are actually considering. If $\mathcal{C}$ is the category of topological spaces, then this should be replaced by the topological product $\times$ throughout the definition.

Definition 3.2. An operad in the category $\mathcal{C}$ (or a $\mathcal{C}$-*operad*) is a symmetric sequence $\mathcal{P}$ in $\mathcal{C}$, together with morphisms

$$\circ_j : \mathcal{P}(r) \otimes \mathcal{P}(s) \to \mathcal{P}(r+s-1)$$

called *partial composition* and an element $1_{\mathcal{P}} \in \mathcal{P}(1)$, which we call identity. The partial composition morphisms need to satisfy the following three axioms:

(1) *Identity.* For $\sigma \in \mathcal{P}(r)$, we have

$$\begin{aligned} \sigma \circ_j 1_{\mathcal{P}} &= \sigma, && \text{for all } 1 \leqslant j \leqslant r, \\ 1_{\mathcal{P}} \circ_1 \sigma &= \sigma \end{aligned}$$

(2) *Equivariance.* For any $\sigma \in S_s$, and any $\mu \in \mathcal{P}(r), \nu \in \mathcal{P}(s)$, we have

$$\mu \circ_j \nu^{\sigma} = (\mu \circ_j \nu)^{\sigma'},$$

where $\sigma' \in S_{r+s-1}$ acts like σ on the block $\{j, \dots, j+s-1\}$ and identically on the rest. Similarly, for $\sigma \in S_r$, we have

$$\mu^{\sigma} \circ_j \nu = (\mu \circ_{\sigma(j)} \nu)^{\sigma''},$$

where $\sigma'' \in S_{r+s-1}$ acts by translating the block $\{j, j+1, \dots, j+s-1\}$ to $\{\sigma(j), \sigma(j)+1, \dots \sigma(j)+s-1\}$ and on $\{1,2,\dots,r+s-1\}\backslash\{j, j+1, \dots, j+s-1\}$ as σ, but with values in $\{1,2,\dots,r+s-1\}\backslash\{\sigma(j), \sigma(j)+1, \dots \sigma(j)+s-1\}$.

(3) *Associativity.* For $\lambda \in \mathcal{P}(r)$, $\mu \in \mathcal{P}(s)$ and $\nu \in \mathcal{P}(t)$, we have

$$\begin{cases} (i) \quad (\lambda \circ_i \mu) \circ_{i+j-1} \nu = \lambda \circ_i (\mu \circ_j \nu), & \text{for } 1 \leqslant i \leqslant l,\ 1 \leqslant j \leqslant m, \\ (ii) \quad (\lambda \circ_i \mu) \circ_{k+m-1} \nu = (\lambda \circ_k \nu) \circ_i \mu, & \text{for } 1 \leqslant i < k \leqslant l \end{cases}$$

We will mainly work with operads in the category of topological spaces, which we call *topological operads*, and operads in the category of dg vector spaces, which we will call *dg operads*.

At first glance, the idea behind the axioms of Definition 3.2 might not be clear, and they may look somewhat arbitrary. An intuitive way to think of an operad is as a sequence of spaces, where the rth space $\mathcal{P}(r)$ is a collection of "operations" with r arguments. The action by S_r can then be viewed simply as permuting the arguments of an operation, and partial composition as inserting an operation with s arguments into the jth argument of an operation with r arguments, resulting in an operation with $r+s-1$ arguments. Under this light the axioms in Definition 3.2. become more intuitive.

Next, we define a morphism of operads:

Definition 3.3. A morphism of $\mathcal{C}$-operads $f : \mathcal{P} \to \mathcal{Q}$ is a sequence of $\mathcal{C}$-morphisms $f_r : \mathcal{P}(r) \to \mathcal{Q}(r)$, which preserves the identity, the symmetric action and the partial composition:

- $f_1(1_{\mathcal{P}}) = 1_{\mathcal{Q}}$,
- $f_r(p^{\mu}) = f_r(p)^{\mu}$, for $p \in \mathcal{P}(r)$ and $\mu \in S_r$,
- $f_{r+s-1}(p \circ_j q) = f_r(p) \circ_j f_s(q)$, where $p \in \mathcal{P}(r)$, $q \in \mathcal{P}(s)$ and $1 \leqslant j \leqslant r$.

Remark 3.4. When considering morhpisms of operads later in the paper, we will often abuse notation a bit and simply denote all components of the morphisms by the same symbol, dropping the indices. Note also that a morphism of operads is simply a morphism of symmetric sequences, which also respects the operadic composition and identity.

3.2. **Examples of operads.** Let us illustrate the definitions we have made so far with some examples. These will not be of any major importance for the remainder of the paper, so they can be safely skipped by the reader familiar with operads.

Example 3.5. The endomorphism operad and algebras over an operad.

Let $\mathcal{C}$ be a symmetric monoidal category, and let $X \in \mathcal{C}$ be an object. We can then define the *endomorphism operad* $\mathrm{End}(X)$ by

$$\mathrm{End}(X)(r) = \mathrm{Hom}_{\mathcal{C}}(X^{\otimes r}, X),$$

with the obvious symmetric action and the identity simply being the identity morphism $X \to X$. The partial composition is given by composition of morphisms.

If $\mathcal{P}$ is an operad in the category $\mathcal{C}$, we say that an object $X \in \mathcal{C}$ is a $\mathcal{P}$-*algebra* (or an algebra over $\mathcal{P}$), if it is equipped with a morphism of operads $\mathcal{P} \to \mathrm{End}(X)$. In the intuitive view of operads, as consisting of collections of operations, we may view this as a way of endowing an object of the category with the operations from the operad.

The interconnection between operads and their algebras is fundamental. Often an operad is interesting precisely because a question about its algebras can be reframed as a problem concerning the underlying operad. This is the case in the following examples. For these, the setting is the category of vector spaces, over some field $\mathbb{K}$.

Example 3.6. The associative operad Ass.

The associative operad, in the category of vector spaces, is the operad whose algebras are precisely the associative algebras. We construct this operad as follows: The rth space is the free vector space

$$\mathsf{Ass}(r) := \mathbb{K}S_r.$$

The intuition for this is that given r elements in an associative algebra, there is one way to multiply them for each permutation in S_r. The symmetric action is given by multiplication from the right, and the partial composition $\circ_j$ is induced by the map

$$S_r \times S_s \to S_{r+s-1}$$

given by sending (σ, τ) to the permutation defined by composing σ'' (as defined in Definition 3.2(2)) with the permutation in S_{r+s-1} that acts like τ on $\{\sigma(j), \dots, \sigma(j) + s - 1\}$ and identically on $\{1, \dots, r + s - 1\} \backslash \{\sigma(j), \dots, \sigma(j) + s - 1\}$.

Example 3.7. The commutative operad Com.

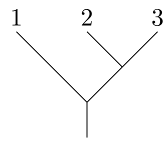
Similarly, the commutative operad is the operad whose algebras are the commutative algebras. The structure of this operad is even simpler than that of Ass. We construct it from the spaces

$$\mathsf{Com}(r) = \mathbb{K},$$

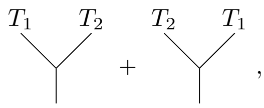
with the trivial action by S_r and composition simply given by multiplication in $\mathbb{K}$. In a similar way of thinking as in the previous example, the intuition here is that there is only one way to multiply r elements in a commutative algebra.

Example 3.8. The Lie operad Lie.

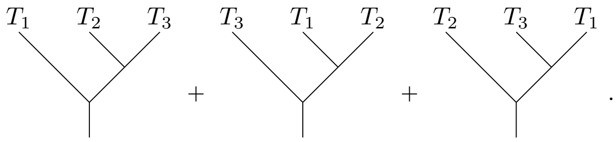
As the name implies, this is the operad whose algebras are the Lie algebras. Since the way to "multiply" r elements in a Lie algebra is determined by how they are bracketed, we can encode this structure using binary trees. For example, the bracketing $[x_1, [x_2, x_3]]$ can be represented by the tree



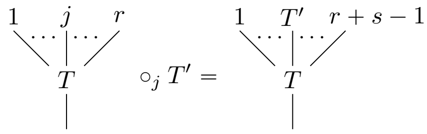
The space $\mathsf{Lie}(r)$ is simply the space of binary r-trees (trees with r leaves), but modulo anti-symmetry



where T_1 and T_2 are binary trees who together have r leaves, and the Jacobi relation



The symmetric action is given by permuting the labels on the leaves of the trees, while the partial composition is given by grafting trees, i.e. $T \circ_j T'$ consists of attaching the root of T' to the jth leaf of T:



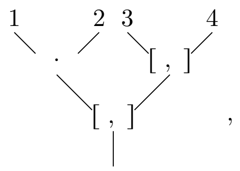
where the labels on the leaves are modified in the obvious way. The identity is simply the trivial tree.

Example 3.9. The Poisson operad Pois.

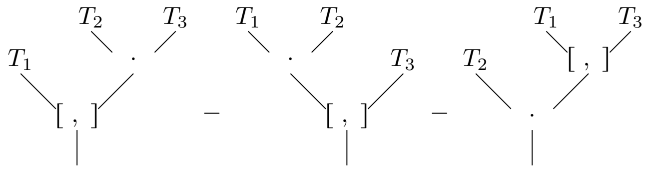
A Poisson algebra is a vector space which both has the structure of an associative algebra and a Lie algebra, where the two structures are related by the Leibniz rule:

$$[x, y \cdot z] = [x, y] \cdot z + y \cdot [x, z].$$

This can be expressed by saying that $[x, -]$ is a derivation with relation to the associative product. The Poisson operad is precisely the *operadic composite* of the associative operad and the Lie operad (see [LV12, Chapter 5.1.4]), but let us give a more intuitive construction. One way is to start from the definition of the Lie operad, but instead consider trees with two types of inner vertices (i.e. vertices that are neither leaves nor root): one type representing the operation $\cdot$, of the associative algebra structure, and one representing the bracket operation $[\ ,\ ]$ of the Lie algebra structure. An example of such a tree is



which represents an operation like $(x_1, x_2, x_3, x_4) \mapsto [x_1 \cdot x_2, [x_3, x_4]]$ in some Poisson algebra. We let $\mathsf{Pois}(r)$ be the quotient of the space of such trees with r leaves, with the Jacobi relation and the antisymmetry relation (with all inner vertices of type [,]), but also by the relation



which represents the Leibniz rule. The symmetric action is once again given by permuting the labels on leaves, and the composition by grafting trees.

Remark 3.10. Here we have defined the Poisson operad in the non-graded setting. In the graded setting, we also introduce a degree on the trees, given by their number of internal vertices, i.e. vertices that are neither leaves nor the root. In this setting we also introduce an underlying dimension n, so that we get one operad Pois_n for each positive integer. For a definition of this operad, see for example [Sinha10, Sections 2 and 5].

3.3. **Homology and algebraic models of operads.** Suppose that $\mathcal{P}$ is a dg operad or a topological operad. By the Künneth theorem, we have an isomorphism

$$H_\bullet(\mathcal{P}(r) \otimes \mathcal{P}(s)) \cong H_\bullet(\mathcal{P}(r)) \otimes H_\bullet(\mathcal{P}(s)),$$

so the partial composition maps $\mathcal{P}(r) \otimes \mathcal{P}(s) \to P(r+s-1)$ induce partial composition maps on homology, which makes the homology of an operad into a dg operad (with zero differential on all spaces).

If $f : \mathcal{P} \to \mathcal{P}'$ is a morphism of dg operads, it follows that we get an induced map $f_* : H_\bullet(\mathcal{P}) \to H_\bullet(\mathcal{P}')$. If f_* is an isomorphism, we say that f is a quasi-isomorphism of operads. In the category of dg operads we call the quasi-isomorphisms *weak equivalences* of operads, in analogy with the definition for dg vector spaces. In the same manner, we say that two dg operads $\mathcal{P}$ and $\mathcal{P}'$ are weakly equivalent if there exists a zigzag

$$\mathcal{P} \xrightarrow{\sim} \cdots \xleftarrow{\sim} \mathcal{P}'$$

of weak equivalences between them, where we define a zigzag in the analogous way as we did for dg vector spaces. If $\mathcal{P}$ and $\mathcal{P}'$ are weakly equivalent we also say that $\mathcal{P}$ is a *model* for $\mathcal{P}'$, and vice versa.

Now let $\mathcal{T}$ be a topological operad. We want to define an algebraic model of $\mathcal{T}$, analogously to the definition we did for dg vector space models of topological spaces, but we need to be a bit careful here. This is because we do not a priori know that the spaces $C_\bullet(\mathcal{T}(r))$ assemble to form an operad, as applying the functor $C_\bullet$ to the partial composition $\circ_j$ in $\mathcal{T}$ maps only gives us maps

$$C_\bullet(\mathcal{T}(r) \times \mathcal{T}(s)) \xrightarrow{C_\bullet(\circ_j)} C_\bullet(\mathcal{T}(r+s-1)).$$

To make this into an operad, we use the Eilenberg-Zilber map

$$\kappa : C_\bullet(\mathcal{T}(r)) \otimes C_\bullet(\mathcal{T}(s)) \to C_\bullet(\mathcal{T}(r) \times \mathcal{T}(s)),$$

and define partial composition maps in $C_\bullet(\mathcal{T})$ as the compositions $C_\bullet(\circ_j) \circ \kappa$, which makes $C_\bullet(\mathcal{T})$ into a dg operad. We can thus say that a dg operad $\mathcal{P}$ is a dg operad model of the topological operad $\mathcal{T}$ if it is weakly equivalent to $C_\bullet(\mathcal{T})$ as a dg operad.

If the homology $H_\bullet(\mathcal{P})$ of a dg operad $\mathcal{P}$ is weakly equivalent to $\mathcal{P}$ itself, we say that the operad is *formal*. Similarly, if the homology $H_\bullet(\mathcal{P})$ of a topological operad $\mathcal{T}$ is weakly equivalent to $C_\bullet(\mathcal{T})$, we say that it is formal.

In this paper we are interested in models for the framed little n-disks operad. We will, however, take the model defined by Khoroshkin and Willwacher in [KW17] as given and only show how it is constructed, so for this reason we will never actually apply this definition. We only introduce it here so that when we later refer to something as a model for the framed little n-disks operad, it is clear what is intended.

3.4. **Actions on operads and semi-direct products.** We shall now look at some constructions for operads that will become important when discussing the framed little n-disks operad and later its algebraic models. The first is that of a semi-direct product of a topological operad and a group acting on that operad. We assume that G is a topological group, by giving it the discrete topology if none other is given.

Definition 3.11. If G is a group and $\mathcal{P}$ a topological operad, a group action of G on $\mathcal{P}$ is a sequence of actions $G \times \mathcal{P}(r) \to \mathcal{P}(r)$, that are continuous, S_r-equivariant and respect the operadic identity as well as the composition maps. The last two criteria can be explicitly written

$$g \cdot 1 = 1,$$

for any $g \in G$ and $1 \in \mathcal{P}(1)$, and

$$g \cdot (p \circ_j q) = (g \cdot p) \circ_j (g \cdot q),$$

for any $g \in G$ and $p \in \mathcal{P}(r)$, $q \in \mathcal{P}(s)$.

Definition 3.12. Given a group action by G on the operad $\mathcal{P}$, we can construct the semi-direct product $\mathcal{P} \circ G$, from the spaces

$$(\mathcal{P} \circ G)(r) = \mathcal{P}(r) \times G^{\times r},$$

where the action by S_r is extended from that on $\mathcal{P}$ simply by permuting the G-factors, the composition is given by

$$(p; g_1, \ldots, g_r) \circ_j (q; h_1, \ldots, h_s) = (p \circ_j (g_j q); g_1, \ldots, g_{j-1}, g_j h_1, \ldots, g_j h_s, g_{j+1}, \ldots, g_r),$$

and the identity element is simply $(1; e)$, where $e \in G$ is the group identity.

The verification that this indeed satisfies the axioms of an operad is elementary, so we leave it as an exercise.

In a similar fashion we can construct the semi-direct product of a operad in the category of (graded) vector spaces and a cocommutative Hopf algebra which acts on it. Let $\mathcal{H}$ be a cocommutative Hopf algebra and let $\mathcal{P}$ be an operad in the category of graded vector spaces.

Definition 3.13. An action by a cocommutative Hopf algebra $\mathcal{H}$ on a dg operad $\mathcal{P}$ is a sequence of linear maps $\mathcal{H} \otimes \mathcal{P}(r) \to \mathcal{P}(r)$, $(h, p) \mapsto h \cdot p$, of degree zero, which are S_r-equivariant, satisfy $(h_1 h_2) \cdot p = h_1 \cdot (h_2 \cdot p)$ and $h \cdot (p \circ_j q) = (h' \cdot p) \circ_j (h'' \cdot q)$. The last condition can be illustrated by the commutative diagram

$$\begin{tikzcd}
\mathcal{P}(r) \otimes \mathcal{P}(s) \arrow[r, "\circ_j"] \arrow[d, "\Delta(h)\cdot"'] & \mathcal{P}(r+s-1) \arrow[d, "h\cdot"] \\
\mathcal{P}(r) \otimes \mathcal{P}(s) \arrow[r, "\circ_j"] & \mathcal{P}(r+s-1)
\end{tikzcd}$$

where $\Delta(h)$ acts on $(p,q) \in \mathcal{P}(r) \otimes \mathcal{P}(s)$ as $(h'p, h''q)$.

Note that since there is a summation implicit in the Sweedler notation, this would not be well defined in for example the category of topological spaces.

Definition 3.14. Given an action by a Hopf algebra $\mathcal{H}$ on an operad $\mathcal{P}$ in the category of (graded) vector spaces, we define the semi-direct product $\mathcal{P} \circ \mathcal{H}$ as the operad assembled from the spaces

$$(\mathcal{P} \circ \mathcal{H})(r) = \mathcal{P}(r) \otimes \mathcal{H}^{\otimes r},$$

with the S_r-action and identity element analogous to those in Definition 3.12, and where composition is defined by

$$(p; h_1, \ldots, h_r) \circ_j (q; k_1, \ldots, k_s) := (p \circ_j (h_j' \cdot q); h_1, \ldots, h_{j-1}, h_j'' \cdot k_1, \ldots h_j^{(s+1)} \cdot k_s, h_{j+1}, \ldots, h_r)$$

We once again leave the verification that this satisfies the axioms of an operad to the reader.

Remark 3.15. The construction of Definition 3.14 may be viewed as an algebraic version of the topological construction in Definition 3.12. If G is a connected Lie group, then its homology is a cocommutative Hopf algebra. Given a topological operad $\mathcal{T}$ and a connected Lie group G acting on $\mathcal{T}$, applying homology thus gives us an action by the Hopf algebra $H_\bullet(G)$ on the dg operad $H_\bullet(\mathcal{T})$, and the homology of the semi-direct product $\mathcal{T} \circ G$ is the semi-direct product $H_\bullet(\mathcal{T}) \circ H_\bullet(G)$.

Lastly, we define an action by a dg Lie algebra on a dg operad.

Definition 3.16. Let (L, d) be a dg Lie algebra and $(\mathcal{P}, d)$ be a dg operad. An action by L on $\mathcal{P}$ is a sequence of linear maps $L \otimes \mathcal{P}(r) \to \mathcal{P}(r)$, $(x, p) \mapsto x \cdot p$, of degree zero such that:

(1) the Lie bracket is preserved:
$$[x, y] \cdot p = x \cdot (y \cdot p) - (-1)^{|x||y|} y \cdot (x \cdot p),$$
(2) each map is an operadic derivation:
$$x \cdot (p \circ_j q) = (x \cdot p) \circ_j q + (-1)^{|x||p|} p \circ_j (x \cdot q),$$
(3) each map preserves the differentials on L and $\mathcal{P}$:
$$d(x \cdot p) = dx \cdot p + (-1)^{|x|} x \cdot dp.$$

This definition will not be used to construct any sort of semi-direct product, but will be of key importance in Sections 5.4, 6, 8 and 9. Recall that the universal enveloping algebra of a dg Lie algebra is a Hopf algebra. This results in the following connection between actions by Lie algebras and Hopf algebras on operads:

Proposition 3.17. Suppose that L is a dg Lie algebra acting on the dg operad $\mathcal{P}$. Then this action extends to the universal enveloping algebra of L as an action of dg Hopf algebras $\mathcal{U}L \circlearrowright \mathcal{P}$ given by $(x_1 x_2 \cdots x_k) \cdot p = x_1 \cdot (x_2 \cdot (\cdots x_k \cdot p) \cdots)$.

We leave the proof of this proposition as an exercise to the reader, for brevity. Note that the universal enveloping algebra has the deconcatenation coproduct:

$$\Delta(x_1 x_2 \cdots x_k) = \sum_{I \subseteq [k]} \varepsilon(\sigma_I; x_1, \ldots x_k) x_I \otimes x_{[k] \setminus I},$$

where $[k] = \{1, 2, \dots, k\}$, $I = \{i_1, i_2, \dots, i_l\} \subseteq [k]$, $x_I = x_{i_1} x_{i_2} \cdots x_{i_l}$ and σ_I is the permutation in S_k that maps $[k]$ to $I \sqcup ([k] \backslash I)$ (in that order).

Now let us finally move on to our operad of interest. We will construct the framed little n-disks operad from its unframed version, so we begin by defining that.

3.5. **The original little n-disks operads.** The little n-disks operad was introduced by Boardman, Vogt and May in the study of iterated loop spaces in the early 1970's and was one of the earliest operads to be defined. In [May72], May proved that any n-fold loop space is an algebra over this operad, and that any connected algebra over this operad has the weak homotopy type of an n-fold loop space. We shall denote the little n-disks operad by $\mathcal{D}_n$.

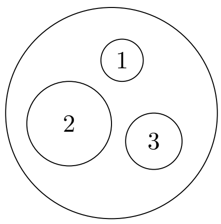
Definition 3.18. We assemble $\mathcal{D}_n$ from the subspaces

$$\mathcal{D}_n(r) \subset \mathrm{Emd}\left(\bigsqcup_r D^n, D^n\right),$$

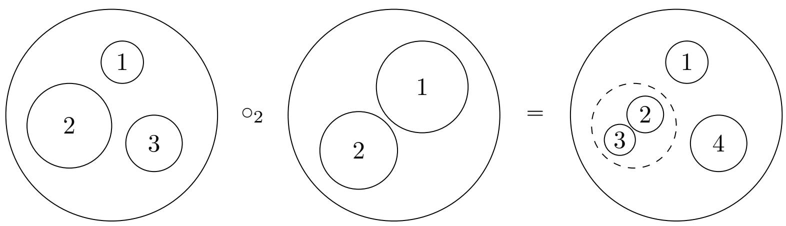
of embeddings of r n-dimensional disks into the n-disk itself, where we assume that the embeddings are *rectilinear*. The identity is the identity embedding of the disk into itself, while the symmetric action is given by permuting the embeddings. The composition $(f_1, \dots, f_r) \circ_j (g_1, \dots, g_s)$ is given by composing with the jth embedding f_j

$$(f_1, \dots, f_r) \circ_j (g_1, \dots, g_s) = (f_1, \dots, f_{j-1}, f_j \circ g_1, \dots, f_j \circ g_s, f_{j+1}, \dots, f_r).$$

That the embeddings are rectilinear means that the "little disks" are only permitted to be scaled and translated. A typical element of $\mathcal{D}_2(3)$ can thus be illustrated like:

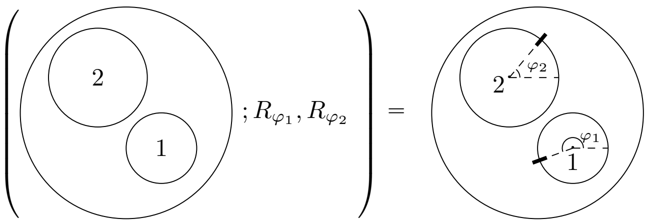


The symmetric action may then be viewed as "permuting the labels" on the disks. The jth partial composition can be illustrated by insertion into the jth disk. Once again, it is easiest to illustrate this with an example in the 2-dimensional case:

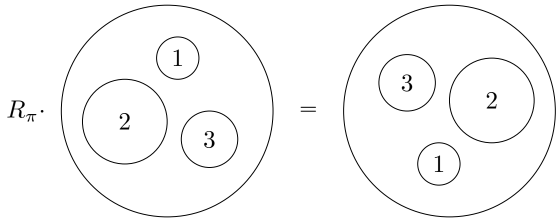


Recall from example 6 that the homology of an operad is also an operad. It can be proven (see for example [Sinha10]) that the homology of $\mathcal{D}_n$ is precisely the nth Poisson operad Pois_n, mentioned in Remark 3.10.

3.6. **The framed little n-disks operad.** If we loosen the criterion that the embeddings in the operad need to be rectilinear, and also permit rotations, we get the *framed* little n-disks operad. We can thus represent an element by one from the original operad, together with an element from $SO(n)$ associated to each embedding: $(p; g_1, \ldots, g_r)$, where $p \in \mathcal{D}_n(r)$ and $g_1, \ldots, g_r \in SO(n)$. For $n = 2$ we can illustrate this with the example



where R_φ is the rotation matrix associated to the angle φ. The symmetric action and identity are the same as in $\mathcal{D}_n$ (with the addition that we permute the associated rotations accordingly with the labels), but when composing disks, we also need to compose rotations. One way of defining this structure is by taking the semi-direct product with relation to the action by $SO(n)$ on $\mathcal{D}_n$ given by rotating the centers of the embedded disks around the center of the big disk, but keeping the orientations of the little disks themselves fixed. Note that if these are rotated as well, the resulting element is not an element in the operad, since the embeddings are required to be rectilinear. In the case $n = 2$, we for example have:



The composition is then given by

$$(p; g_1, \ldots, g_r) \circ_j (q; h_1, \ldots, h_s) = (p \circ_j (g_j \cdot q); g_1, \ldots, g_{j-1}, g_j h_1, \ldots, g_j h_s, g_{j+1}, \ldots, g_r),$$

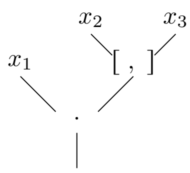
just as desired. The symmetric action and identity are also the same as in $\mathcal{D}_n$, so we define:

Definition 3.19. The framed little n-disks operad is the semi-direct product

$$\mathcal{D}_n^{fr} := \mathcal{D}_n \circ SO(n),$$

with relation to the action by $SO(n)$ on $\mathcal{D}_n$ defined above.

Since $\mathcal{D}_n^{fr} = \mathcal{D}_n \circ SO(n)$, it follows that the homology of the framed little n-disks operad is $\mathsf{Pois} \circ H_\bullet(SO(n))$, with relation to the induced action by $H_\bullet(SO(n))$ on Pois. In arity r the homology is thus $\mathsf{Pois}(r) \otimes H_\bullet(SO(n))^{\otimes r}$. We can view an element of $H_\bullet(\mathcal{D}_n^{fr})(r)$ as a tree from $\mathsf{Pois}(r)$, with its leaves decorated by elements from $H_\bullet(SO(n))$, so for example



,

where x_1, x_2 and x_3 lie in $H_\bullet(SO(n))$. The composition is given by combining grafting with the induced action.

4. The special orthogonal group

We have used the special orthogonal group in the construction of $\mathcal{D}_n^{fr}$ and throughout the remainder of the paper, we will consistently use this group and its classifying space. For this reason we dedicate this section to this group. We will start with describing the homology of $SO(n)$, as well as that of its classifying space $BSO(n)$. For the rest of the paper, let us denote $G := SO(n)$, since this will be the only group we consider.

4.1. The (co)homology of $SO(n)$.

A computation of the cohomology of G can be found for example in [Fung12, Section 1.4]. As a graded algebra, it is

$$H^\bullet(G) = \begin{cases} \Lambda(\mathbb{R}\{p_3, p_7, \ldots, p_{2n-3}\}) & n \text{ odd}, \\ \Lambda(\mathbb{R}\{p_3, p_7, \ldots, p_{2n-5}, E\}) & n \text{ even}, \end{cases}$$

where we use $\mathbb{R}S$ to denote the free vector space on a set S. The generators p_{4i-1} are called *Pontryagin* classes and E is called the *Euler class*. Since the dimension is finite in every degree, and we are considering coefficients in a field, it follows by the universal coefficient theorem that it is isomorphic to the homology of the group. We shall use the same symbols for the generators in homology as those in cohomology (we will only consider the homology of G from now on, so it should cause no confusion).

As always, the homology has the structure of a cocommutative coalgebra. The generators are *primitive*, so $\Delta(p_{4i-1}) = 1 \otimes p_{4i-1} + p_{4i-1} \otimes 1$, where Δ denotes the coproduct. Since G is a connected Lie group, the homology additionally has a product, induced from the group multiplication in G, and which turns the homology into a cocommutative Hopf algebra. As an algebra, it is an exterior algebra, since the generators are of odd degree.

Since G is a topological group, it is weakly equivalent to the loop space of its classifying space: ΩBG [Hatcher02, Proposition 4.66]. We can use this to describe the homology in a nice way that connects it to the homotopy of the group. The real homology of a loop space is the universal enveloping algebra of the real homotopy (considered as an abelian graded Lie algebra), so

$$H_\bullet(G) = \mathcal{U}(\pi^{\mathbb{R}}(G)) = \Lambda\pi^{\mathbb{R}}(G),$$

where the free graded commutative algebra is equal to the universal enveloping algebra in this case, as the Lie algebra is abelian. We will later use this relation between the homology and homotopy of G, when constructing Khoroshkin and Willwacher's model for the framed little n-disks operad.

4.2. The (co)homology of $BSO(n)$.

The cohomology of BG can be computed using *equivariant cohomology* (we shall see that the cohomology is of finite type once again, so it is isomorphic to the homology as a dg vector space), see for example [Pestun16, Equation 2.13].

For odd n a maximal torus of $SO(n)$ is the subgroup of block-matrices of the form

$$\begin{pmatrix} R_{\varphi_1} & & & \\ & \ddots & & \\ & & R_{\varphi_k} & \\ & & & 1 \end{pmatrix},$$

for $n = 2k+1$, where

$$R_{\varphi_i} = \begin{pmatrix} \cos\varphi_i & \sin\varphi_i \\ -\sin\varphi_i & \cos\varphi_i \end{pmatrix}$$

and the Weyl group is $S_k \rtimes S_2^{\times k}$. Using this, one may compute the cohomology of BG to be the polynomial algebra

$$H^\bullet(BG) = \mathbb{R}[\tilde{p}_4, \ldots, \tilde{p}_{2n-2}],$$

for odd n, where $\tilde{p}_{4i}$ is a generator of degree $4i$. Thus the homology is the polynomial coalgebra with the same underlying vector space and the deconcatenation coproduct.

If n is even the Weyl group is the subgroup of $S_k \rtimes S_2^{\times k}$ where an even number of the S_2-components are required to be non-identity. Using this, the cohomology can be determined to be the polynomial algebra

$$H^\bullet(BG) \cong \mathbb{R}[\tilde{p}_4, \ldots, \tilde{p}_{4k-4}, \tilde{E}],$$

where $\tilde{E}$ is of degree $2k$ and satisfies $\tilde{E}^2 = \tilde{p}_{4k}$. The homology is once again the coalgebra on the same underlying vector space, with the deconcatenation coproduct.

4.3. **The Koszul complex $H_\bullet(G) \otimes H_\bullet(BG)$.** Since both $H_\bullet(G)$ and $H_\bullet(BG)$ are cocommutative coalgebras, we can define a new cocommutative coalgebra by taking the tensor product $H_\bullet(G) \otimes H_\bullet(BG)$. For brevity we shall use the notation $\mathcal{H} := H_\bullet(G) \otimes H_\bullet(BG)$ throughout the paper. Since both $H_\bullet(G)$ and $H_\bullet(BG)$ have zero differential, we initially view their tensor product $\mathcal{H}$ as having zero differential as well, before equipping it with the *twisted* differential introduced in Definition 4.1. We make our construction for odd n here, for brevity, but it is easy to see that the analogous construction can be made in the even case as well.

Definition 4.1. Let $\pi : H_\bullet(BG) \to \mathbb{R}\{p_3, \ldots, p_{2n-3}\}$ be the composition of the projection of $H_\bullet(BG)$ onto the subspace $\mathbb{R}\{\tilde{p}_4, \ldots, \tilde{p}_{2n-2}\}$ spanned by the generators, with the degree -1 linear map $\mathbb{R}\{\tilde{p}_4, \ldots, \tilde{p}_{2n-2}\} \to \mathbb{R}\{p_3, \ldots, p_{2n-3}\}$ given by $\tilde{p}_{4i} \mapsto p_{4i-1}$. Furthermore, let $\iota : \mathbb{R}\{p_3, \ldots, p_{2n-3}\} \to H_\bullet(G)$ be the natural inclusion. If we denote the product $H_\bullet(G)$ in by μ, we can now define a map $\mathcal{H} \to \mathcal{H}$ by

$$d := -(\mu \otimes 1) \circ (1 \otimes \iota\pi \otimes 1) \circ (1 \otimes \Delta)$$

If $\alpha \otimes \beta \in \mathcal{H}$, we thus have

$$d : \alpha \otimes \beta \mapsto -(-1)^{|\alpha|} \alpha \cdot \iota\pi(\beta') \otimes \beta'',$$

where the sign factor $(-1)^{|\alpha|}$ is due to the degree -1 map $\iota\pi$ having to jump over α.

Proposition 4.2. The map d defined above makes $(\mathcal{H}, d)$ into an acyclic chain complex, i.e.

$$H_\bullet(\mathcal{H}, d) \cong \mathbb{R}.$$

Proof. We shall prove this more generally, where we follow (with some modifications) the proof of [LV12, Proposition 3.4.8]. Suppose that V is a graded vector space with basis $\{\alpha_1, \ldots, \alpha_n\}$, where all generators have odd degree. Let A be the free graded commutative algebra ΛV and C be the cofree graded cocommutative coalgebra $\Lambda^c V[1]$, which in this case is just the polynomial coalgebra on the generators $\{s\alpha_1, \dots, s\alpha_n\}$ with the deconcatenation coproduct.

Let $\pi : C \to V[1] \to V$ be the natural projection (of degree -1) and $\iota : V \to A$ be the inclusion. On elements of V, this map is thus $\pi(v) = s^{-1}v$. Define the *Koszul complex* $(A \otimes C, d)$ to be the dg vector space $A \otimes C$ with the differential defined by

$$d(\alpha \otimes \beta) = -(-1)^{|\alpha|}\alpha \cdot \iota\pi(\beta') \otimes \beta''.$$

Since π is zero outside of the linear summand of C, this can more explicitly be written

$$d(x_1x_2\cdots x_p \otimes y_1y_2\cdots y_q) = -\sum_{i=1}^{p}(-1)^{|x_1|+\cdots+|x_p|}x_1x_2\cdots x_p(s^{-1}y_i) \otimes y_1\cdots\hat{y}_i\cdots y_q,$$

where $x_1, \dots, x_p \in V$ and $y_1, \dots, y_q \in V[1]$. When applying the double differential, for each $i \neq j$, we get two different terms

$$-(-1)^{|x_1|+\cdots+|x_p|+|s^{-1}y_i|}x_1x_2\cdots x_p(s^{-1}y_i)(s^{-1}y_j) \otimes y_1\cdots\hat{y}_i\cdots\hat{y}_j\cdots y_q$$

and

$$-(-1)^{|x_1|+\cdots+|x_p|+|s^{-1}y_j|}x_1x_2\cdots x_p(s^{-1}y_j)(s^{-1}y_i) \otimes y_1\cdots\hat{y}_i\cdots\hat{y}_j\cdots y_q,$$

where $(-1)^{|x_1|+\cdots+|x_p|+|s^{-1}y_i|} = (-1)^{|x_1|+\cdots+|x_p|+|s^{-1}y_j|}$, since $|s^{-1}x_i|$ and $|s^{-1}x_j|$ are both off odd degree. These terms thus differ only by the order of $s^{-1}x_i$ and $s^{-1}x_j$, and since these have odd degree in A, they differ by a negative sign and thus cancel, which means that this map squares to zero.

To prove the the complex is acyclic, we define $h : C \otimes A \to C \otimes A$ by

$$h(x_1x_2\cdots x_p \otimes y_1y_2\cdots y_q) = -\sum_{i=1}^{p}(-1)^{|x_1|+\cdots+|x_{i-1}|}x_1x_2\cdots\hat{x}_i\cdots x_p \otimes (sx_i)y_1\cdots y_q$$

and

$$h(a) = a,$$

for a in $\mathbb{R}$. Then we have

$$\begin{aligned}
&dk(x_1x_2\cdots x_p \otimes y_1y_2\cdots y_q)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{|x_{i+1}|+\cdots+|x_p|+|s^{-1}y_j|}x_1\cdots\hat{x}_i\cdots x_p(s^{-1}y_j) \otimes (sx_i)y_1\cdots\hat{y}_j\cdots y_q\\
&+ \sum_{j=1}^{q} x_1\cdots x_p \otimes y_1\cdots y_q\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{|x_{i+1}|+\cdots+|x_p|+|s^{-1}y_j|}x_1\cdots\hat{x}_i\cdots x_p(s^{-1}y_j) \otimes (sx_i)y_1\cdots\hat{y}_j\cdots y_q\\
&+ qx_1\cdots x_p \otimes y_1\cdots y_q
\end{aligned}$$

Similarly, we get

$$\begin{aligned}
&hd(x_1x_2\cdots x_p \otimes y_1y_2\cdots y_q)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{q}(-1)^{|x_{i+1}|+\cdots+|x_p|}x_1\cdots\hat{x}_i\cdots x_p y_j \otimes x_i y_1\cdots\hat{y}_j\cdots y_q + px_1\cdots x_p \otimes y_1\cdots y_q
\end{aligned}$$

The terms in the sums of these two expressions differ by $(-1)^{|s^{-1}y_j|} = -1$. Thus we have $(dh + hd)(x_1 \cdots x_p \otimes y_1 \cdots y_q) = (p+q)x_1 \cdots x_p \otimes y_1 \cdots y_q$. If $x_1 \cdots x_p \otimes y_1 \cdots y_q$ is a cycle, we thus have

$$x_1 \cdots x_p \otimes y_1 \cdots y_q = d\left(\frac{1}{p+q} h(x_1 \cdots x_p \otimes y_1 \cdots y_q)\right),$$

so it is also a boundary. If $p = q = 0$, i.e. if the element is a scalar, then clearly it is a cycle, by the definition of d. It is also clear that it is not a boundary. Thus we have $H_\bullet(C \otimes A, d) \cong \mathbb{R}$. ■

The dg coalgebra $\mathcal{H}$ will be used in Section 9, where we will decorate certain vertices in our graphs with its elements.

5. The Maurer-Cartan equation

Throughout Section 7-9, we will make great use of something called *Maurer-Cartan elements* in certain dg Lie algebras. For this reason, we dedicate this section to studying the properties of such elements.

5.1. **Maurer-Cartan elements in a dg Lie algebra.**

Definition 5.1. A Maurer-Cartan element in a dg Lie algebra (L, d) is an element $m \in L$ of degree -1, satisfying the *Maurer-Cartan equation:*

$$dm + \frac{1}{2}[m, m] = 0.$$

The property of being Maurer-Cartan has several nice consequences.

Proposition 5.2. If m is a Maurer-Cartan element in (L, d), then $d_m = d + [m, \cdot]$ defines a differential on L.

Proof. It is clear that d_m is linear, by the linearity of d and the Lie bracket. Since $|m| = -1$ it also has degree -1. That $[m, \cdot]$ is a derivation is precisely what is expressed by the graded Jacobi identity. We thus only need to prove that it squares to zero. For $x \in L$, we have

$$\begin{aligned} d_m^2(x) &= d^2x + d[m, x] + [m, dx] + [m, [m, x]] \\ &= [dm, x] + (-1)^{-1}[m, dx] + [m, dx] + [m, [m, x]] \\ &= -\frac{1}{2}[[m, m], x] + [m, [m, x]] \\ &= 0 \end{aligned}$$

where we, apart from the Maurer-Cartan equation, have used that d is a derivation, and then that $[[m, m], x] = 2[m, [m, x]]$, by the graded Jacobi identity. Thus $d_m^2 = 0$ and we are done. ■

We call this the differential *twisted by* m, and similarly the dg Lie algebra with the underlying space L, but differential d_m, the Lie algebra twisted by m. We denote this twisted Lie algebra by $(L, d)^m$, or (L^m, d_m). In a very similar fashion we can also prove the following proposition:

Proposition 5.3. Suppose that we have an action by the dg Lie algebra L on a dg operad $\mathcal{P}$, as defined in Section 3. If m is a Maurer-Cartan element in (L, d) and d is the differential in $\mathcal{P}$, then $d_m = d + m\cdot$ defines a new differential on $\mathcal{P}$.

Proof. Once again it is clear that d_m is linear and of degree -1. Since $m\cdot$ is an operadic derivation, by assumption, we only need to verify that $d_m^2 = 0$. For $p \in \mathcal{P}$ we have

$$\begin{aligned}(d + m\cdot)^2(p) &= d^2p + d(m \cdot p) + m \cdot (dp) + m \cdot (m \cdot p)\\ &= dm \cdot p - m \cdot dp + m \cdot dp + \frac{1}{2}[m, m] \cdot p\\ &= \left(dm + \frac{1}{2}[m, m]\right) \cdot p\\ &= 0\end{aligned}$$

where we, apart from the Maurer-Cartan equation, have used that the action preserves the differential and that

$$[m, m] \cdot p = m \cdot (m \cdot p) - (-1)^{|m|} m \cdot (m \cdot p) = 2m \cdot (m \cdot p),$$

since the action preserves the Lie bracket. ■

Just as for the Lie algebra, we can define the twisted operad $\mathcal{P}^m$ with this twisted differential, i.e. $\mathcal{P}^m := (\mathcal{P}, d + m\cdot)$. Using these two previous propositions, we prove the following:

Proposition 5.4. Suppose we have an action of dg Lie algebras by L on the operad $\mathcal{P}$. If m is a Maurer-Cartan element of (L, d), then this is also an action by L^m on $\mathcal{P}^m$.

Proof. Note that since the twisting only affects the differential, we only need to check that the action preserves the twisted differentials. For $p \in \mathcal{P}$ and $x \in L$ we have

$$\begin{aligned}(d + m\cdot)(x \cdot p) &= d(x \cdot p) + m \cdot (x \cdot p)\\ &= dx \cdot p + (-1)^{|x|} x \cdot dp + [m, x] \cdot p + (-1)^{|x|} x \cdot (m \cdot p)\\ &= (dx + [m, x]) \cdot p + (-1)^{|x|} x \cdot (dp + m \cdot p)\\ &= d_m x \cdot p + (-1)^{|x|} x \cdot d_m p.\end{aligned}$$

Thus the action preserves the twisted differentials as well, which proves the proposition. ■

5.2. **Quillen's functors $\mathcal{C}_*$ and $\mathcal{L}$.** We shall now introduce the functors $\mathcal{C}_*$ and $\mathcal{L}$, following the example of [Quillen96, Appendix B6]. These functors are also treated in [FHT01, Chapter 22] The main fact that interests about these functors is that they have an important connection to Maurer-Cartan elements that we shall use later on. We start with the functor $\mathcal{C}_*$, that goes from dg Lie algebras to cocommutative dg coalgebras.

Definition 5.5. Let (L, d) be a dg Lie algebra. We define

$$\mathcal{C}_*(L) = \bigoplus_{k \geqslant 1} \Lambda^k(L[1]),$$

where $\Lambda^k(L[1])$ is the component of $\Lambda L[1]$ with wordlength k. We equip this with the deconcatenation coproduct, making it into a coalgebra. We equip this coalgebra with the differential $D_{\mathcal{C}} = d_0 + d_1$, where d_0 is induced by the differential on L:

$$d_0(sx_1 sx_2 \cdots sx_k) = -\sum_{i=1}^{k} (-1)^{|sx_1| + \cdots + |sx_{i-1}|} sx_1 sx_2 \cdots s(dx_i) \cdots sx_k,$$

and d_1 is induced by the bracket in L:

$$d_1(sx_1 sx_2 \cdots sx_k) = -\sum_{1 \leqslant i < j \leqslant k} (-1)^{|x_i|} (-1)^{n_{ij}} s[x_i, x_j] sx_1 \cdots \hat{sx_i} \cdots \hat{sx_j} \cdots sx_k,$$

where $n_{ij} = |sx_i|(|x_1|+\cdots+|x_{i-1}|)+|sx_j|(|sx_1|+\cdots\widehat{|sx_i|}+\cdots|x_{j-1}|)$. This makes $(\mathcal{C}_*(\mathfrak{g}), D_{\mathcal{C}})$ into a cocommutative quasi-cofree dg coalgebra[1].

Next, we introduce the functor $\mathcal{L}$:

Definition 5.6. Let (C, d) be a coaugmented dg coalgebra, and define the Lie algebra

$$\mathcal{L}(C) = \mathrm{FreeLie}(\bar{C}[-1]),$$

where the free lie algebra on a vector space V is the Lie algebra generated by V under the commutator bracket in the tensor algebra $T(V)$. Equip this with the differential $D_{\mathcal{L}} = d_0 + d_1$, where d_0 is the negative of the differential induced by d (analogously as above), and d_1 is induced by the coproduct in C and the Lie bracket (commutator) in $\mathcal{L}(C)$:

$$d_1(c) = -(-1)^{|sc'|}\frac{1}{2}[sc', sc''],$$

for $c \in C$. This is extended to $\mathcal{L}(C)$ in the unique way.

These two functors are in fact adjoint, which we shall see by connecting them with the Maurer-Cartan elements of the dg Lie algebra $\mathrm{Hom}(C, L)$.

5.3. **Quillen's functors and Maurer-Cartan elements.** The dg Lie algebra $\mathrm{Hom}(C, L)$, of linear maps from C to L, has the bracket

$$[f, g] := \mu \circ (f \otimes g) \circ \Delta,$$

where Δ is the coproduct of C and μ is the bracket of L. The Maurer-Cartan equation in this dg Lie algebra reads

$$d_L f(x) + f(d_C x) + (-1)^{|x'|}\frac{1}{2}[f(x'), f(x'')] = 0$$

Let us denote the set of Maurer-Cartan elements in this dg Lie algebra by $MC(C, L)$. First we shall prove the following:

Proposition 5.7. There is a bijective correspondence between elements of $MC(C, L)$ and maps of dg coalgebras $C \to C_*(L)$.

Proof. Suppose that $f : C \to C_*(L)$ is a coalgebra map. The condition to be a map of dg coalgebras is that

$$f d_C = d_{C_*} f,$$

since $C_*(L)$ is cofree, this is true if and only if $\pi f d_C = \pi d_{C_*} f$, where π is the natural projection $C_*(L) \to L$. Let f' be the composite πf. Then the condition for f to be a map of dg coalgebras is that, for $x \in C$,

$$f' d_C(x) = -d_L f'(x) - (-1)^{|x'|}\frac{1}{2}[f'(x'), f'(x'')],$$

which is precisely the Maurer-Cartan condition from above. Note that the half in the last term comes from the fact that $i < j$ in the part d_1 of the differential on $\mathcal{C}_*(L)$.

We have now shown that if f is a map of dg coalgebras $C \to \mathcal{C}_*(L)$, we get a Maurer-Cartan element of $\mathrm{Hom}(C, L)$. Conversely if m is a Maurer-Cartan element of $\mathrm{Hom}(C, L)$, i.e. a

[1]Quasi-cofree dg coalgebra means that it is cofree as a coalgebra, but not as a dg coalgebra

linear map $C \to L$, it extends to a map of coalgebras $C \to \mathcal{C}_*(L)$, since $\mathcal{C}_*(L)$ is quasi cofree, and which is a dg coalgebra map because of the Maurer-Cartan condition. ■

Thus we have a bijective correspondence between the sets

$$MC(C, L) \leftrightarrow \mathrm{Hom}_{dgc}(C, \mathcal{C}_*(L)).$$

Next, we show the following similar correspondence:

Proposition 5.8. There is a bijective correspondence between elements of $MC(C, L)$ and dg Lie algebra maps $\mathcal{L}(C) \to L$.

Proof. Since $\mathcal{L}(C)$ is a free Lie algebra, a map $C \to L$ extends uniquely to a map of Lie algebras $\mathcal{L}(C) \to L$. Thus we only need to show that if f is a Lie algebras map $\mathcal{L}(C) \to L$, then the condition that $fd_{\mathcal{L}} = d_L f$ agrees with the Maurer-Cartan condition. Once again, since $\mathcal{L}(C)$ is free, we only need to look at what happens on $x \in C$. The condition becomes

$$d_L f(x) = fd_{\mathcal{L}}(x) = -fd_C(x) - (-1)^{|x'|}\frac{1}{2}f([x', x'']) = -fd_C(x)x - (-1)^{|x'|}\frac{1}{2}[f(x'), f(x'')],$$

which is precisely the Maurer-Cartan condition. ■

We thus have bijective correspondences

$$\mathrm{Hom}_{dgl}(\mathcal{L}(C), L) \leftrightarrow MC(C, L) \leftrightarrow \mathrm{Hom}_{dgc}(C, \mathcal{C}_*(L)).$$

Proving that the correspondence $\mathrm{Hom}_{dgl}(\mathcal{L}(C), L) \leftrightarrow \mathrm{Hom}_{dgc}(C, \mathcal{C}_*(L))$ is natural is a simple, but tedious, verification, so we will omit it.

Corollary. If we set $C = \mathcal{C}_*(g)$ and let L' be some other dg Lie algebra, we get a bijective correspondence

$$\mathrm{Hom}_{dgl}(\mathcal{L}\mathcal{C}_*(L), L') \leftrightarrow MC(\mathcal{C}_*(L), L').$$

We will use this correspondence when constructing Khoroshkin and Willwacher's model for the framed little n-disks operad in Section 8. There we shall also use the following fact about the composite functor $\mathcal{L}\mathcal{C}_*$:

Proposition 5.9. The natural projection map $\mathcal{L}\mathcal{C}_*(\mathfrak{g}) \to \mathfrak{g}$ is a quasi-isomorphism, so $\mathcal{L}\mathcal{C}_*(\mathfrak{g})$ is a resolution of $\mathfrak{g}$.

Proof. See [FHT01, Theorem 22.9].

5.4. **A resolution of $H_\bullet(SO(n))$.** In the construction of Khoroshkin and Willwacher's model for the framed little n-disks operad, it will be necessary to take a resolution of $H_\bullet(G)$. For this, we use the construction introduced above. Let $\pi^{\mathbb{R}}(G)$ be the homotopy, viewed as an abelian dg Lie algebra, with the zero differential. Then $\mathcal{L}\mathcal{C}_*(\pi^{\mathbb{R}}(G))$ is a resolution of $\pi^{\mathbb{R}}(G)$. Since the universal enveloping algebra functor preserves quasi-isomorphisms [FHT01, Theorem 21.7], it follows that

$$\widehat{H}_\bullet(G) := \mathcal{U}\mathcal{L}\mathcal{C}_*(\pi^{\mathbb{R}}(G))$$

is a resolution of $\mathcal{U}(\pi^{\mathbb{R}}(G))$. But $\mathcal{U}(\pi^{\mathbb{R}}(G)) = H_\bullet(G)$, so it follows that $\widehat{H}_\bullet(G)$ is a resolution of $H_\bullet(G)$. Note that since $\pi^{\mathbb{R}}(G)$ is an abelian Lie algebra, we have $\mathcal{U}(\pi^{\mathbb{R}}(G)) = \Lambda(\pi^{\mathbb{R}}(G))$. Thus $\mathcal{C}_*(\pi^{\mathbb{R}}(G)) = H_\bullet(BG)$ and

$$\widehat{H}_\bullet(G) = \mathcal{U}\mathcal{L}\mathcal{C}_*(\pi^{\mathbb{R}}(G)) = \mathcal{U}\mathrm{FreeLie}(\overline{H_\bullet(BG)}[-1])$$

But the universal enveloping algebra of a free Lie algebra is just the tensor algebra, so $\widehat{H}_\bullet(G)$ is simply the tensor algebra on $\overline{H_\bullet(BG)}[-1]$.

Remark 5.10. We may identify p_{4i-1} in $H_\bullet(G)$ with $s^{-1}\tilde{p}_{4i}$ in $H_\bullet(BG)[-1]$. As we will see in Section 8, the top Pontryagin class $s^{-1}\tilde{p}_{2n-2}$ in $H_\bullet(BG)[-1]$, for odd n, will play an important role in our models for the framed little n-disks operad. For this reason we will simply use the notation $p^k = s^{-1}\tilde{p}^k_{2n-2}$ for those generators in $\widehat{H}_\bullet(G)$.

6. Some tools from homological algebra

In this section we prove some lemmas from homological algebra that we will be of use in our proofs in Section 9. The first two lemmas are necessary to prove Proposition 6.6, which we will be using in the later sections when proving that certain maps are quasi-isomorphisms. After proving these lemmas, we exemplify the use of Proposition 6.6 by looking at the chain complex $\widehat{H}_\bullet(G)\otimes H_\bullet(BG)$, which we prove is a resolution of the complex $\mathcal{H} = H_\bullet(G)\otimes H_\bullet(BG)$, defined in Section 4. After this, we prove that taking invariants by a finite group action commutes with homology, which is a result we need in Section 9, since our graph complexes will be invariant subspaces of bigger vector spaces.

Lemma 6.1. Let (V, d_V) and (W, d_W) be dg vector spaces, and $f : V \to W$ be a map of dg vector spaces. Define the cone on f as

$$\mathrm{Cone}(f) = (V \oplus W[-1], D = (d_V, f - d_W)).$$

This defines a differential on the cone, and furthermore $H(\mathrm{Cone}(f)) = 0$ if and only if f is a quasi-isomorphism.

Proof. In matrix form we have

$$D^2 = \begin{pmatrix} d_V & 0 \\ f & -d_W \end{pmatrix}^2 = \begin{pmatrix} d_V^2 & 0 \\ fd_V - d_W f & d_W^2, \end{pmatrix} = 0$$

since d_V, d_W are differentials and f is a map of dg vector spaces, so

$$fd_V = d_W f. \tag{2}$$

To prove the second part of the definition, note that we have a short exact sequence

$$0 \longrightarrow W[-1] \longrightarrow \mathrm{Cone}(f) \longrightarrow V \longrightarrow 0.$$

This gives us a long exact sequence in homology (see fore example [May99, Chapter 12.4])

$$\cdots \longrightarrow H_q(W[-1]) \longrightarrow H_q(\mathrm{Cone}(f)) \longrightarrow H_q(V) \xrightarrow{\partial} H_{q-1}(W[-1]) \longrightarrow \cdots.$$

The connecting map sends $[v]$ in $H_q(V)$ to $[w]$ in $H_{q-1}(W[-1])$, where w is some element in $W[-1]$ such that $(0, w) \in \mathrm{Cone}(f)$ is a a boundary of the form $D(v, w')$. We thus have $(d_v v, fv - d_w w') = (0, w)$. Thus $[w] = [fv - d_w w'] = [fv]$, so ∂ is simply the map induced by f between $H_q(V) \to H_{q-1}(W[-1])$. If the cone is acyclic, this implies that ∂ is an isomorphism, which means precisely that f is a quasi-isomorphism. Conversely, if f is a quasi-isomorphism, then ∂ is an isomorphism, which means that $H_q(\mathrm{Cone}(f)) = 0$ for all $q \geqslant 0$, and thus the cone is acyclic. ■

To state the next lemma, we need the following definitions:

Definition 6.2. Let U be a graded vector space. A filtration $U \supseteq F_0U \supseteq F_1U \supseteq \cdots$ is said to be *complete* if the natural map from U to the inverse limit $\varprojlim(U/F_pU)$, is an isomorphism of graded vector spaces. The inverse limit $\varprojlim(U/F_pU)$ is called the completion of U with relation to this filtration, so we may say that a filtration of U is a complete if the natural map from U to the completion with relation to that filtration is an isomorphism of graded vector spaces.

Definition 6.3. Suppose that (U,d) is a dg vector space. Let $U = F_0U \supset F_1U \supset \cdots$ be a complete filtration of U that is preserved by the differential, in the sense that $dF_pU \subset F_pU$. We then define the *associated graded vector space* to U by

$$\mathrm{gr}(U) := \prod_p F_pU/F_{p+1}U$$

Note that since d preserves the filtration, we get an induced differential d_0, making $(\mathrm{gr}(U), d_0)$ into a dg vector space.

Remark 6.4. In the following lemma (and at several points further on) we will use an alternative characterization of Definition 6.2: a filtration of U is complete if any series of the form

$$\sum_{p\geqslant 0} v_p,$$

where $v_p \in F_pU$, is convergent. Proving this is elementary, so we will leave it as an exercise. Note that by this characterization it follows that no non-zero element of U can be contained in F_pU, for all $p \geqslant 0$.

Lemma 6.5. If $(\mathrm{gr}(U), d_0)$ is acyclic, then so is (U,d).

Proof. If a is an element of F_pU, let us denote its equivalence class in $F_pU/F_{p+1}U$ by $[a]$. First, note that if $(\mathrm{gr}(U), d_0)$ is acyclic, it follows that if $a \in F_pU$ and $da \in F_{p+1}U$ (i.e. $d_0[a] = 0$), then there exists $b \in F_pU$ such that $db = a + c$, where $c \in F_{p+1}U$ (so $d_0[b] = [a]$).

Now suppose $a \in U = F_0U$ is such that $da = 0$. By the previous paragraph, it follows that there exists some $b_0 \in U$ such that $db_0 - a = c_1 \in F_1U$. It thus holds that $dc_1 = 0$. By induction we may thus choose b_p such that

$$a - d\sum_{i=0}^{p} b_p = c_{p+1} \in F_{p+1}U.$$

By Remark 6.4 it must therefore hold that $a - d\sum_{i\geqslant 0} b_p = 0$. Thus a is a boundary, which proves that U is acyclic. ■

We will now use Lemmas 6.1 and 6.5 to prove the following proposition, which is the most important of this section:

Proposition 6.6. Let F_pV and F_pW be complete, differential preserving filtrations on the dg vector spaces (V, d_V) and (W, d_W), respectively. Suppose that $f : (V, d_V) \to (W, d_W)$ is a map of dg vector spaces that preserves that filtration, i.e. $f(F_pV) \subset F_pW$. We then get an induced map of dg vector spaces $f_0 : \mathrm{gr}(V) \to \mathrm{gr}(W)$. If this is a quasi-isomorphism, then so is f.

Proof. We clearly have

$$\mathrm{gr}(V) \oplus \mathrm{gr}(W[-1]) = \prod_p (F_pV/F_{p+1}V) \oplus \prod_p (F_pW[-1]/F_{p+1}W[-1])$$

$$\cong \prod_p \left(\frac{F_pV \oplus F_pW[-1]}{F_{p+1}V \oplus F_{p+1}W[-1]} \right) = \mathrm{gr}(V \oplus W[-1])$$

It is furthermore clear that the differential on the cone $\mathrm{gr}(V) \oplus \mathrm{gr}(\mathrm{W[-1]})$ (i.e. $(d_{V,0}, f_0 - d_{W,0})$ is the same as the differential induced on $\mathrm{gr}(V \oplus W[-1])$ by the differential on the cone $V \oplus W[-1]$ (i.e. $(d_V, f - d_W)_0$). Thus

$$\mathrm{Cone}(f_0) = \mathrm{gr}(\mathrm{Cone}(f)).$$

If f_0 is a quasi-isomorphism, it follows by Lemma 6.1 that $H(\mathrm{Cone}(f_0)) = 0$, which by the above equation and Lemma 6.5 implies that $H(\mathrm{Cone}(f)) = 0$. But then, also by Lemma 6.1, it follows that f is a quasi-isomorphism, and we are done. ■

Remark 6.7. Note that taking the homology commutes with direct products:

$$H_\bullet \left(\prod_p (V_p, d_p) \right) = \prod_p H(V_p, d_p).$$

This means that when applying Proposition 6.6, it is sufficient to prove that every component of f is a quasi-isomorphism to prove that f_0 is a quasi-isomorphism.

An intuitive way to view this lemma is that it allows us to choose a filtration that "filters" away part of the differential on a space when we want to prove that a map is a quasi-isomorphism. A nice example of this is the following, which we will have use for in Section 9:

Example 6.8. A resolution of $H_\bullet(G) \otimes H_\bullet(BG)$.

Recall the coalgebra $\mathcal{H} = H_\bullet(G) \otimes H_\bullet(BG)$ defined in Section 4. In Section 8 we will see that when n is odd, it will be necassary to resolve $H_\bullet(G)$ with $\widehat{H}_\bullet(G)$ to construct Khoroshkin and Willwacher's model for the framed little n-disks operad. In Section 9 this will force us to replace $\mathcal{H}$ with $\hat{\mathcal{H}} := \widehat{H}_\bullet(G) \otimes H_\bullet(BG)$ when studying the case where n is odd.

The differential on $\hat{\mathcal{H}}$ has two parts. The first is defined analogously to the differential on $\mathcal{H}$, but instead use the projection

$$\pi : H_\bullet(BG) \to \mathbb{R}\{p_4, \ldots, p_{2n-2}, p_4^2, \ldots, p_{2n-2}^2, p_4^3, \ldots\}[-1]$$

composed with the inclusion of this into $H_\bullet(BG)[-1] \subseteq \widehat{H}_\bullet(G)$. We can explicitly describe this map by

$$a \otimes b \mapsto a.\iota\pi(b') \otimes b'',$$

for $a \in \widehat{H}_\bullet(G)$ and $b \in H_\bullet(BG)$. Verifying that this is indeed a differential is analogous to the proof for $\mathcal{H}$. The differential also has an additional piece coming from the differential on $\widehat{H}_\bullet(G)$. We want to show that $\hat{\mathcal{H}}$ is a resolution of $\mathcal{H}$, i.e. that the projection $f : \hat{\mathcal{H}} \to \mathcal{H}$ is a quasi-isomorphism. We already know that the map respects the differential on $\widehat{H}_\bullet(G)$. Note also that the map clearly respects the twisted differential, since the only part of this differential that survives after the projection is precisely the differential as defined on $\mathcal{H}$. We will now apply Proposition 6.6, so the first step is to define appropriate filtrations of $\hat{\mathcal{H}}$ and $\mathcal{H}$.

To do this, note first that we have a well-defined "word-length" of monomials in $H_\bullet(BG)$. In $\widehat{H}_\bullet(G)$, we combine the word-length of its $H_\bullet(BG)[-1]$-generators, with the length of the words in the associative algebra structure. The first part of the differential on $\hat{\mathcal{H}}$ increases the word-length in the $\widehat{H}_\bullet(G)$-component, and the second part of the differential (i.e. the differential from $\widehat{H}_\bullet(G)$) preserves this length. Thus the filtration where $F_p\hat{\mathcal{H}}$ is the subspace spanned by elements with word-length at least p in the $\widehat{H}_\bullet(G)$-component is preserved by the differential. It is also clear that any series

$$\sum_{p\geqslant 0} v_p,$$

with $v_p \in F_p\hat{\mathcal{H}}$, is convergent, since any term has a finite word length, and can thus only occur in the sum a finite number of times. We can use the same analogous filtration in $\mathcal{H}$. Since the projection f maps an element in the $\widehat{H}_\bullet(G)$-component to either zero or a an element with the same word-length, it preserves the filtration as well. Thus all hypotheses of Proposition 6.6. are fulfilled, so we only need to show that f_0 is a quasi-isomorphism. But this is now almost trivial, since the first part of the differential in $\hat{\mathcal{H}}$ increases the word-length, which means that this part is "filtered" by the filtration. The induced differential on each component of $\mathrm{gr}(\hat{\mathcal{H}})$ is thus induced completely by the second part of the differential, and on $\mathrm{gr}(\mathcal{H})$ it is zero.

Let us make this more precise. As a graded vector space, $\widehat{H}_\bullet(G)$ can be decomposed as a direct sum $\bigoplus_i \widehat{H}_\bullet(G)_i$, where the ith space is the subspace of elements of word-length i. Since homology commutes with direct sums, it follows that the projection $\widehat{H}_\bullet(G) \to H_\bullet(G)$ is a quasi-isomorphism on every such component. The map f_0 on the pth component of $\mathrm{gr}(\hat{\mathcal{H}})$ is precisely the tensor product between this component map and the identity on $H_\bullet(BG)$. Since these are quasi-isomorphisms on the components, it follows that f_0 is a quasi-isomorphism, which means that the original projection f is as well.

The graph complexes we consider in Section 8 are all invariant subspaces of a bigger space, under a certain finite group action. When considering their homology, the following proposition will be useful:

Proposition 6.9. Let (V, d) be a dg vector space and G be finite group acting on V by $(g, x) \mapsto gx$. Then

$$H_\bullet(V^G, d) = H_\bullet(V, d)^G,$$

where V^G denotes the subspace of V of invariants of the group action and $H_\bullet(V, d)^G$ is the subspace of invariants under the induced action $(g, [x]) \mapsto g[x] := [gx]$.

Proof. Let x be an element of $Z(V^G, d)$, i.e. it satisfies $gx = x$ and $dx = 0$. Then $g[x] = [x]$, so $[x] \in H^\bullet(V, d)^G$. We thus have a natural map

$$f : Z(V^G, d) \to H_\bullet(V, d)^G.$$

We want to prove that this map is surjective, and that its kernel is the subspace of invariant boundaries $B(V^G, d) := \{x \in V^G \mid x = dy\}$.

First, note that since G is finite we have the group averaging map

$$p(x) = \frac{1}{|G|}\sum_{g\in G} gx.$$

By definition of an action, the map $x \mapsto gx$ is linear and commutes with the differential in V. Thus we also have $dp = pd$. Furthermore $gp(x) = f(x)$ for any $g \in G$, so $p(x) \in V^G$.

Now suppose $[x] \in H_\bullet(V, d)^G$, which means that $[gx] = [x]$. It follows that $[p(x)] = [x]$. Since $p(x) \in V^G$, it follows that f is surjective.

Next, we want to show that $\ker f = B(V^G, d)$. Suppose that $f(x) = 0$. Then $x \in Z(V^G, d) \cap B(V, d)$. Since $x \in B(V, d)$ we have $x = dy$, for some $y \in V$. Furthermore, since $x \in V^G$, we have $p(x) = x$. Thus

$$x = p(x) = p(dy) = dp(y),$$

which means that $x \in B(V^G, d)$, since $p(y) \in V^G$ for any $y \in V$. We have thus shown that $\ker f \subset B(V^G, d)$, but clearly $B(V^G, d) \subset \ker f$, so we are done. ■

Now we have the sufficient tools for our proofs, so let us finally move on to discuss graph complexes.

7. Graph complexes

As stated in the introduction, a graph complex is a dg vector space in which the elements are linear combinations or series of some sort of graphs. Let us first specify what our notion of a graph is in this paper. We shall make this definition a bit more general than what is initially necessary, for reasons that will become clear in Section 7.2. Since we are mainly interested in the case where n is odd, we shall work under this assumption throughout this section. For even n, the definitions and proofs are analogous, and the differences are outlined in Remark 7.13 at the end of the section.

Definition 7.1. A graph is a $(k+3)$-tuple $(V_1, V_2, \dots, V_k, E, s, t)$, where $V_1, \dots, V_k$ are disjoint and ordered finite sets, E_Γ is another disjoint finite set, and $s, t : E \to V_1 \sqcup V_2 \sqcup \cdots \sqcup V_k$ are functions. We call the elements of V_i, for $1 \leqslant i \leqslant k$, *vertices of type i* and those of E *edges*. For an edge $e \in E$, we call $s(e)$ and $t(e)$ the *source* and *target* of the edge e, respectively.

We are thus considering directed graphs, that may have several types of different vertices. Until Section 7.2 we shall only consider graphs with one type of vertices.

In this section, we introduce two examples of graph complexes; the complexes GC_n and graphs_n, which were both defined by Kontsevich. We will start by defining the graph *operad* Gra_n, under the assumption that n is odd. The definitions follow [Willwacher12].

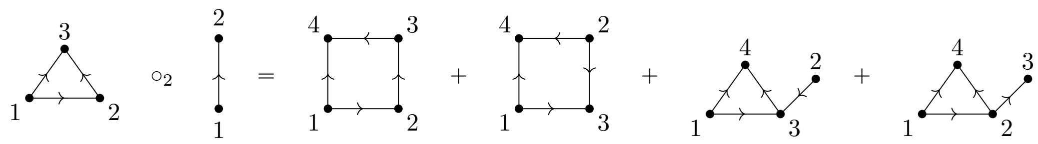
First, let $\mathsf{dgra}_{N,k}$ be the set of graphs of the form (V, E, s, t), with N vertices and k edges. Let $S_2^{\times k}$ act on the set of edges by flipping their direction and let sgn_2 be the one-dimensional sign representation of S_2. We use this to define a graded vector space $\mathsf{Gra}_n(N)$ by

$$\mathsf{Gra}_n(N) = \bigoplus_{k \geqslant 0} (\mathbb{R}\mathsf{dgra}_{N,k} \otimes_{S_2^{\times k}} \mathrm{sgn}_2^{\otimes k})[k(n-1)].$$

The space $\mathsf{Gra}_n(N)$ is thus the the space of linear combinations of such graphs, where we give edges the degree $n-1$ and flip the sign of a graph if we reverse the direction of an edge. Often, when drawing a graph from $\mathsf{Gra}_n(N)$, we will in omit the directions on edges for simplicity, and leave the absolute sign of the graph undetermined.

Remark 7.2. Note that since the sign of a graph is flipped if we flip the direction of an edge, a graph is automatically zero if it contains a "tadpole", i.e. an edge with common source and target.

Definition 7.3. The vector spaces $\mathsf{Gra}_n(N)$ assemble to form the graph operad Gra_n. The identity in this operad is simply the one vertex graph without edges. The symmetric action is given by permuting the vertex order and operadic composition $\gamma \circ_j \nu$ is given by inserting the graph ν into the jth vertex of the graph γ and summing over all ways of reconnecting the edges incident to j to vertices in ν:



where we use that flipping the orientation of an edge and transposing the labels on the two adjacent vertices are both flips the sign, so doing both at the same leaves the sign unaltered. We will now use this operad to construct the graph complex GC_n.

7.1. **The graph complex GC_n.** We will construct the graded vector space GC_n from the spaces $\mathsf{Gra}_n(N)$ in the operad. The complex GC_n is not an operad, but we will use the operadic composition from $\mathsf{Gra}_n(N)$ to give GC_n the structure of a differential graded Lie algebra. This Lie algebra is in fact *the Lie algebra associated to the operad* Gra_n (see [LV12, Chapter 5.4.3]), but we shall prove that it is a Lie algebra separately from this, as it illustrates the definitions we have made so far in a nice way. It is, however, worth noting that this is a special case of a more general construction.

Let sgn_N be the one-dimensional sign representation of S_N and note that S_N acts on $\mathsf{Gra}_n(N)$ by permuting the numbering of the vertices. We use this to define

$$\mathsf{GC}_n := \prod_{N\geqslant 0} ((\mathsf{Gra}_n(N) \otimes_{S_N} \mathrm{sgn}_N)[n(1-N)])^{S_N}.$$

This is thus the space of invariants under the action of the permutation group on the vertices of graphs. The tensor product with the sign representation indicates that when we permute the order on the edges of a graph, we change the sign accordingly. Note that since we are taking the direct product here, and not the direct sum, we are allowing infinite series of graphs in GC_n. Note also that by the definition of $\mathsf{Gra}_n(N)$ and GC_n the degree of a graph γ in GC_n is

$$|\gamma| = k_\gamma(n-1) - n(N_\gamma - 1),$$

where k_γ is the number of edges and N_γ is the number of vertices in γ.

Remark 7.4. The act of taking invariants in the definition of GC_n should be viewed as a way of making the vertices "indistinguishable" from each other. For this reason, we will draw a graph in GC_n simply as an graph with unlabelled vertices, but must keep in mind that this represents a sum of such graphs over all ways of labeling the vertices, and with a specific direction on all edges. As in Gra_n we will often leave the absolute sign undetermined, by leaving out the orientation of edges when this is not relevant.

We will later give the structure of a graph complex, by adding a differential. But first, let us give it the structure of a graded Lie algebra, using the operadic composition from Gra_n. We define the bracket as:

$$[\gamma, \nu] = \gamma \bullet \nu - (-1)^{|\gamma||\nu|} \nu \bullet \gamma,$$

where $\nu \bullet \gamma$ denotes the sum over all ways of inserting the graph γ into a vertex of ν, i.e. $\gamma \bullet \nu = \sum_i \gamma \circ_i \nu$.

Remark 7.5. Note that if γ and ν are graphs, then

$$\deg(\gamma \bullet \nu) = (k_\gamma + k_\nu)(n-1) - n((N_\gamma + N_\nu - 1) - 1) = \deg \gamma + \deg \nu.$$

Proposition 7.6. This bracket satisfies the axioms for a Lie bracket on GC_n.

Remark 7.7. Before proving this, take note of the sign in the bracket. We will use the convention that if we insert ν into a vertex of γ, the ordering on the new graph puts the vertices of ν *after* those of γ (the opposite convention of course works just as well), where the first vertex of the inserted graph takes the place of the inserted graph in the ordering. In the coming proofs, sign arguments from the ordering of vertices will be ubiquitous. For example, if the graphs μ and ν are inserted into two different vertices of γ, we can inserted either of them first. If μ is inserted first, the ordering of vertices will first have those of γ, then $N_\mu - 1$ vertices from μ and lastly $N_\nu - 1$ from ν. If ν is inserted first, the $N_\mu - 1$ vertices from μ must "jump over" those from ν, which means that the sign between the two cases will differ by $(-1)^{(N\mu-1)(N_\nu-1)}$. But

$$\begin{aligned}|\nu||\mu| = (k_\nu(n-1) - n(N_\nu - 1))(k_\mu(n-1) - n(N_\mu - 1)) &\equiv (n^2(N_\mu - 1)(N_\nu - 1)) \bmod 2 \\ &\equiv ((N_\mu - 1)(N_\nu - 1)) \bmod 2\end{aligned}$$

since n is odd. Thus when switching the numbering of the inserted graphs, the sign changes by $(-1)^{|\nu||\mu|}$. By the same reasoning, if the underlying graphs of some terms of $\nu \bullet \gamma$ and $\gamma \bullet \nu$ are the same, they will only differ by the sign $(-1)^{|\gamma||\nu|}$, which gives an intuitive idea of why we should have this sign in the bracket. We will see in the following proof that the sign is in fact necessary for the bracket to be a Lie bracket.

Proof of the proposition. For graphs γ, ν, μ we need to verify that the following two relations hold:

(1) Antisymmetry: $[\gamma, \nu] = -(-1)^{|\gamma||\nu|}[\nu, \gamma]$,
(2) Jacobi identity: $(-1)^{|\gamma||\mu|}[\gamma, [\nu, \mu]] + (-1)^{|\nu||\gamma|}[\nu, [\mu, \gamma]] + (-1)^{|\mu||\nu|}[\mu, [\gamma, \nu]] = 0$.

1. This follows by direct computation:

$$\begin{aligned}-(-1)^{|\gamma||\nu|}[\nu, \gamma] &= -(-1)^{|\gamma||\nu|}(\nu \bullet \gamma - (-1)^{|\gamma||\nu|}\gamma \bullet \nu) \\ &= \gamma \bullet \nu - (-1)^{|\gamma||\nu|}\nu \bullet \gamma \\ &= [\gamma, \nu].\end{aligned}$$

2. Here things get a bit messier. We have

$$(-1)^{|\gamma||\mu|}[\gamma, [\nu, \mu]] + (-1)^{|\nu||\gamma|}[\nu, [\mu, \gamma]] + (-1)^{|\mu||\nu|}[\mu, [\gamma, \nu]]$$

$$\begin{aligned}&= (-1)^{|\gamma||\mu|}\gamma \bullet (\nu \bullet \mu) - (-1)^{|\mu|(|\nu|+|\gamma|)}\gamma \bullet (\mu \bullet \nu) - (-1)^{|\nu||\gamma|}(\nu \bullet \mu) \bullet \gamma + (-1)^{|\nu|(|\gamma|+|\mu|)}(\mu \bullet \nu) \bullet \gamma \\ &+ (-1)^{|\nu||\gamma|}\nu \bullet (\mu \bullet \gamma) - (-1)^{|\gamma|(|\mu|+|\nu|)}\nu \bullet (\gamma \bullet \mu) - (-1)^{|\mu||\nu|}(\mu \bullet \gamma) \bullet \nu + (-1)^{|\mu|(|\nu|+|\gamma|)}(\gamma \bullet \mu) \bullet \nu \quad (*) \\ &+ (-1)^{|\mu||\nu|}\mu \bullet (\gamma \bullet \nu) - (-1)^{|\nu|(|\gamma|+|\mu|)}\mu \bullet (\nu \bullet \gamma) - (-1)^{|\gamma||\mu|}(\gamma \bullet \nu) \bullet \mu + (-1)^{|\gamma|(|\mu|+|\nu|)}(\nu \bullet \gamma) \bullet \mu\end{aligned}$$

It may not be obvious that this expression is zero at first look, but to see why this is we split it into the three parts: one part consists of those terms beginning with γ, the second with those beginning in ν and the last those beginning in μ. Let us take a look at the first part by itself:

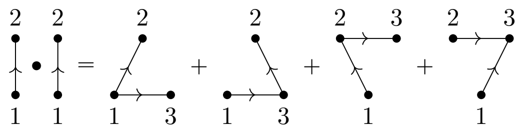
$$(-1)^{|\gamma||\mu|}\gamma \bullet (\nu \bullet \mu) - (-1)^{|\mu|(|\gamma|+|\nu|)}\gamma \bullet (\mu \bullet \nu) - (-1)^{|\mu||\gamma|}(\gamma \bullet \nu) \bullet \mu + (-1)^{|\mu|(|\nu|+|\gamma|)}(\gamma \bullet \mu) \bullet \nu$$

All terms of the first term occur also in the third term, but with the opposite sign. The remaining terms of the third term are those where μ are inserted to some vertex not from ν. These are all contained in the last term as well, but they differ by the sign $(-1)^{|\mu||\nu|}$ because of the different order on the vertices. Thus these cancel. The remaining terms of the last term are precisely those of the second term. Since those have the opposite sign, the entire expression is thus zero. Applying the same argument to the other two parts of $(*)$, it follows that it is zero, which means that the Jacobi-identity is satisfied. ■

We now use the Lie bracket to define our differential on GC_n. First, let us equip it with the zero differential, which trivially makes it into a dg Lie algebra. Let $\delta_0 = \frac{1}{2}$ •——•. The coefficient and index is added for reasons that will become clear later in Section 8. This graph has degree $1 \cdot (n-1) - n(2-1) = -1$ and we want to show that it is a Maurer-Cartan element in $(\mathsf{GC}_n, 0)$.

Proposition 7.8. The graph δ_0 satisfies $[\delta_0, \delta_0] = 0$.

Proof. First note that since $|\delta_0| = -1$, we have $[\delta_0, \delta_0] = -\frac{1}{2}\delta_0 \bullet \delta_0$. But, fixing a numbering and orientation, we get



All terms in the sum are zero. To see this, note that for every term, we get the same graph by interchanging the labels on the vertices on the ends, and either reversing the direction on both edges or none. But as long as we do the same thing to both edges the sign doesn't change. Switching the label does, however, so the graph must be zero. Thus we have $[\delta_0, \delta_0] = 0$. ■

Since the differential is zero, this implies that δ_0 is indeed a Maurer-Cartan element in $(\mathsf{GC}_n, 0)$. It follows by Proposition 5.2 that we can twist the differential by δ_0, and define $d : \mathsf{GC}_n \to \mathsf{GC}_n$ by:

$$d\gamma := [\delta_0, \gamma] = \delta_0 \bullet \gamma - (-1)^{|\gamma|}\gamma \bullet \delta_0$$

and this defines a new differential on GC_n. This is the standard differential on GC_n that we will consider going forth.

Remark 7.9. An intuitive way to think about this differential is that it consists of all ways of "splitting" a vertex into two, connected by a new edge. This is precisely what happens in the second term above. Note that in $\gamma \bullet \delta_0$ the new vertex comes last in the order, while in $\delta_0 \bullet \gamma$ it comes first in the order. The terms of $\gamma \bullet \delta_0$ where a univalent vertex is produced thus differs from the corresponding term of $\delta_0 \bullet \gamma$ by precisely the sign $(-1)^{|\delta_0||\gamma|} = (-1)^{|\gamma|}$, so we get cancellation. Thus the differential cannot produce a univalent vertex.

7.2. The graph complex graphs_n.

Having defined GC_n we move on to define our second graph complex of interest: graphs_n. This complex is constructed in a completely analogous fashion to GC_n, but unlike GC_n, graphs_n is an operad in the category of (differential) graded vector spaces. We start by defining the spaces $\mathsf{graphs}_n(r)$ from which it is assembled.

Structure of the graded vector spaces. First, let $\mathsf{dgra}_{r,N,k}$ be the set of graphs (V_1, V_2, E, s, t) with two types of vertices, out of which r are of the first kind and N are of the second, and with k edges. We shall call the first type of vertices *external* and the second type *internal*. As before, $S_2^{\otimes k}$ acts on the edges. We endow the set of *all* vertices with an order and let

$$\mathsf{Gra}_n(N,r) = \bigoplus_{k\geqslant 0} (\mathbb{R}\mathsf{dgra}_{r,N,k} \otimes_{S_2^{\times k}} \mathrm{sgn}_2^{\otimes k})[k(n-1)]$$

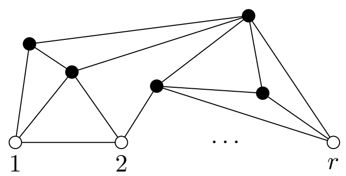
be the space of graphs with r external and N internal vertices, where once again the sign of a graph depends on the direction of its edges, and we give edges degree $n-1$.

As before, the group S_{N+r} acts by relabeling on the vertices in $\mathsf{Gra}_n(N,r)$. Using this, we define the spaces of the operad graphs_n as

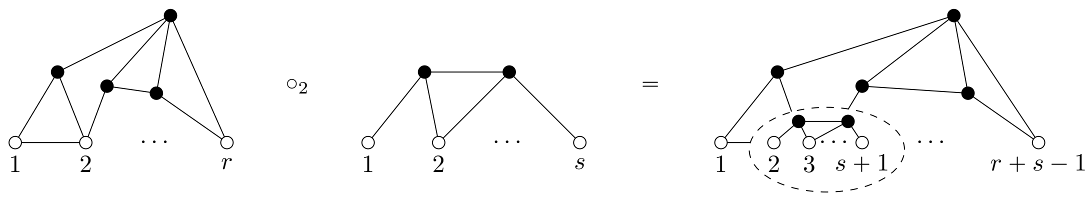
$$\mathsf{graphs}_n(r) := \prod_{N\geqslant 0} ((\mathsf{Gra}_n(N,r) \otimes_{S_{N+r}} \mathrm{sgn}_{N+r})[n(1-N-r)])^{S_N},$$

where the invariants are taken with respect to the action by S_N on the internal vertices. We thus make the internal vertices of a graph in $\mathsf{graphs}_n(r)$ "indistinguishable". Note in particular that $\mathsf{graphs}_n(0) = \mathsf{GC}_n$, i.e. we can view the graphs of GC_n as graphs with only internal vertices.

We illustrate an element of $\mathsf{graphs}_n(r)$ as a graph with undirected edges and where the internal vertices are unlabeled black dots, just as for GC_n, and the external vertices are numbered circles, as in the following example:



Operadic structure. We assemble the spaces $\mathsf{graphs}_n(r)$ into the operad graphs_n. The operadic structure is given by insertion insertion into external vertices, and summation over ways of reconnecting incoming edges. The space $\mathsf{graphs}_n(r)$ consists of all linear combinations of graphs with r external vertices, so the operadic composition $\circ_j$, i.e. insertion into the jth external vertex, does indeed go from $\mathsf{graphs}_n(r) \otimes \mathsf{graphs}_n(s) \to \mathsf{graphs}_n(r+s-1)$. For example:



Just as we did for GC_n, we will initially consider this as a graph complex with the zero differential and then use Proposition 5.3 to construct our differential, by twisting by the Maurer-Cartan element δ_0 from GC_n. For this we need an action by $(\mathsf{GC}_n, 0)$ on $(\mathsf{graphs}_n, 0)$, which is what we will construct in the following subsection. This construction is taken from [Willwacher12].

7.3. **The action by GC_n on graphs_n.** We construct our action as follows:

Let $\gamma \in \mathsf{GC}_n$ and $\Gamma \in \mathsf{graphs}_n$. Construct the graph γ_1 in graphs_n by marking vertex 1 in γ as external (note that since γ is the sum over all possible ways to number the vertices on its underlying graph, we are not singling one vertex out as special here). The action by γ on Γ is then defined as:

$$\gamma \cdot \Gamma = \gamma_1 \circ_1 \Gamma - (-1)^{|\gamma||\Gamma|}(\Gamma \bullet \gamma + \sum_j \Gamma \circ_j \gamma_1),$$

where $\circ_j$ denotes the operadic composition in graphs_n and we use $\bullet$ to denote the insertion of γ into internal vertices of Γ, as in the definition of the bracket in GC_n.

Proposition 7.10. The map $\mathsf{GC}_n \otimes \mathsf{graphs}_n(r) \to \mathsf{graphs}_n(r)$ defined by $\gamma \otimes \Gamma \mapsto \gamma \cdot \Gamma$ as above defines a dg Lie algebra action on the operad graphs_n, with respect to the zero differentials on GC_n and graphs_n.

Proof. To prove that this is an action, we need to show that it respects the differentials, that it is an operadic derivation and finally that it respects the Lie bracket in GC_n. Here we consider both graphs_n and GC_n with the zero differential, so the first property is fulfilled trivially.

Remark 7.11. Recall that when inserting a graph in GC_n, we assumed that its vertices are put *last* in the order. Here we extend this convention to both external and internal insertion.

The action is an operadic derivation. Here we need to show that if Γ, Γ are graphs in graphs_n and γ is a graph in GC_n, then

$$\gamma \cdot (\Gamma \circ_j \Gamma') = (\gamma \cdot \Gamma) \circ_j \Gamma' + (-1)^{|\gamma||\Gamma|} \Gamma \circ_j (\gamma \cdot \Gamma').$$

We have

$$\gamma \cdot (\Gamma \circ_j \Gamma') = \gamma_1 \circ_1 (\Gamma \circ_j \Gamma') - (-1)^{|\gamma|(|\Gamma|+|\Gamma'|)} \left((\Gamma \circ_j \Gamma') \bullet \gamma + \sum_i (\Gamma \circ_j \Gamma') \circ_1 \gamma_1 \right) \tag{3}$$

while

$$(\gamma \cdot \Gamma) \circ_j \Gamma' = (\gamma_1 \circ_1 \Gamma) \circ_j \Gamma' - (-1)^{|\gamma||\Gamma|} \left((\Gamma \bullet \gamma) \circ_j \Gamma' + \sum_i (\Gamma \circ_i \gamma_1) \circ_j \Gamma' \right) \tag{4}$$

and

$$(-1)^{|\gamma||\Gamma|} \Gamma \circ_j (\gamma \cdot \Gamma')$$

$$= (-1)^{|\gamma||\Gamma|} \Gamma \circ_j (\gamma_1 \circ_1 \Gamma') - (-1)^{|\gamma|(|\Gamma|+|\Gamma'|)} \left(\Gamma \circ_j (\Gamma' \bullet \gamma) + \sum_i \Gamma \circ_j (\Gamma' \circ_i \gamma_1) \right) \tag{5}$$

Let us match the terms in these expressions. That the second term of (3) is equal to the sum of the second terms of (4) and (5) is clear, since we can either insert into an internal vertex from Γ or from Γ'. The signs are also correct, by our vertex order convention. The same is true for the third term of (3), the third term of (5) and all terms of third term of (4) except for the one where i is equal to j. This term is instead cancelled by those terms in the first term of (5) where at least some edge of Γ is connected to a vertex of $\gamma_1 \circ_1 \Gamma$ coming from γ_1. The remaining terms of the first term of (5), together with the first term of (4) correspond precisely to the last term of (3). Thus our action is indeed an operadic derivation.

The action preserves the Lie bracket. Recall that the Lie bracket on GC_n is given by

$$[\gamma,\nu]=\gamma\bullet\nu-(-1)^{|\gamma||\nu|}\nu\bullet\gamma,$$

What we need to show is that

$$(\gamma\bullet\nu-(-1)^{|\gamma||\nu|}\nu\bullet\gamma)\cdot\Gamma=\gamma\cdot(\nu\cdot\Gamma)-(-1)^{|\gamma||\nu|}\nu\cdot(\gamma\cdot\Gamma) \tag{6}$$

Let us unpack both sides. We have:

$$(\gamma\bullet\nu-(-1)^{|\gamma||\nu|}\nu\bullet\gamma)\cdot\Gamma$$

$$\begin{aligned}&=(\gamma\bullet\nu)_1\circ_1\Gamma-(-1)^{|\Gamma|(|\gamma|+|\nu|)}\left(\Gamma\bullet(\gamma\bullet\nu)+\sum_j\Gamma\circ_j(\gamma\bullet\nu)_1\right)\\&-(-1)^{|\gamma||\nu|}(\nu\bullet\gamma)_1\circ_1\Gamma+(-1)^{|\gamma||\nu|+|\Gamma|(|\gamma|+|\nu|)}\left(\Gamma\bullet(\nu\bullet\gamma)+\sum_j\Gamma\circ_j(\nu\bullet\gamma)_1\right)\end{aligned} \tag{7}$$

Furthermore

$$\gamma\cdot(\nu\cdot\Gamma)-(-1)^{|\gamma||\nu|}\nu\cdot(\gamma\cdot\Gamma)$$

$$\begin{aligned}&=\gamma_1\circ_1(\nu_1\circ_1\Gamma)-(-1)^{|\nu||\Gamma|}\left(\gamma_1\circ_1(\Gamma\bullet\nu)+\sum_j\gamma_1\circ_1(\Gamma\circ_j\nu_1)\right)\\&+(-1)^{|\gamma|(|\Gamma|+|\nu|)}\Bigg((-1)^{|\nu||\Gamma|}\Bigg((\Gamma\bullet\nu)\bullet\gamma+\sum_j(\Gamma\circ_j\nu_1)\bullet\gamma+\sum_j(\Gamma\bullet\nu)\circ_j\gamma_1\\&\qquad+\sum_{i,j}(\Gamma\circ_j\nu_1)\circ_i\gamma_1\Bigg)-(\nu_1\circ_1\Gamma)\bullet\gamma-\sum_j(\nu_1\circ_1\Gamma)\circ_j\gamma_1\Bigg)\\&-(-1)^{|\gamma||\nu|}\nu_1\circ_1(\gamma_1\circ_1\Gamma)+(-1)^{|\gamma|(|\Gamma|+|\nu|)}\left(\gamma_1\circ_1(\Gamma\bullet\gamma)+\sum_j\nu_1\circ_1(\Gamma\circ_j\gamma_1)\right)\\&-(-1)^{|\nu||\Gamma|}\Bigg((-1)^{|\gamma||\Gamma|}\Bigg((\Gamma\bullet\gamma)\bullet\nu+\sum_j(\Gamma\circ_j\gamma_1)\bullet\nu+\sum_j(\Gamma\bullet\gamma)\circ_j\nu_1\\&\qquad+\sum_{i,j}(\Gamma\circ_j\gamma_1)\circ_i\nu_1\Bigg)-(\gamma_1\circ_1\Gamma)\bullet\nu-\sum_j(\gamma_1\circ_1\Gamma)\circ_j\nu_1\Bigg)\end{aligned} \tag{8}$$

First, we note that

$$(-1)^{|\gamma||\nu|}\Gamma\bullet(\nu\bullet\gamma)-\Gamma\bullet(\gamma\bullet\nu)=(-1)^{|\gamma||\nu|}(\Gamma\bullet\nu)\bullet\gamma-(\Gamma\bullet\gamma)\bullet\nu.$$

The terms on the right left side clearly occur on the right hand side as well, with the same sign. The remaining terms on the right hand side are those where ν and γ are inserted into different vertices of Γ, which means that they cancel.

Next, we note that

$$\sum_j\left((-1)^{|\gamma||\nu|}\Gamma\circ_j(\nu\bullet\gamma)_1-\Gamma\circ_j(\gamma\bullet\nu)_1\right)$$

$$= \sum_j \left((-1)^{|\gamma||\nu|} \Big((\Gamma \circ_j \nu_1) \bullet \gamma + \sum_i (\Gamma \circ_j \nu_1) \circ_i \gamma_1 \Big) - (\Gamma \bullet \gamma) \circ_j \nu_1 \right.$$
$$\left. -(\Gamma \circ_j \gamma_1) \bullet \nu - \sum_i (\Gamma \circ_j \gamma_1 \circ_i \nu_1 + (-1)^{|\gamma||\nu|} (\Gamma \bullet \nu) \circ_j \gamma_1 \right).$$

To see this, note that all the terms of the first term in the left hand side occur in the first and second terms of the right hand side, with the same sign. What remains of those two terms, are those cases in the first term where γ is inserted into some internal vertex of Γ. This term is, however, cancelled by the third term. We can use the same line of argument for the second term of the left hand side and the remaining terms of the right hand side.

Similarly, we have

$$(\gamma \bullet \nu)_1 \circ_1 \Gamma - (-1)^{|\gamma||\nu|} (\nu \bullet \gamma)_1 \circ_1 \Gamma$$
$$= \gamma_1 \circ_1 (\nu_1 \circ_1 \Gamma) + (-1)^{|\nu||\Gamma|} \Big((\gamma_1 \circ_1 \Gamma) \bullet \nu - \gamma_1 \circ_1 (\Gamma \bullet \nu) \Big)$$
$$-(-1)^{|\gamma||\nu|} \left(\nu_1 \circ_1 (\gamma_1 \circ_1 \Gamma + (-1)^{|\gamma||\Gamma|} \Big(\nu_1 \circ_1 \Gamma) \bullet \gamma - \gamma_1 \circ_1 (\Gamma \bullet \gamma) \Big) \right).$$

All terms of the first term in the left hand side are once again included in the first two terms of the right hand side, with the correct sign. The remaining terms are those where ν is inserted into an internal vertex of Γ in $\gamma_1 \circ_1 \Gamma$. But these are cancelled by the third term. We can use the same line of reasoning for the second term on the left hand side and the three last terms in the right hand side.

Now we have shown that all terms in (7) also occur in (8). We still have some remaining terms in (8), however, and need to show that these cancel. It is fortunately clear that

$$(\nu_1 \circ_1 \Gamma) \circ_j \gamma_1 = \nu_1 \circ_1 (\Gamma \circ_j \gamma_1),$$

and similarly that

$$(\gamma_1 \circ_1 \Gamma) \circ_j \nu_1 = \gamma_1 \circ_1 (\Gamma \circ_j \nu_1),$$

so the remaining terms do indeed cancel. This means that the action preserves the Lie bracket, and thus that we indeed have a dg Lie algebra action by $(\mathsf{GC}_n, 0)$ on $(\mathsf{graphs}_n, 0)$. ■

It now follows by Proposition 5.3 that we can define a new differential, which we will simply denote d, on graphs_n, by twisting by the Maurer-Cartan element δ_0. The differential is thus:

$$d\Gamma = \delta_0 \cdot \Gamma = (\delta_0)_1 \circ_1 \Gamma - (-1)^{|\Gamma|} \left(\Gamma \bullet \delta_0 + \sum_j \Gamma \circ_j (\delta_0)_1 \right).$$

By Proposition 5.4, the action we defined above also respects the twisted differentials (which we will from now on consider as the standard differentials) on GC_n and graphs_n, so it lifts to an action $(\mathsf{GC}_n, d) \circlearrowright (\mathsf{graphs}_n, d)$, with $d = [\delta_0, \cdot]$ on GC_n.

Remark 7.12. Note that just as in GC_n, the twisted differential in graphs_n amounts exactly to summing over all ways of splitting a vertex in a graph, where an external vertex is always split into one external and one internal vertex. Once again, the last term serves to cancel all the terms where a univalent vertex is produced. Note also that in $\mathsf{graphs}_n(0) = \mathsf{GC}_n$, the differential is precisely $d\Gamma = [\delta_0, \Gamma]$. This action thus gives us a clean way of introducing the vertex splitting differential on graphs_n. It will also play a key role in the following section, where we first use it to construct Khoroshkin and Willwacher's model for the framed little

n-disks operad, and then extend it to a larger Lie algebra acting on a larger graph complex, which we will then prove is another model for the same operad.

Remark 7.13. In this section, we have made all definitions and proofs under the assumption that n is odd, for the sake of brevity and since this is the case we are mainly interested in. If we assume that n is even, the definitions are almost identical, apart from how we define our graphs and their signs in the graph complexes. For odd n the we assumed that the graphs in GC_n and graphs_n have directed edges and an order on their vertices, and that their sign depends on the direction of edges and the order of its internal vertices. For even n, the we instead start with graphs with undirected vertices (so that we replace the functions s and t by one function assigning to each edge a pair of vertices) and instead of ordering the vertices, we impose an order on the *edges*. The degree of a graph is defined in the same way, but since n is even this means that vertices have even degree and edges have odd degree. All proofs in this section thus work almost verbatim in the even case, if we replace vertices by edges in the sign arguments. Note that when n is even, tadpoles are permitted in graphs, but by similar reasoning a graph is zero if it contains a multiple edge between two vertices, or more than one tadpole at the same vertex.

The graph complex graphs_n, with the vertex splitting differential, was originally introduced by Maxim Kontsevich in his paper [Kontsevich99], where he uses it to prove that the original little n-disks operad is *formal* for all $n \geqslant 1$, which means that it is modeled by its homology, i.e. the Poisson operad. Explicitly, he uses graphs_n as an intermediate in a zigzag between the two operads. Thus graphs_n is a model for the original little n-disks operad.

8. Khoroshkin and Willwacher's model for the framed little n-disks operad

The graph complex graphs_n is a model for the original little n-disks operad, but we are interested in its framed version. The first model for this operad was found by Thomas Willwacher and Anton Khoroshkin in their paper [KW17]. In this section we review the construction of this model[2].

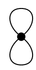
Recall that $\mathcal{D}_n^{fr}$ is the semi-direct product $\mathcal{D}_n \circ G$. We saw in Section 3.3 that in the algebraic setting, we can form an semi-direct product of a cocommutative Hopf algebra $\mathcal{H}$ and a dg operad $\mathcal{P}$ given an action by $\mathcal{H}$ on $\mathcal{P}$. Since we have a dg Operad model graphs_n for $\mathcal{D}_n$, the idea is to find an appropriate Hopf algebra acting on this graph complex, so that their semi-direct product is a model for $\mathcal{D}_n^{fr}$. Since the homology $H_\bullet(G)$ is a cocommutative Hopf algebra, this is a natural candidate. Willwacher and Khoroshkin show that for even n it is possible to define an action by $H_\bullet(G)$ on graphs_n such that the semi-direct product is indeed a model for $\mathcal{D}_n^{fr}$. This does unfortunately not work for odd n, however. Recall that in Section 5.4, we constructed a resolution $\widehat{H}_\bullet(G)$ of $H_\bullet(G)$. By replacing $H_\bullet(G)$ with this resolution, we can define an analogous action in the odd case, which results in a model for $\mathcal{D}_n^{fr}$.

[2]A caveat here is that in [KW17], the authors work in the dual setting, where instead of looking at chains on the operad $\mathcal{D}_n^{fr}$, they model so called PA-forms (PA is short for *piecewise algebraic*) on $\mathcal{D}_n^{fr}$, which has the structure of a *co*operad. Strictly speaking, the model that we are going to construct is therefore the operad with the dual structure to their model. They also consider cooperads in the category of commutative dg algebras, so called Hopf cooperads, instead of just in dg vector spaces. Dually, we could therefore add to our models in this paper the structure of dg coalgebras, which we have however chosen to omit because of time constraints.

We shall see that in both the even and odd cases the Hopf algebra action is defined using a certain Maurer-Cartan element in the dg Lie algebra $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$, which we call the *completed tensor product* of $H^\bullet(BG)$ and GC_n. We may for example construct this by taking the completion of $H^\bullet(BG)\otimes\mathsf{GC}_n$ with relation to the filtration where $F_p(H^\bullet(BG)\otimes\mathsf{GC}_n)$ is spanned by graphs with at least p vertices and edges. The elements of the completed tensor product can then be viewed as possibly infinite series of graphs in GC_n, each with a coefficient in $H^\bullet(BG)$. This is, however, isomorphic to $\mathrm{Hom}(H_\bullet(BG),\mathsf{GC}_n)$ as a dg Lie algebra. We shall therefore identify these two dg Lie algebras from now on. Let us first look at the simpler case, where n is even.

Proposition 8.1. For even n, the element $m = \tilde{E}$ , where $\tilde{E}$ is the Euler class in $H^\bullet(BG)$, is a Maurer-Cartan element in $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$.

Proof. The differential on $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ is only nonzero on the GC_n-component and the differential of  is zero since graphs with multiple edges are zero when n is even, and the differential in GC_n cannot produce univalent vertices. Since the degree of the tadpole graph is odd, we also have

$[$  $,$  $] = 2$  $\bullet$  $=$  $= 0.$

Thus m is indeed a Maurer-Cartan element in $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$. ■

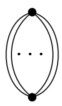
By Proposition 5.7 this Maurer-Cartan element corresponds to a dg coalgebra map $\mathcal{C}_*(\pi^{\mathbb{R}}(G)) \to \mathcal{C}_*(\mathsf{GC}_n)$, which in this case restricts to a linear map $\pi^{\mathbb{R}}(G) \to \mathsf{GC}_n$ defined by

$x \mapsto \begin{cases} \alpha\,\text{(tadpole)} & \text{if } x = \alpha E, \\ 0 & \text{otherwise,} \end{cases}$ 

where $\alpha \in \mathbb{R}$. Since $\pi^{\mathbb{R}}(G)$ is abelian and is equipped with the zero differential, and we have already proven that both the differential of the tadpole graph and its bracket with itself are both zero, this is a map of dg Lie algebras. Composing with the action by GC_n on graphs_n defined in the previous section, we get a dg Lie algebra action by $\pi^{\mathbb{R}}(G)$ on graphs_n. By Proposition 3.17 this extends to a Hopf algebra action by $\mathcal{U}\pi^{\mathbb{R}}(G) = H_\bullet(G)$ on graphs_n. It is the semi-direct product $\mathsf{graphs}_n \circ H_\bullet(G)$ with respect to this action that Khoroshkin and Willwacher prove is a model for the framed little n-disks operad, for even n. In fact, they also prove that this complex is a model for the homology of $\mathcal{D}_n^{fr}$ in this case, which means that the operad is in fact formal for all even n.

Now let us look at the case where n is odd.

Proposition 8.2. For odd n, the element

(9) $$m = \sum_{k\geqslant 1} \frac{p_{2n-2}^k}{4^k}\frac{1}{2(2k+1)!}$$  $(2k+1 \text{ edges})$

is a Maurer-Cartan element in $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$.

Remark 8.3. Before proving this, note that in this case we need to use the completed the tensor product, since the Maurer-Cartan element is an infinite series.

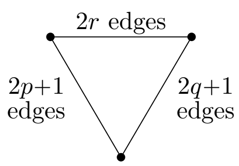
Remark 8.4. For the remainder of the paper we will use m both to denote this Maurer-Cartan element, as well as the Maurer-Cartan element from Proposition 8.1. Which is intended should be clear from context, as we will consider the even and odd cases separately.

Proof. For brevity, let us denote the graph with $2k+1$ vertices and the coefficient $c_k := \frac{1}{4^k \cdot 2(2k+1)!}$ by δ_k, so that $m = \sum_{k \geqslant 1} p_{2n_2}^k \delta_k$. Note that this agrees with our earlier definition of δ_0. For consistency, we will use this notation throughout this proof. We have

$$\begin{aligned} dm + \frac{1}{2}[m,m] &= \sum_{k \geqslant 1} \left([\delta_0, p_{2n-2}^k \delta_k] + \frac{1}{2} \sum_{\substack{i+j=k \\ 0<i,j}} [p_{2n-2}^i \delta_i, p_{2n-2}^j \delta_j] \right) \\ &= \sum_{k \geqslant 1} \sum_{\substack{i+j=k \\ 0 \leqslant i \leqslant j}} [p_{2n-2}^i \delta_i, p_{2n-2}^j \delta_j] \\ &= \sum_{k \geqslant 1} \sum_{\substack{i+j=k \\ 0 \leqslant i \leqslant j}} p_{2n-2}^{i+j} (\delta_j \bullet \delta_i + \delta_i \bullet \delta_j), \end{aligned}$$

where we in the last step use that $|\delta_k| = (2k+1)(n-1) - n$ (which is odd), for any k, which means that we get a positive sign in the bracket. Since all δ_k have an odd number of vertices, all graphs in the sum are of the form

(10)



for some non-negative integers p, q and r. If $p = q$ the graph is zero. To see this, fix an order on the vertices and a direction on all edges. We can, without loss of generality, assume that the edges to the right and left are directed the same way in relation to the bottom vertex. Flipping the order of the two upper vertices and the direction on the edges between them switches the sign, since the number of edges is even, but results in the same graph, which means that it must be zero.

If $p \neq q$, this graph appears in the two terms $\delta_{p+r} \bullet \delta_q$ and $\delta_{q+r} \bullet \delta_p$ and only in these two terms. By fixing an order on the vertices and direction on the edges, it is easily verified that the signs are opposite in these two terms. Thus we only need to verify that the coefficients are the same. Note that if we insert δ_q into δ_{p+r}, we can reconnect the vertices as in (10) in precisely $\binom{2(p+r)+1}{2r}$ ways. Thus the coefficient in the term $\delta_{p+r} \bullet \delta_q$ is

$$\begin{aligned} c_{p+r} c_q \binom{2(p+r)+1}{2r} &= \frac{1}{4^{p+r} \cdot 2(2(p+r)+1)!} \frac{1}{4^q \cdot 2(2q+1)!} \frac{(2(p+r)+1)!}{(2r)!(2p+1)!} \\ &= \frac{1}{4^{p+q+r+1}(2p+1)!(2q+1)!(2r)!}. \end{aligned}$$

Similarly, the coefficient in $\delta_{q+r} \bullet \delta_p$ is

$$\begin{aligned} c_{q+r} c_p \binom{2(q+r)+1}{2r} &= \frac{1}{4^{q+r} \cdot 2(2(q+r)+1)!} \frac{1}{4^p \cdot 2(2p+1)!} \frac{(2(q+r)+1)!}{(2r)!(2q+1)!} \\ &= \frac{1}{4^{p+q+r+1}(2p+1)!(2q+1)!(2r)!}, \end{aligned}$$

which proves that we get cancellation and thus that m is indeed a Maurer-Cartan element. ■

Since p_{2n-3}^k is zero for $k \geqslant 1$ in $H_\bullet(G)$, we can see that this element does not give us a well-defined map from $H_\bullet(G)$. This is the why we need the resolution $\widehat{H}_\bullet(G)$. By the corollary to Propositions 5.4 and 5.5, the Maurer-Cartan element m corresponds to a map of dg Lie algebras from $\mathcal{LC}_*(\pi^{\mathbb{R}}(G))$ to GC_n. Since $\mathcal{LC}_*(\pi^{\mathbb{R}}(G))$ is the free Lie algebra on $\overline{H_\bullet(BG)}[-1]$, we can describe this map explicitly by

$$x \mapsto \begin{cases} \delta_k & \text{if } x = s^{-1}\tilde{p}_{2n-4}^k \text{ for } k > 0 \\ 0 & \text{otherwise,} \end{cases} \tag{11}$$

for $x \in \overline{H_\bullet(BG)}[-1]$, using the notation from the previous proof. By composition with the action by GC_n on graphs_n this map gives us an action by the dg Lie algebra $\mathcal{LC}_*(\pi^{\mathbb{R}}(G))$ on graphs_n, which by Proposition 3.17 lifts to a Hopf algebra action

$$\mathcal{ULC}_*(\pi^{\mathbb{R}}(G)) = \widehat{H}_\bullet(G) \circlearrowright \mathsf{graphs}_n.$$

Note that since $\widehat{H}_\bullet(G)$ is the free associative algebra on $\overline{H_\bullet(BG)}[-1]$, the map $\widehat{H}_\bullet(G) \to \mathcal{U}\mathsf{GC}_n$ that induces this action is also completely described by (11). Let us denote the multiplication in $\widehat{H}_\bullet(G)$ by ".", to distinguish an element like $\tilde{p}_{4i}^2$ in $H_\bullet(BG)$ with a product $p_{4i}.p_{4i}$ in the free associative algebra. We can now use this action to form the semi-direct product with relation to this action, and this is precisely how we define Khoroshkin and Willwacher's graph complex for odd n.

Definition 8.5. For odd n, we define the Khoroshkin and Willwacher's graph complex as the semi-direct product

$$\mathsf{graphs}_n \circ \widehat{H}_\bullet(G).$$

Since the composition in $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ is "twisted" by the action defined by the Maurer-Cartan element $m \in H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$, we will denote it by $\circ_j^m$, to distinguish it from the original composition. If the jth external vertex of graph Γ in $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ is decorated by some $x \in \widehat{H}_\bullet(G)$, and Γ' is some other graph, twisted composition $\Gamma \circ_j^m \Gamma'$ consists of acting with x' on Γ' before composing. By (11) this action is zero unless x' is a scalar, in which case it is simply scalar multiplication, or x' is a product of the form $p^{i_1}.p^{i_2}.\cdots.p^{i_k}$, where p is the top Pontryagin class in $H_\bullet(BG)[-1]$. In this case, the action is

$$(p^{i_1}.p^{i_2}.\cdots.p^{i_k}) \cdot \Gamma = \delta_{i_1} \cdot (\delta_{i_2} \cdot \cdots (\delta_{i_k} \cdot \Gamma') \cdots). \tag{12}$$

In [KW17] it is proven that $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ is a model for $\mathcal{D}_n^{fr}$, for all odd n. It is also proven that the operad is not formal in this case. The model $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ is thus the only model we have in this case, which means that we might be able to define other models that are more practical than this one. This is the goal of the next section, where we define a new complex, which is "better" than $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$, in the sense that it comes with an action by a Lie algebra which is bigger than GC_n. In this way it models more of the derivations on the framed little n-disks operad than the previous model does. Most of the next section will then be spent trying to prove that this new graph complex is actually a model for $\mathcal{D}_n^{fr}$.

9. A new model for the framed little n-disks operad

When defining $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$, we used the Maurer-Cartan element m to "twist" the operadic composition by the action by $\widehat{H}_\bullet(G)$ on graphs_n. When we define our new model, we will instead use this action to twist the differential. We start this section by defining this complex, which we will denote by graphs_n^{dec}. This will be done in two steps. First, we define the underlying vector spaces and the operadic structure, in a similar way to how we defined graphs_n. We then prove that there exists an action by the dg Lie algebra $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ on this complex, which means that we can twist the differential by our Maurer-Cartan element m. The remainder of the section is then spent on proving that the complex with this twisted differential is weakly equivalent to the complex $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$, by constructing an explicit zigzag between them. Thus, we prove that graphs_n^{dec}, with its twisted differential, is a also a model for the framed little n-disks operad.

9.1. **The graph complex graphs_n^{dec}.** As the name suggests, the graphs of graphs_n^{dec} will look like those of graphs_n, but with decorated vertices. The external vertices will be decorated by elements from $H_\bullet(G)$, while the internal vertices will be decorated by elements from $\mathcal{H} := H_\bullet(G) \otimes H_\bullet(BG)$. The reason we include $H_\bullet(BG)$ in these internal decorations is so that we can utilize the duality between $H_\bullet(BG)$ and $H^\bullet(BG)$ when defining our action by $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$. We will define this complex for both even and odd n. Even though we already have formality in the even case, we will prove that this complex is a model in this simpler case as well, to illustrate the idea.

To construct the rth space of the operad graphs_n^{dec} for odd n, we begin by considering the tensor product

$$\mathsf{Gra}_n(N, r) \otimes \mathcal{H}^{\otimes N} \otimes H_\bullet(G)^{\otimes r},$$

where we view the tensor factors of $\mathcal{H}$ as decorations on the internal vertices of the graphs in $\mathsf{Gra}_n(N, r)$, and the factors of $H_\bullet(G)$ as decorations on external vertices. Furthermore, we extend the order on vertices in $\mathsf{Gra}_n(N, r)$ to the decorations, so that for any graph the set of vertices and decorations is ordered. This is done for sign reasons, just as the order on vertices in graphs_n. With this, we make the following definition:

Definition 9.1. We assemble the operad graphs_n^{dec} from the spaces

$$\mathsf{graphs}_n^{dec}(r) := \prod_{N\geqslant 0} (([\mathsf{Gra}_n(N, r) \otimes \mathcal{H}^{\otimes N} \otimes H_\bullet(G)] \otimes_{S_{2(N+r)}} \mathrm{sgn}_{2(N+r)})[n(1 - N - r)])^{S_N},$$

where the last tensor product is once again taken with respect to the action by $S_{2(N+r)}$ on the order of the vertices and decorations, so that permuting this order changes the sign of the graph accordingly. The S_N-invariants are taken with respect the action on the order on internal vertices. Similarly as in graphs_n, the symmetric action in $\mathsf{graphs}_n^{dec}(r)$ is defined by permuting the labels, as well as the decorations, on external vertices. The identity element is the graph with only one external vertex, decorated by 1.

In order to define the operadic composition, let Γ, Γ' be decorated graphs of $\mathsf{graphs}_n^{dec}(r)$ and $\mathsf{graphs}_n^{dec}(s)$, respectively, where the jth external vertex of Γ is decorated by some x in $H_\bullet(G)$ and Γ' has $s+N$ vertices in total. We define the operadic composition by taking the coproduct $\Delta^{s+N-1}(x) = x' \otimes \cdots \otimes x^{(s+N)}$ and multiplying *all* vertices of Γ' with the components of this coproduct from the left, identifying $H_\bullet(G)$ with the subspace $H_\bullet(G) \otimes \mathbb{R} \subseteq \mathcal{H}$ when multiplying with the internal decoratoins. We then insert this graph into the jth external vertex of Γ. That this satisfies the axioms of operadic composition is easily verified using the cocommutativity and associativity of the coproduct in $H_\bullet(G)$, together with the bialgebra condition for $H_\bullet(G)$ that $\Delta(xy) = \Delta(x)\Delta(y)$.

Remark 9.2. For even n we define the complex analogously, but replacing vertices with edges in the order.

In analogy with graphs_n, we will start by considering graphs_n^{dec} without the vertex splitting differential. Note here that we do however have a non-zero term in the differential, coming from the differential on $\mathcal{H}$. We define this simply as the sum of the differential on $\mathcal{H}$ over all internal vertices, with an extra sign factor in each term that depends on where the decoration is in the order, as the map needs to jump over all vertices and decorations that come before.

The next step is to prove that we have an action by $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$, which we do consider with the zero differential, so that we can define a new differential by Maurer-Cartan twisting.

Remark 9.3. Note that when we consider a decorated graph, we can use the bilinearity of the tensor product to write the element as a sum of terms with the same underlying graph, but where the decorations on all vertices are either basis vectors or 1. Since we are working in the category of dg vector spaces, all maps we consider are linear, which means that when determining what a map does to a decorated graph, we may always assume that the decorations are of this form, without loss of generality. Note also that this applies to $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ as well as to graphs_n^{dec}.

9.2. **An action by $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ on graphs_n^{dec}.** We want to define an action by $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ on graphs_n^{dec}, analogously to our action by GC_n on graphs_n. Recall that this action is defined as

$$\gamma \cdot \Gamma = \gamma_1 \circ_1 \Gamma - (-1)^{|\gamma||\Gamma|}\left(\Gamma \bullet \gamma + \sum_j \Gamma \circ_j \gamma_1\right). \tag{13}$$

We will define our new action in the same way on the level of graphs, but also taking decorations into account. This will be done by using the cap product $H^\bullet(BG)\otimes H_\bullet(BG) \to H_\bullet(BG)$, which can be written

$$\alpha \cap x = \alpha(x')x'',$$

for α in $H^\bullet(BG)$ and $x \in H_\bullet(BG)$. We modify the three parts of (13) as follows:

1) In the first part of the action we insert Γ into the external vertex of γ_1, so there is no decoration that we can naturally act on α with. We instead take the cap product with the coefficient of the graph. As in the second part, this means that this part of the differential will only be non-trivial if α is scalar, in which case we just multiply it with the coefficient.
2) Let $\alpha\gamma \in H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$, where $\alpha \in H^\bullet(BG)$ and $\gamma \in \mathsf{GC}_n$, such that γ has k vertices. Let v be the vertex in Γ at which we want to insert our graph and suppose it is decorated by $x \otimes y$, where $x \in H_\bullet(G)$ and $y \in H_\bullet(BG)$. We act on this decoration by taking the cap product with α, so the decoration becomes $x \otimes (\alpha \cap y)$. After this, we take the $(k-1)$th coproduct $\Delta^{k-1}(x\otimes(\alpha \cap y))$ in $\mathcal{H}$ of this new decoration, insert γ and decorate its vertices with the k components of this coproduct.
3) The third part can be modified in the same way, if we identify $H_\bullet(G)$ with the subspace $H_\bullet(G) \otimes \mathbb{R} \subseteq \mathcal{H}$. This means that if α is non-scalar, the cap product will be zero, so this component of the action will only be non-zero if $\alpha\gamma \in \mathsf{GC}_n$, in which case we simply insert the graph into an external vertex, and distribute its decoration onto the inserted graph's vertices using the coproduct.

Remark 9.4. Note that since α is an element of $H^\bullet(BG)$, it is of even degree, which means that we do not need to include it in sign factors.

Remark 9.5. Similarly as before, we use the convention that when inserting a graph from GC_n internally, we place its vertices after the vertices and decorations of the graph into which it is inserted, and before if the insertion is external. With this convention, it is clear that the proof for the action by GC_n on graphs_n extends to this case as well, on the level of graphs. Thus we only need to verify that corresponding graphs get the same decorations.

Proposition 9.6. The action by $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$ on graphs_n^{dec} above is a dg Lie algebra action on a dg operad.

Proof. We divide the proof into three parts, where we first prove that the action preserves the Lie bracket, then that it is an operadic derivation and lastly that it is compatible with differentials.

The action preserves the Lie bracket. Let $\alpha\gamma, \beta\nu \in H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$. The bracket in $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$ is given by

$$[\alpha\gamma, \beta\nu] = \alpha\beta[\gamma, \nu].$$

We need to prove that acting by this on a decorated graph Γ is the same as

$$\alpha\gamma \cdot (\beta\nu \cdot \Gamma) - (-1)^{|\gamma||\nu|}\beta\nu \cdot (\alpha\gamma \cdot \Gamma).$$

Since we already know the correspondence of terms in the Lie bracket of GC_n, we only need to look at the decorations and check that acting by the product $\alpha\beta$ is the same thing as acting with β first and then α. Since the first and last term of the action are zero whenever α or β are non-scalar, verifying this is only non-trivial in the terms where we insert internally twice (in the other cases, it follows directly from the associativity of the coproduct). To check this, note that the product $\alpha\beta$ evaluated on x in $H_\bullet(BG)$ is $(\alpha\beta)(x) = \alpha(x')\beta(x'')$, where the last product is taken in $\mathbb{R}$. In particular this means that

$$(\alpha\beta) \cap x = (\alpha\beta)(x')x'' = \alpha(x')\beta(x'')x''', \tag{14}$$

because of the associativity of the coproduct. Now suppose that γ has l vertices, ν has k vertices and that the vertex in Γ where we insert is decorated by $x \otimes y \in H_\bullet(G) \otimes H_\bullet(BG)$. Then when we insert $\beta\nu$, it gets decorated by

$$\begin{aligned}\Delta^{k-1}(x \otimes \beta \cap y) &= \Delta^{k-1}(x \otimes \beta(y')y'') \\ &= \beta(y')\Delta^{k-1}(x \otimes y'') \\ &= \beta(y')\left(x' \otimes y'', \cdots, x^{(k)} \otimes y^{(k+1)}\right).\end{aligned}$$

If we then insert $\alpha\gamma$ at the ith vertex of ν, cocommutativity and associativity of the coproduct implies that $\nu \bullet \gamma$ gets the decorations

$$\begin{aligned}&\beta(y')\left(x' \otimes y'', \cdots, \Delta^{l-1}(x^{(i)} \otimes \alpha \cap y^{(i+1)}), \cdots, x^{(k)} \otimes y^{(k+1)}\right) \\ &= \beta(y')\alpha(y^{(i+1)})\left(x' \otimes y'', \cdots, \Delta^{l-1}(x^{(i)} \otimes y^{(i+2)}), \cdots x^{(k)} \otimes y^{(k+2)}\right) \\ &= \beta(y')\alpha(y^{(i+1)})\left(x' \otimes y'', \cdots, x^{(i-1)} \otimes y^{(i)}, x^{(i)} \otimes y^{(i+2)}, \cdots, x^{(k+l)} \otimes y^{(k+l+2)}\right) \\ &= \beta(y')\alpha(y'')\left(x' \otimes y''', \cdots, x^{(k+l)} \otimes y^{(k+l+2)}\right).\end{aligned}$$

If we instead insert $\alpha\beta(\nu\bullet\gamma)$ directly, using (14) and cocommutativity, we get the decorations

$$\begin{aligned}\Delta^{k+l-2}(x\otimes(\alpha\beta)\cap y) &= \Delta^{k+l-2}(x\otimes\alpha(y')\beta(y'')y''')\\ &= \alpha(y')\beta(y'')\Delta^{k+l-2}(x\otimes y''')\\ &= \alpha(y')\beta(y'')\left(x'\otimes y''',\cdots,x^{(k+l)}\otimes y^{(k+l+2)}\right)\\ &= \beta(y')\alpha(y'')\left(x'\otimes y''',\cdots,x^{(k+l)}\otimes y^{(k+l+2)}\right).\end{aligned}$$

Thus the decorations are identical, which means that this action does indeed preserve the Lie bracket.

The action is an operadic derivation. Once again we only need to look at what happens to decorations. Let $\Gamma\in\mathsf{graphs}_n^{dec}(k)$, $\Gamma'\in\mathsf{graphs}_n^{dec}(l)$ and $\alpha\gamma\in H_\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$. We want to check that

$$\alpha\gamma\cdot(\Gamma\circ_j\Gamma') = (\alpha\gamma\cdot\Gamma)\circ_j\Gamma' + (-1)^{|\Gamma||\gamma|}\Gamma\circ_j(\alpha\gamma\cdot\Gamma')$$

Equality in the first term of the action is trivial, since the action on the decorations at internal insertion is independent from the external composition. Since the last part of the action is only nonzero if α is scalar, and this part does not affect the decorations on the graph we act on, the same correspondence as in the proof of Proposition 7.10. holds. We thus only need to look at the correspondence between the terms in the second part of the action and the remainder $\Gamma\circ_j(\gamma_1\circ_1\Gamma')$ of the last part.

Firstly, we have correspondence between the terms of $(\Gamma\circ_j\Gamma')\circ_i\gamma_1$ where i is one of the external vertices other than $j,j+1,\ldots,j+l-1$ and those of $(\Gamma\circ_i\gamma_1)\circ_j\Gamma'$ where $i\neq j$. This one is clear, since the insertions of Γ' and γ_1 are independent. The second correspondence is between the terms of $(\Gamma\circ_j\Gamma')\circ_i\gamma_1$ where $j\leqslant i\leqslant j+l-1$ and the terms of $\Gamma\circ_j(\Gamma'\circ_1\gamma_1)$. Suppose that Γ' has l vertices, with $H_\bullet(G)$-decorations $y_1,\ldots,y_l$ and γ has m vertices and that the jth external vertex of Γ is decorated by x. That the decorations agree here follows from the associativity and cocommutativity of the coproduct and the bialgebra condition $\Delta(xy)=\Delta(x)\Delta(y)$, which together imply that:

$$\begin{aligned}&((x'y_1)',\cdots,(x'y_1)^{(m)},x''y_2,\cdots,x^{(l)}y_l)\\ &=(x'y_1',\cdots,x^{(m)}y_1^{(m)},x^{(m+1)}y_2,\cdots x^{(m+l)}y_l).\end{aligned}$$

The last correspondence, which is between the $(\Gamma\circ_j\gamma_1)\circ_j\Gamma'$ and $\Gamma\circ_j(\gamma_1\circ_1\Gamma')$, holds by the same reasoning. Thus the action is an operadic derivation.

Compatibility with differentials. We consider $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ with the zero differential and graphs_n^{dec} with the differential from $\mathcal{H}$, which we shall denote by $d_\mathcal{H}$, so it only remains to verify that

$$d_\mathcal{H}(\alpha\gamma\cdot\Gamma) = (-1)^{|\gamma|}\alpha\gamma\cdot d_\mathcal{H}\Gamma. \tag{15}$$

Since only internal vertices are decorated by elements of $\mathcal{H}$, the last two parts of the action satisfy this trivially.

Suppose some internal vertex of Γ is decorated by $x\otimes y$, where $x\in H_\bullet(G)$ and $y\in H_\bullet(BG)$, and that γ has r vertices. If we first apply the differential $d_\mathcal{H}$, we get

$$d_\mathcal{H}(x\otimes y) = x\iota\pi(y')\otimes y''.$$

Next we apply α to this to get

$$\alpha \cap (d_{\mathcal{H}} x \otimes y) = \alpha(y''')x\iota\pi(y') \otimes y'',$$

using associativity and cocommutativity. After inserting γ, we decorate with the components of $\Delta^{r-1}(\alpha(y''')x\iota\pi(y') \otimes y'')$, which are

$$\begin{aligned}&\alpha\left(y^{(r+2)}\right)\left((x\iota\pi(y'))' \otimes y'', \dots, (x\iota\pi(y'))^{(r)} \otimes y^{(r+1)}\right)\\ &= \alpha\left(y^{(r+2)}\right)\left(x'\iota\pi(y')' \otimes y'', \dots, x^{(r)}\iota\pi(y')^{(r)} \otimes y^{(r+1)}\right).\end{aligned}$$

But $\iota\pi(x')$ is primitive in $H_\bullet(G)$, so it follows that this is precisely

$$\sum_{i=1}^{r} \alpha\left(y^{(r+2)}\right)\left(x' \otimes y'', \dots, x^{(i)}\iota\pi(y') \otimes y^{(i+1)}, \dots, x^{(r)} \otimes y^{(r+1)}\right).$$

If we instead act by $\alpha\gamma$ first, we get the decorations

$$\alpha(y')\left(x' \otimes y'', \dots, x^{(r)} \otimes y^{(r+1)}\right)$$

applying $d_{\mathcal{H}}$ to each of the vertices gives us the decorations

$$\begin{aligned}&\sum_{i=1}^{r} \alpha(y')(x' \otimes y'', \dots, x^{(i)}\iota\pi(y^{(i)}) \otimes y^{(i+1)}, \dots, x^{(r)} \otimes y^{(r+2)})\\ &= \sum_{i=1}^{r} \alpha(y^{(r+2)})(x' \otimes y'', \dots, x^{(i)}\iota\pi(y') \otimes y^{(i+1)}, \dots, x^{(r)} \otimes y^{(r+1)}).\end{aligned}$$

We thus get the same decorations, up to sign. Since the vertices of γ come after the vertices and decorations of Γ in both the left and right hand sides of (15), $d_{\mathcal{H}}$ has to jump over as many objects in the order on both sides. By the definition of our action the sign on the left hand side is thus $(-1)^{|\gamma||\Gamma|}$, while on the right hand side it is $(-1)^{|\gamma||d_{\mathcal{H}}\Gamma|} = (-1)^{|\gamma|(|\Gamma|-1)}$. They thus differ by the sign $(-1)^{|\gamma|}$, which means that (15) holds. We hence have a well defined action by $(H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n, 0)$ on $(\mathsf{graphs}_n^{dec}, d_{\mathcal{H}})$. ■

Since the graph δ_0 is a Maurer-Cartan element with relation to the zero differential in GC_n it follows by Proposition 5.4 that this action extends to an action by $(H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n, [\delta_0, \cdot])$ on $(\mathsf{graphs}_n^{dec}, \delta_0\cdot)$. We will denote the vertex splitting differential $\delta_0\cdot$ on graphs_n^{dec} simply by d.

Remark 9.7. Note that this proof works just as well for odd and even n, so this action exists in both cases.

We can now use our Maurer-Cartan element to twist the differential. It is this graph complex graphs_n^{dec}, together with this twisted differential, that we want to prove is a new model for the framed little n-disks operad. This will be the main result of this thesis:

Theorem 9.8. For all n, the operad graphs_n^{dec} is weakly equivalent to Khoroshkin and Willwacher's model for the framed little n-disks operad.

We will prove this by constructing an explicit zigzag between this complex and Khoroshkin and Willwacher's complex. To illustrate the idea, we will begin with the even case, which is simpler.

9.3. **The zigzag between graphs_n^{dec} and $\mathsf{graphs}_n \circ H_\bullet(G)$ for even n.** In this case, it will suffice to define one intermediate complex between graphs_n^{dec} and $\mathsf{graphs}_n \circ H_\bullet(G)$ in our zigzag. Specifically, the zigzag will have the form

$$(\mathsf{graphs}_n^{dec}, d_{\mathcal{H}} + d + m\cdot, \circ_j) \xleftarrow{f} (\mathsf{graphs}_n^{dec}, d_{\mathcal{H}} + d, \circ_j^m) \xrightarrow{\pi} \mathsf{graphs}_n \circ H_\bullet(G),$$

where $d_{\mathcal{H}}$ is the differential on the decorations from $\mathcal{H}$, d is the vertex splitting differential, $\circ_j$ is the ordinary operadic composition of graphs_n and $\circ_j^m$ is the composition twisted by the Maurer-Cartan element m (modified so that all internal decorations on the inserted graph are multiplied by the components of the coproduct).

First of all we need to verify that this intermediate complex is actually well defined. The verification that the composition satisfies the axioms of operadic composition is trivial, so we will omit it. Since the underlying vector space is the same as for graphs_n^{dec}, we know that the differential squares to zero. We do, however, need to verify that the differential is actually a derivation, with respect to the twisted composition. Since the composition does not affect the $H_\bullet(BG)$-component of the internal decorations, it is clear that this is the case for $d_{\mathcal{H}}$, so it is sufficient to verify it for the vertex splitting part of the differential.

Proposition 9.9. Vertex splitting is a derivation with respect to the twisted differential $\circ_j^m$.

Proof. We want to check that

$$d(\Gamma \circ_j^m \Gamma') = d\Gamma \circ_j^m \Gamma' + (-1)^{|\Gamma|}\Gamma \circ_j^m d\Gamma'.$$

As usual the sign is needed because in $d(\Gamma \circ_j^m \Gamma')$, the map d needs to jump over the vertices of Γ when applying it to a vertex coming from Γ', while this is not the case in $\Gamma \circ_j^m d\Gamma'$. We can, without loss of generality, assume that the jth external vertex of Γ is decorated by E. In this case, the twisted composition gets two terms, one where we simply sum over all vertices of Γ' and multiply each vertex by E and then insert into Γ. That the differential respects this term is trivial. The other term consists of acting by E on Γ' and then inserting this. The action by E consists of summing over all ways of adding an edge in Γ' between non-adjacent vertices. But since the vertices are assumed to be non-adjacent, it is clear that vertex splitting is a derivation with respect to this term as well. ■

We can now move on to define the maps in our zigzag. We start with the map to the right, which is the simplest. We define this map as the tensor product of the identities on the underlying graph and external decorations, with the counits on the internal decorations, i.e. the projections $\mathcal{H} \to \mathbb{R}$, which we know are quasi-isomorphisms, since $\mathcal{H}$ is acyclic. Let us denote this map by π. We can describe π on basis elements as mapping graphs that have trivial internal decorations to themselves, and graphs that have non-trivial internal decorations to zero.

Proposition 9.10. The map $\pi : (\mathsf{graphs}_n^{dec}, d_{\mathcal{H}} + d, \circ_j^m) \to \mathsf{graphs}_n \circ H_\bullet(G)$ is a morphism of dg operads.

Proof. First we need to show that π respects the operadic composition. Suppose that Γ and Γ' are graphs that are trivially decorated on internal vertices. Suppose the jth external vertex of Γ is decorated with $x \in H_\bullet(G)$. When composing Γ and Γ', we first act by x' on Γ', which does not affect the decorations, since it is either zero, trivial, or the action by the tadpole. We then decorate Γ' by the components of the coproduct of x''. The terms where internal vertices get non-trivial decorations are mapped to zero, so what remains is precisely the terms of $\pi(\Gamma) \circ_j^m \pi(\Gamma')$. If either of Γ or Γ' has a non-trivial internal decoration, then so does their composition, so $\pi(\Gamma) \circ_j^m \pi(\Gamma') = \pi(\Gamma \circ_j^m \Gamma') = 0$. Thus π respects the operadic composition.

Next we need to verify that it respects the differentials. Suppose that Γ is a graph with trivial internal compositions, so that $\pi(\Gamma) = \Gamma$. Then $d_{\mathcal{H}}$ is zero on Γ, while vertex splitting may produce non-trivially decorated internal vertices when splitting an external vertex. These terms are, however, mapped to zero, and what remains are precisely the terms of the differential in $\mathsf{graphs}_n \circ H_\bullet(G)$. If Γ instead has some non-trivial internal decorations, then so does $d\Gamma$, while $d_{\mathcal{H}}\Gamma$ is either zero or results in a new graph with a nontrivial internal decoration. Thus $d\pi = \pi(d + d_{\mathcal{H}}) = 0$ in this case. ■

Now it only remains to prove that π is also a quasi-isomorphism. To prove this, we utilize our machinery from Section 6 for the first time.

Proposition 9.11. The map π is a quasi-isomorphism.

Proof. We are going to apply Proposition 6.6, so we begin by defining filtrations on our two spaces. We define $F_p(\mathsf{graphs}_n^{dec})$ as the space spanned by graphs with at least p vertices, and $F_p(\mathsf{graphs}_n \circ H_\bullet(G))$ in the same way. Since all terms of the differentials either increase or fix the number of vertices in a graph, it is clear that the differentials preserve the filtration. Since π either maps a graph to zero or to a graph with the same underlying graph, it is also clear that π also preserves the filtrations. Furthermore, since any graph has a finite number of vertices, it follows that the filtration is complete, using Remark 6.4. Thus we have reduced the problem to looking at the induced map π_0, between the associated spaces. In both these spaces, the pth factor is the space spanned by graphs with exactly p vertices. Thus the term of the differential on the associated space induced by d is zero.

We can decompose the pth factor of $\mathrm{gr}(\mathsf{graphs}_n^{dec})$ into a direct product of spaces of the form

$$\left(\mathbb{R}\{\Gamma\}) \otimes \mathcal{H}^N \otimes H_\bullet(G)^{\otimes r} \otimes_{S_{2p}} \mathrm{sgn}_{2p}\right)^{S_N},$$

where Γ is some graph with N internal vertices and r external vertices, such that $N + r = p$. Since homology commutes with direct products it suffices to show that the map induced by π_0 on each such subspace is a quasi-isomorphism. By Proposition 6.9. it furthermore suffices to show that it is a quasi-isomorphism on the space

$$\mathbb{R}\{\Gamma\} \otimes \mathcal{H}^N \otimes H_\bullet(G)^{\otimes r} \otimes_{S_{2p}} \mathrm{sgn}_{2p}.$$

But the induced map on this space is simply the tensor product of identities, with the counit $\mathcal{H} \to \mathbb{R}$, which we know is a quasi-isomorphism. By the Künneth theorem it thus follows that π_0 is a quasi-isomorphism, which by Proposition 6.6. implies that the same is true for π. ■

We have now filled in the first, and simplest, map in our zigzag. Despite having the same underlying vector spaces, connecting the two different copies of graphs_n^{dec} with each other is not as simple, since they both have different operadic composition and differential. We will denote the map between them by f. To distinguish these two complexes, let us for brevity use the notation

$$\mathsf{graphs}_n^{dec,1} := (\mathsf{graphs}_n^{dec}, d_{\mathcal{H}} + d + m\cdot, \circ_j)$$
$$\mathsf{graphs}_n^{dec,2} := (\mathsf{graphs}_n^{dec}, d_{\mathcal{H}} + d, \circ_j^m)$$

We will define $f : \mathsf{graphs}_n^{dec,2} \to \mathsf{graphs}_n^{dec,1}$ as a composite of maps that act at one vertex each. If $x \in H_\bullet(G)$, let us abuse notation a bit and use m to denote the map $H_\bullet(G) \to \mathcal{U}\mathsf{GC}_n$ defined by the Maurer-Cartan element m. Explicitly, recall that this map is given by

$$x \mapsto \begin{cases} \alpha & \text{if } x = \alpha E, \\ \alpha & \text{if } x = \alpha, \\ 0 & \text{otherwise,} \end{cases} \qquad \text{for } \alpha \in \mathbb{R}.$$

Now let v be some vertex of a decorated graph in $\mathsf{graphs}_n^{dec,2}$ and suppose that v is decorated by $x \otimes y$ in $\mathcal{H}$. We define the map f_v by taking the coproduct of x, acting by $m(x')$ from the right at v (i.e. by insertion) and decorate the resulting vertices by taking the iterated coproduct of $\tilde{x} \otimes x''$. Since $m(x')$ has at most one vertex, however, we never actually need to do the last step. Note that unless x contains E as a factor, this map is simply the identity. If it contains E as a factor, we get one term which is the identity and one term where we "replace" the E with a tadpole. Finally, we define f as to composite $f_{v_1} f_{v_2} \cdots f_{v_{N+r}}$ over all $N + r$ vertices. Is is clear that the maps all commute, so the order of the composition does not matter. Note that we can also write f as a sum, where we sum over all ways of adding tadpoles at vertices decorated with an E. It is clear that f is linear and since E has the same degree as an edge, it is also of degree 0.

Proposition 9.12. The map f is a map of symmetric sequences of dg vector spaces.

Proof. It is clear that the map f is equivariant with relation to the S_r-action on the component spaces. We thus only need to prove that

$$f(d + d_{\mathcal{H}}) = f(d + d_{\mathcal{H}} + m\cdot). \tag{16}$$

Note that since we can factorize f as a composition of maps acting at only one vertex and each term of the differential is a sum of maps acting at one vertex at a time, it is sufficient to prove that each map f_v satisfies (16). Let us first prove that

$$f_v d_{\mathcal{H}} = (d_{\mathcal{H}} + m\cdot) f_v \tag{17}$$

Suppose that $x \otimes y$ is an element of $\mathcal{H}$, such that $y = y_1 y_2 \cdots y_k$ does not contain $\tilde{E}$ as a factor, and x does not contains E as a factor. Suppose first that v is decorated by $x \otimes y$. Since f_v is just the identity when the decoration does not contain E and $m\cdot$ is zero if it does not contain $\tilde{E}$, it is clear that the identity holds in this case. Suppose next that v is decorated by $x \otimes Ey$. Since $m\cdot$ is zero in this case as well, (17) is trivially fulfilled once again.

Suppose now that the decoration is $x \otimes \tilde{E}^p y$, for some $p > 0$:

$$x \otimes \tilde{E}^p y \quad .$$

We can, without loss of generality, assume that the decoration comes first in the order. Let us denote $y_1 \cdots \hat{y}_i \cdots y_k$ by $y_{\hat{\imath}}$. Applying $d_{\mathcal{H}}$ to this then gives us

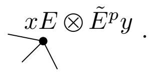
$$-p \; xE \otimes \tilde{E}^{p-1} y \; - \sum_{i=1}^{k} xy_i \otimes \tilde{E}^p y_{\hat{\imath}} \; ,$$

and if we apply f_v to this we get

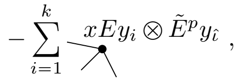
$$-p \; x \otimes \tilde{E}^{p-1} y \; - p \; xE \otimes \tilde{E}^{p-1} y \; - \sum_{i=1}^{k} xy_i \otimes \tilde{E}^p y_{\hat{\imath}} \; . \tag{18}$$

If we instead start by applying f_v, this is just the identity since there is no E-factor. Applying $d_{\mathcal{H}} + m\cdot$ directly, however, gives us precisely (18), up to sign. The only difference is that in the first term, we also get an extra sign factor as the tadpole is placed last in the order. In order to order it in the same way as in (18), it must however jump back over everything, which results in the same factor. Thus the signs agree.

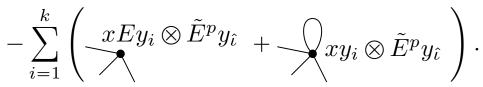
Lastly, suppose that the decoration is $xE \otimes \tilde{E}^p y$:



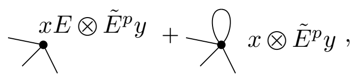
If we first apply $d_{\mathcal{H}}$, we get



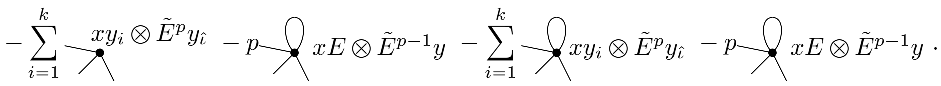
since $E^2 = 0$. Applying f_v to this gives us



If we instead start by applying f_v, we get



and if we the apply $d_{\mathcal{H}} + m\cdot$ to this, we get

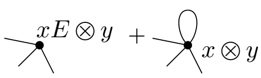


In the second term, m needs to jump over all the edges and decorations and the tadpole is then placed last in the order. In the last term, $d_{\mathcal{H}}$ needs to jump over all edges and decorations before this one, and then over x. To move the tadpole to the end it needs to jump over everything coming after this decoration. Thus the two terms differ by the sign $(-1)^{|E|} = -1$, so they cancel. Hence we get the same thing as when applying $d_{\mathcal{H}} f_v$. We have exhausted all cases, so (18) does indeed holds for all possible decorations of v.

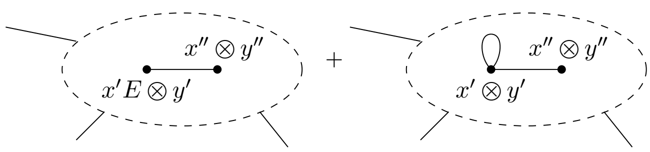
Next we prove that f commutes with the vertex splitting differential:

$$fd = df.$$

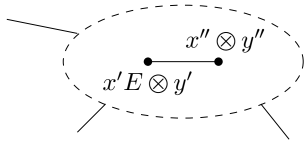
First suppose v is decorated by $x \otimes y$, where x does not contain E as a factor. Then f is just the identity, so it clearly commutes with vertex splitting. Suppose therefore that v is decorated by $xE \otimes y$. Applying f results in the terms



Now applying d gives us the terms



If we instead start by applying d we get the terms



and applying f to this gives the exact same thing as we got from df. Thus it follows that f is indeed a morphism of symmetric sequences of dg vector spaces. ■

Before we prove that f is a quasi-isomorphism, we also need to show that it respects the operadic structure.

Proposition 9.13. The map f is a morphism of dg operads.

Proof. Here we need to show that if Γ and Γ' are decorated graphs, then

$$f(\Gamma) \circ_j f(\Gamma') = f(\Gamma \circ_j^m \Gamma'). \tag{19}$$

It is clear that if v is a vertex of Γ, different from the jth external vertex, then

$$f_v(\Gamma) \circ_j \Gamma' = f_v(\Gamma \circ_j^m \Gamma'),$$

Thus we only need to look at what happens at the jth external vertex of Γ and in the inserted graph Γ'. This means that we may assume that Γ only has one external vertex and no internal vertices. We may also assume that this vertex is decorated by $E \in H_\bullet(G)$, because E is the only element of $H_\bullet(G)$ resulting in non-trivial terms of f and $\circ_j^m$.

We thus want to show that

$$f\left(\overset{\circ}{E}\right) \circ_1 f(\Gamma') = f\left(\overset{\circ}{E} \circ_1^m \Gamma'\right). \tag{20}$$

We have

$$f\left(\overset{\circ}{E}\right) = \overset{\circ}{E} + \tag{21}$$ 

Thus the left hand side consists of terms where we first apply f to Γ' and then

(1) multiply some decoration by E, or
(2) add a tadpole at some vertex, or
(3) add an edge between two non-adjacent vertices.

Let us now look at the right hand side. The action by E on Γ' amounts to summing over all ways of adding an edge between non-adjacent vertices. In

$$\overset{\circ}{E} \circ_1^m \Gamma' \tag{22}$$

we thus get terms where we do this, and then apply f. This corresponds precisely to the third case above. The remaining terms of (20) amount to multiplying some vertex of Γ' with E before composing. Applying f to this corresponds precisely to cases 1 and 2 above. Thus we have equality. ■

Now it only remains to prove that this is a quasi-isomorphism, which in this case is even more trivial than for the map π.

Proposition 9.14. The map f is a quasi-isomorphism.

We define filtrations on the two spaces, by letting the pth space in the filtration be spanned by graphs with at least p tadpoles. Since neither f nor any terms of the differentials can decrease the number of tadpoles in a graph (unless it maps it to zero), the filtrations are preserved by both of these. Since any graph has a finite number of tadpoles, this filtration is also complete. Since the twisted term $m\cdot$ of the differential on one of the spaces is either zero or increases the number of tadpoles, it follows that this part of the differential is zero on the associated space. But since the rest of the differential is identical on the two spaces, and they have the same underlying vector space, it follows that the associated spaces are equal. Since f can be written as a sum of the identity on the underlying spaces and terms that add tadpoles to a graph, it follows that f_0 is simply the identity map. Thus f_0 is trivially an isomorphism, which implies that f is as well, by Proposition 6.6. ■

We have thus proven Theorem 9.8 for all even n. Now let us move on to the case where n is odd, which is more complicated, but where we will use the same general idea.

9.4. The zigzag between graphs_n^{dec} and $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ for odd n.

The zigzag that we will construct in this case has the form

$$(\mathsf{graphs}_n^{dec}, d_1 + m\cdot + d_2, \circ_j) \xleftarrow{\pi} (\mathsf{graphs}_n^{\widehat{dec}}, d_1 + m\cdot + d_2 + d_3, \circ_j) \xleftarrow{f} (\mathsf{graphs}_n^{\widehat{dec}}, d_1 + d_2 + d_3, \circ_j^m)$$

$$\xrightarrow{\pi'} (\mathsf{graphs}_n \circ \widehat{H}_\bullet(G), d_1 + d_3, \circ_j^m),$$

where d_1 denotes the regular vertex splitting differential, d_2 is the differential on internal decorations, d_3 is the differential on external decorations and on the $\widehat{H}_\bullet(G)$-component of internal decorations. Note here that we include an additional intermediate graph complex in the zigzag. This is because in graphs_n^{dec}, we do not resolve $H_\bullet(G)$ in our decorations, while this is the case in $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$.

As a graded vector space, we define $\mathsf{graphs}_n^{\widehat{dec}}$ in the same way as we defined graphs_n^{dec}, but replacing $H_\bullet(G)$ with $\widehat{H}_\bullet(G)$ for the external decorations, and replacing $\mathcal{H}$ with $\hat{\mathcal{H}}$ for the internal decorations. Furthermore, we can define an action by $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ on this complex in the same way as we defined the action on graphs_n^{dec}. Proving that this action is well-defined is done almost verbatim as the proof of that action, apart from showing that it respects the differential on decorations, which we need to verify. The proof for the differential on $\hat{\mathcal{H}}$, which we here denote by d_2, is identical to the proof we have already made, so we will omit this. In this case, however, we also have a differential defined on $\widehat{H}_\bullet(G)$, which we need to take into account. We will denote this part of the differential by d_3. On a decoration $x_1.x_2.\cdots.x_k \in \widehat{H}_\bullet(G)$, d_3 acts like

$$d_3(x_1.x_2.\cdots.x_k)$$

$$= -\frac{1}{2}\sum_{i=1}^{k}(-1)^{|x_1|+\cdots+|x_{i-1}|}(x_1.\cdots.x_i'.x_i''.\cdots.x_k - (-1)^{|x'||x''|}x_1.\cdots.x_i''.x_i'.\cdots.x_k),$$

where (x_i', x_i'') is the coproduct from $H_\bullet(BG)$ (i.e. the deconcatenation coproduct). When acting by d_3 on a graph, note that we also get a sign factor from the degree -1 map having to jump over all vertices and decorations coming before x in the order.

Proposition 9.15. The action by $H^\bullet(BG)\widehat{\otimes}\mathsf{GC}_n$ on $\mathsf{graphs}_n^{\widehat{dec}}$, defined analogously as the action defined in the beginning of Section 9.2, respects the differential d_3 on $\widehat{H}_\bullet(G)$.

Proof. Recall that here we endow the Lie algebra $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$ with the zero differential. We want to check that if $\alpha\gamma$ is an element of $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$ and Γ is a decorated graph in $\mathsf{graphs}_n^{\widehat{dec}}$, then

$$d_3(\alpha\gamma \cdot \Gamma) = (-1)^{|\gamma|}\alpha\gamma \cdot d_3\Gamma. \tag{23}$$

Since we are looking at the differential on $\hat{H}_\bullet(G)$, we can without loss of generality assume that $\alpha = 1$. Suppose that some vertex, which can be either external or internal, is decorated by $x \in \widehat{H_\bullet(G)}$, where $x = x_1.x_2.\cdots.x_k$ for $x_1, \dots, x_k \in \overline{H_\bullet(BG)}[-1]$. It is clear that up to sign we get the same thing if we apply the differential before or after the last term of the action, since this term does not affect the decorations. Since d_3, which is of degree -1, needs to jump over the vertices of γ in the right hand side, the sign is also correct. We thus only need to look at the first and second terms of the action. The sign arguments are the same for these, so we will omit it.

Let us first look at the term where we apply d_3 to the ith component of x, and then insert the graph γ. We have

$$x_1.x_2.\cdots.d_3x_i.\cdots.x_k = -x_1.x_2.\cdots.x_i'.x_i''.\cdots.x_k + (-1)^{|x_i'||x_i''|}x_1.x_2.\cdots.x_i''.x_i'.\cdots.x_k,$$

where x_i' and x_i'' are the components of the coproduct of x_i in $H_\bullet(BG)[-1]$. Since all elements of $\overline{H_\bullet(BG)[-1]}$ have odd degree, the sign $(-1)^{|x_i'||x_i''|}$ is negative. We now insert γ and take the coproduct in $\hat{H}_\bullet(G)$, i.e. the deconcatenation product, of these terms to decorate the vertices inserted graph. Note that for every term of this coproduct where x_i' and x_i'' are separated, there is an corresponding term that cancels this coming from the second term, since we get another sign factor $(-1)^{|x'||x''|}$ that is due to the one of the factors having to jump over the other again. Thus this is the exact same thing as first taking the deconcatenation product of $x_1.x_2.\cdots, x_k$ and then applying d_3 to x_i, which is what we do when applying d_3 to $\gamma \cdot \Gamma$. Thus it follows that (23) holds. ∎

This means that we have an action by $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_N$ on $(\mathsf{graphs}_n^{\widehat{dec}}, d_2 + d_3, \circ_j)$, so the differential $d_1 + m\cdot = (\delta_0 + m)\cdot$ as above is well defined.

The third space $(\mathsf{graphs}_n, d_1 + d_2 + d_3, \circ_j^m)$, has the same underlying vector space, but the operadic composition is instead defined as in $\mathsf{graphs}_n \circ \hat{H}_\bullet(G)$, with the addition that when we insert a graph externally we also need to modify the decorations of internal vertices using the coproduct of the decoration of the external vertex in question. The first part of the differential is simply vertex splitting, while the second and third parts come from the internal and external decorations once again. Before we move on, we need to verify that this complex is actually well defined. We already know that $d_1 + d_2 + d_3$ is a well-defined differential on $\mathsf{graphs}_n^{\widehat{dec}}$, viewed simply as a dg vector space, so it only remains to verify that it is also an operadic derivation with respect to $\circ_j^m$. We shall do this in two parts. We start with the term d_2.

Proposition 9.16. The differential d_2 in $\mathsf{graphs}_n^{\widehat{dec}}$ is a derivation with respect to the twisted operadic composition $\circ_j^m$.

Proof. We need to show that if Γ and Γ' are decorated, then

$$d_2(\Gamma \circ_j^m \Gamma') = d_2\Gamma \circ_j^m \Gamma' + (-1)^{|\Gamma|}\Gamma \circ_j^m d_2\Gamma'. \tag{24}$$

This part of the differential is defined only on internal vertices, and since it is zero on $\widehat{H}_\bullet(G)$ (identified with the subspace $\widehat{H}_\bullet(G) \otimes \mathbb{R} \subseteq \hat{\mathcal{H}}$), it is zero on the vertices of the graph that may be inserted by the twisted differential before composing. Thus it follows that we can decompose $d_2(\Gamma \circ_j^m \Gamma')$ as a sum

$$\pm d_2\Gamma \circ_j^m \Gamma' \pm \Gamma \circ_j^m d_2\Gamma'.$$

The sign in the first term is positive, since the vertices and decorations of Γ come before those of Γ' in $\Gamma \circ_j^m \Gamma'$. The sign of the second term is thus $(-1)^{|\Gamma|}$, since d_2 has to jump over the vertices and decorations of Γ when applied to the decorations of Γ' in $\Gamma \circ_j^m \Gamma'$. Thus d_2 is indeed a derivation with respect to this composition. ■

Now let us look at for $d_1 + d_3$.

Proposition 9.17. The differential $d_1 + d_3$ in $\mathsf{graphs}_n^{\widehat{dec}}$ is a derivation with respect to the twisted operadic composition $\circ_j^m$.

Proof. Here we need to prove that

$$(d_1 + d_3)(\Gamma \circ_j^m \Gamma') = (d_1 + d_3)\Gamma \circ_j^m \Gamma' + (-1)^{|\Gamma|}\Gamma \circ_j^m (d_1 + d_3)\Gamma'. \tag{25}$$

It is clear that all terms of $d_1\Gamma \circ_j^m \Gamma' + (-1)^{|\Gamma|}\Gamma \circ_j^m d_1\Gamma'$ occur also in $d_1(\Gamma \circ_j^m \Gamma')$. The sign is correct by our usual line of reasoning. The remaining terms on the left hand side are those where a vertex in one the graphs inserted by $\circ_j^m$ is split.

The terms of $\Gamma \circ_j^m d_3\Gamma'$ are also all included in the terms of $d_3(\Gamma \circ_j^m \Gamma')$, and correspond to those terms where d_3 is applied to the decorations coming from Γ'. The terms of $d_3\Gamma \circ_j^m \Gamma'$ where d_3 is applied to other vertices other than the jth external vertex of Γ match precisely with those same terms of $d_3(\Gamma \circ_j^m \Gamma')$ and these also have the same sign. So do the terms where d_3 is applied to the jth external vertex, but the factor of x that d_3 is applied to is not the part of x'. The remaining terms on the right hand side of (25) are those where d_3 is applied to the jth external vertex of Γ before composition and then both new factors of x are inserted into Γ' in the composition. Note that by the same reasoning as in the proof of Proposition 9.15 we may also assume that that the second that is inserted is inserted into the first one. We want to show that these correspond precisely to the remaining terms on the left hand side.

Before proving this, let us make a brief interlude to illustrate the idea with a simple example. Suppose that the $j$the external vertex of Γ is decorated by p^k. Let us look at the term of $d_3\Gamma \circ_j^m \Gamma'$ where we apply d_3 to this decoration, and the insert at some vertex v of Γ'. We have

$$d_3(p^k) = -\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} (p^i.p^j + p^j.p^i)$$

Now note that acting by this is the same thing as acting by

$$-\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} (\delta_i \bullet \delta_j + \delta_j \bullet \delta_i).$$

If we instead look at the terms of $d_1(\Gamma \circ_j^m \Gamma')$ where d_1 splits a vertex coming from the graph δ_k. But this is the same as replacing the action by δ_k by $[\delta_0, \delta_k]$. But we know that we have

$$[\delta_0, \delta_k] = -\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} (\delta_i \bullet \delta_j + \delta_j \bullet \delta_i),$$

from the proof that m is a Maurer-Cartan element. thus the terms agree.

Now we need to generalize this to the case where we have an arbitrary element of $\widehat{H}_\bullet(G)$ decorating the jth external vertex of Γ. Suppose that the jth external vertex of Γ' is decorated by x. We can assume that x' has the form $p^{i_1}.p^{i_2}.\cdots.p^{i_k}$, where p is the desuspension of the top Pontryagin class of $H_\bullet(BG)$, since otherwise that term of the composition is zero.

Now act on Γ' with x' and look at one specific vertex of some inserted graph. This vertex might be shared between several of the inserted graphs, but by induction it follows that all edges of one of the inserted graphs, say δ_{i_l}, meet this vertex. Assume that δ_{i_l} is the last such graph that was inserted. Consider the term where d_1 is applied to this vertex. Since δ_{i_l} was the last graph inserted that contains this vertex, splitting this vertex after the composition is the same thing as splitting vertex before acting by $p^{i_{l+1}}$ and the following elements. If we instead start by applying d_3 to the factor p^{i_l} of x, and then compose, just as in our example we get terms where instead of δ_{i_l}, we insert

$$-\frac{1}{2} \sum_{\substack{i_l'+i_l''=i_l \\ i_l',i_l''>0}} [\delta_{i_l'}, \delta_{i_l''}].$$

But as in the example, since m is a Maurer-Cartan element we have

$$[\delta_0, \delta_{i_l}] = -\frac{1}{2} \sum_{\substack{i_l'+i_l''=i_l \\ i_l',i_l''>0}} [\delta_{i_l'}, \delta_{i_l''}].$$

Conversely, splitting a vertex in δ_{i_l} before inserting δ_{i_l+1} and so on always produces a graph that can be produced by first inserting all the graphs and then splitting. This means that the terms where d_3 is applied to the jth vertex of Γ before composing corresponds exactly to the terms of $d_1(\Gamma \circ_j^m \Gamma')$ where d_1 is applied to a vertex coming from the graphs inserted by the twisted composition.

We have thus showed that on the level of graphs, the equality holds. That the decorations are the same on both sides follows directly from the associativity and cocommutativity of the coproduct. Thus the equality (25) hold and the proof is completed. ■

Now that we know that all graph complexes in our zigzag are indeed well defined, we can move on to define the maps, and prove that they are quasi-isomorphisms. The maps on the ends of the zigzag will be the simplest, so let us start with those. Since we have the same notation for the underlying vector spaces of the two complexes in the middle of the zigzag, for brevity let us use the notation

$$\mathsf{graphs}_n^{\widehat{dec},1} := (\mathsf{graphs}_n^{\widehat{dec}}, d_1 + d_2 + d_3 + m\cdot, \circ_j),$$
$$\mathsf{graphs}_n^{\widehat{dec},2} := (\mathsf{graphs}_n^{\widehat{dec}}, d_1 + d_2 + d_3, \circ_j^m).$$

We start with the map to the left in the zigzag, which we will denote by π, and simply define as the projection $\mathsf{graphs}_n^{\widehat{dec},1} \to \mathsf{graphs}_n^{dec}$, defined by applying the projections $\hat{\mathcal{H}} \to \mathcal{H}$ and $\widehat{H}_\bullet(G) \to H_\bullet(G)$ to the internal and external decorations, respectively, and the identity on the remaining factors of the tensor product. Note that since the projection $\hat{\mathcal{H}} \to \mathcal{H}$ is simply the projection $\widehat{H}_\bullet(G) \to H_\bullet(G)$ on this component, the map π corresponds to applying the projection $\widehat{H}_\bullet(G) \to H_\bullet(G)$ to the $\widehat{H}_\bullet(G)$-component of each decoration.

Since the projection $\pi : \widehat{H}_\bullet(G) \to H_\bullet(G)$ is a map of dg Hopf algebras, it is clear that it commutes with vertex splitting, as well as with the differentials on decorations. Since the composition $\circ_j$ only applies the coproduct and product on decorations, it is also clear that π is a morphism of operads.

Proposition 9.18. The map π is a quasi-isomorphism.

Proof. We prove this in complete analogy with the proof of Proposition 9.11. We define filtrations on both spaces by letting the pth space be the subspace spanned by decorated graphs with at least p vertices. These filtrations are complete and preserved by both differentials as well as the map π. When passing to the map π_0 the associated spaces, the term $d_1 + m\cdot$ of both differentials become zero. Thus only the differentials on decorations remain. By decomposing each factor of the associated spaces to direct products over spaces spanned by one graph, applying Proposition 6.9 and that homology commutes with taking direct products and tensor products, it follows that π_0 is a quasi-isomorphism and thus also that π is, by Proposition 6.6. ■

Let us move next look at the map from $\mathsf{graphs}_n^{\widehat{dec},2}$ to $\mathsf{graphs}_n \circ \widehat{H}_\bullet(G)$ in the zigzag. We will denote this map by π'. We define this in complete analogy with the map π from the previous subsection (i.e. for even n), by taking the tensor product of the counit $\hat{\mathcal{H}} \to \mathbb{R}$ on all decorations. This map thus amounts to applying the counit to all internal decorations of a graph, multiplying the coefficient of the graph with each resulting number. Once again, this implies that the map is nonzero only if all decorations of the graph are 1, when it amounts to "forgetting" the internal decorations. The map is linear by construction, and clearly of degree zero.

Proposition 9.19. The map π' is a quasi-isomorphism of dg operads.

Proof. The proof can be done by copying the proofs for Propositions 9.1-2 almost verbatim, replacing $\mathcal{H}$ with $\hat{\mathcal{H}}$ and $H_\bullet(G)$ by $\widehat{H}_\bullet(G)$. ■

We now come to the final part in this proof, where we finish the zigzag. We will denote the last map by $f : \mathsf{graphs}_n^{\widehat{dec},2} \to \mathsf{graphs}_n^{\widehat{dec},1}$. We once again define it as a composition of maps that act at the different vertices, where each map is defined by applying the coproduct to the component $x \in \widehat{H}_\bullet(G)$ of the decoration at that vertex, acting by x' from the right at that vertex and decorating the graph that is inserted by the components of the iterated coproduct of $x'' \otimes y$, where y is the $H_\bullet(BG)$-component of the original decoration.

Since the Maurer-Cartan element m is more complicated in this case, we can unfortunately not give as simple a description of the components of f as we did in the even case. However, since the map $m : \widehat{H}_\bullet(G) \to \mathcal{U}\mathsf{GC}_n$ is zero unless x' is either a scalar or of the form $p^{i_1}.\cdots.p^{i_k}$, we see that each term of the map is either the identity, or insertion of a graph of the form

$(\delta_{i_1} \bullet \delta_{i_2}) \bullet \cdots) \bullet \delta_{i_k})$. The proof that this map is a quasi-isomorphism of dg operads will be similar to the even case.

Proposition 9.20. The map f is a map of symmetric sequences of dg vector spaces.

Proof. Just as in the even case, it is clear that the symmetric action is preserved by f. Since f is a composite map and the differential is a sum of maps acting at one vertex at a time, is also suffices to look at what happens at one vertex. We need to prove that

$$f_v(d_1 + d_2 + d_3) = (d_1 + d_2 + d_2 + m\cdot)f_v.$$

As in the even case, we will not look at all the parts of the differential at once. The proof that

$$f_v(d_1 + d_3) = (d_1 + d_3)f_v,$$

is practically identical to the proof of Proposition 9.17 and we may use the fact that m is a Maurer-Cartan element to show that applying d_3 to the decoration before applying f_v is the same thing as first applying f_v and then splitting a vertex in the inserted graph. We will therefore omit this part of the proof and move on to show that

$$fd_2 = (d_2 + m\cdot)f. \tag{26}$$

We may assume the the graph only has one internal vertex v. Suppose that v is decorated by $x \otimes y \in \hat{\mathcal{H}}$. Let us denote the map from $\widehat{H}_\bullet(G)$ to $\mathcal{U}GC_n$ by m. If $x' = x'_1. \cdots .x'_k$ and we apply f_v, it means that we insert $(\cdots(m(x'_1) \bullet m(x'_2)) \bullet \cdots) \bullet m(x'_k)$. Let us abuse notation a bit and denote this by $m(x')$. Furthermore, let us denote a graph Γ with k internal vertices decorated by $a_1, \ldots, a_k$ by

$$(\Gamma; a_1, \ldots, a_k).$$

This means that we have

$$\begin{aligned}
fd_2(v\ ; x \otimes y) &= f(v; x.\iota\pi(y') \otimes y'') \\
&= \Big(m(x'); x''.(\iota\pi(y'))' \otimes y'', \ldots, x^{(k+1)}.(\iota\pi(y'))^{(k)} \otimes y^{(k+1)}\Big) \\
&+ \Big(m(x') \bullet m(\iota\pi(y')); x'' \otimes y'', \ldots, x^{(k+1)} \otimes y^{(k+1)}\Big) \\
&= \sum_{i=2}^{k+1} \Big(m(x'); x'' \otimes y'', \ldots, x^{(i)}.\iota\pi(y') \otimes y^{(i)}, \ldots, x^{(k+1)} \otimes y^{(k+1)}\Big) \\
&+ \Big(m(x') \bullet m(\iota\pi(y')); x'' \otimes y'', \ldots, x^{(k+1)} \otimes y^{(k+1)}\Big),
\end{aligned}$$

where we in the last step use that $\iota\pi(y')$ is primitive by definition. Similarly, we get

$$\begin{aligned}
d_2 f(v; x \otimes y) &= d_2\Big(m(x'); x'' \otimes y', \ldots, x^{(k+1)} \otimes y^{(k)}\Big) \\
&= \sum_{i=1}^{k} \Big(m(x'); x'' \otimes y', \ldots, x^{(i+1)}.\iota\pi(y^{(i)} \otimes y^{(i+1)}, \ldots, x^{(k+1)} \otimes y^{(k+1)}\Big) \\
&= \sum_{i=2}^{k+1} \Big(m(x'); x'' \otimes y'', \ldots, x^{(i)}.\iota\pi(y') \otimes y^{(i)}, \ldots, x^{(k+1)} \otimes y^{(k+1)}, \Big).
\end{aligned}$$

Lastly, we want to prove that $(m\cdot)f$ is the same as the second term of fd_2. We note that when m acts at an internal vertex decorated by $x \otimes y$, the only nonzero terms are those where

$y' = \tilde{p^j}$ for some $j \geqslant 0$. But in this case, we insert the graph $m(p^j) = m(\iota\pi(y'))$ at that vertex. Thus we get

$$\begin{aligned} m \cdot f(v; x \otimes y) &= m \cdot \left(m(x'); x'' \otimes y', \ldots, x^{(k+1)} \otimes y^{(k)}\right) \\ &= \left(m(x') \bullet m(\iota\pi(y')); x'' \otimes y'', \ldots, x^{(k+1)} \otimes y^{(k+1)}\right), \end{aligned}$$

so (26) does indeed hold, which means that f is a map symmetric sequences of dg vector spaces. ■

Proposition 9.21. The map f is a morphism of dg operads.

We need to show that if Γ and Γ' are decorated graphs, then

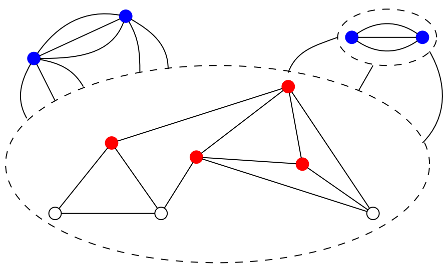
$$f(\Gamma \circ_j^m \Gamma') = f(\Gamma) \circ_j f(\Gamma'). \tag{27}$$

It is clear that if v is a vertex of Γ other than its jth external vertex, then

$$f_v(\Gamma \circ_j^m \Gamma') = f_v(\Gamma) \circ_j \Gamma',$$

so we can assume that Γ only consists of one external vertex decorated by some $x \in \widehat{H}_\bullet(G)$.

First let us introduce some temporary terminology to make make the proof a bit clearer. When we compose $\Gamma \circ_j^m \Gamma'$, we act on Γ' with some element of $\mathcal{U}\mathsf{GC}_n$, whose factors will alternately act from the left or the right. In any term, let us call those vertices of the resulting graph that are either original vertices from Γ', or of graphs inserted into those vertices, *upstairs* vertices, and let us call the remaining vertices *downstairs* vertices. The upstairs vertices thus come from graphs acting from the left and the graphs that are then inserted into these. With the upstairs internal vertices colored blue, and the downstairs internal vertices colored red, a graph $\Gamma \circ_1^m \Gamma'$ (where we've assumed that Γ has only one decorated external vertex) could look something like



Analogously in $\mathsf{graphs}_n^{\widehat{dec},1}$, we will call the vertices of $\Gamma \circ_j \Gamma'$ that come from Γ upstairs vertices, and those coming from Γ' downstairs vertices.

If we delete all edges in $\Gamma \circ_j^m \Gamma'$ that come from those components of x' acting on Γ' from the right, under our assumption that Γ only has one vertex, then the outer part of the graph will be divided into connected components, where we get one connected component for each factor of x' that acted from the right, which consists of all graphs that where then inserted into the vertex added by that factor. If p^{i_l} is some such factor, we will call its corresponding component the outer component generated by p^{i_l}.

Let us first look at $f(\Gamma \circ_j^m \Gamma')$. In the composition, some of the factors of x will act on Γ' from the left and are thus inserted. When then f is applied some more of the factors of x are inserted into some vertices. Let us fix the terms where the factors of both these kinds are $x_{i_1}, \dots, x_{i_k}$, ordered as in x. If x_{i_1} is a factor which is inserted by the composition, it is the last to be inserted before f is applied. If it is instead set to decorate the same vertex where it was inserted in the previous term and the inserted by f, it is the first factor to be inserted when f is applied. Because of the minus sign in the action by GC_n on $\mathsf{graphs}_n^{\widehat{dec}}$, it follows that these terms cancel.

It follows that in $f(\Gamma \circ_j^m \Gamma')$, all terms where some graph acts from the right in the twisted composition is killed by f. It also follows that if some factor of x'' multiplies a downstairs vertex in the composition, then all terms of $f(\Gamma \circ_j^m \Gamma')$ where this factor is inserted is killed by a corresponding term where the factor instead acts from the right in the composition. Furthermore, if the factor multiplies an upstairs vertex in the composition and this factor comes before the factor that generated this outer component, it is killed in the same way. The reason it needs to come after the factor that generated the outer component is that otherwise it cannot be inserted after that factor acts from the left in the twisted composition.

Thus the only nonzero terms of $f(\Gamma \circ_j^m \Gamma')$ are those where x' only acts on Γ' from the left in the composition, and when we then apply f, it only inserts the factors of the decorations that come from Γ' on the downstairs vertices and on the upstairs vertices only those decorations that come after the factor that created that upstairs component, in the order of the factors in x. This means that we get the same element by first applying f to Γ', then acting on this by x' and finally decorating the resulting graph with x''. But this is precisely the definition of $f(\Gamma) \circ f(\Gamma')$. To see this, note that when f is applied to Γ, which we assume has only one vertex, then if a factor is inserted externally, this gives the same result as acting on Γ' from the right, when it is inserted, and if we insert externally it is the same thing as acting from the left on the resulting upstairs vertices. Since we are acting from the left when applying f, it follows that the factors of x' act on Γ' in the same order as if $f(\Gamma \circ_j \Gamma')$. That the signs agree is once again easily verified using our sign conventions, so we will omit this verification. ∎

Proposition 9.22. The map f is a quasi-isomorphism.

Proof. This proof is completely analogous with the corresponding proof for even n. We define filtrations on our two graph complexes, by letting the pth space in both cases be spanned by graphs with at least p edges. Since both f and the differentials either increases the number of edges or keeps it fixed, these filtrations are preserved and as any graph we consider has a finite number of edges the filtration is complete. Since the underlying graded vector spaces are the same, so are the associated spaces. In the associated spaces, the parts of the differentials induced by vertex splitting and $m\cdot$ become zero, because both of these increase the number of edges. Thus the induced differentials on both associated spaces are the same, so they are equal as dg vector spaces as well. Since all non-identity terms of f also increase the number of edges, the induced map f_0 is the identity map, which means that it is in particular a quasi-isomorphism. By Proposition 6.6. it once again follows that f is as well. ∎

We have thus proven Theorem 9.8 for odd n as well, which finishes the proof of the entire theorem.

9.5. **Conclusion.** Let us summarize our results, followed by a brief discussion of their relevance and where they lead. In the beginning of this section, we introduced the graph complex graphs_n^{dec}, which we then proved is equipped with a natural action by the Lie algebra $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$. We have now proven that by twisting the differential on graphs_n^{dec} by the Maurer-Cartan element m, from [KW17], we get a new model for the framed little n-disks operad, for all n. For even n, this is not a particularly interesting result, as we know that $\mathcal{D}_n^{fr}$ is formal in this case and the homology is the simplest model possible for any operad. For odd n, however, we know that the operad is non-formal, which means that finding new models becomes interesting.

What makes graphs_n^{dec} more useful than the original model by Khoroshkin and Willwacher is mainly that it is equipped with an action by the large Lie algebra $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$, unlike the original model by Khorohskin and Willwacher. This means that this Lie algebra maps into the so called *homotopy derivations* of the operad of chains on $\mathcal{D}_n^{fr}$. These are the derivations on a cofibrant-fibrant replacement of the original operad (see [Hirschhorn04, Chapter 8], where it is called a cofibrant-fibrant approximation, for a definition). Explicitly, such a replacement is given by the operadic cobar-bar construction of the operad in this case.

In the current work of Brun and Willwacher, which is soon to be published as [BW18], the model graphs_n^{dec} is extended slightly so that it may be equipped with an action by a Lie algebra even larger than $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$. Explicitly, this dg Lie algebra is given by replacing $H^\bullet(BG)$ with the semi-direct product $\mathrm{Der}(H^\bullet(BG)) \ltimes H^\bullet(BG)$, and replacing GC_n with a semi-direct product $\Lambda\{L\} \ltimes \mathsf{GC}_n$. The generator L is called the *loop order* operator, which acts on graphs in GC_n by multiplying a graph by the loop order of its underlying undirected graph. The loop order of an undirected graph is defined as the minimum number of edges that need to be removed so that the graph becomes acyclic, which is the same thing as the first Betti number of the graph, viewed as a simplical complex. To take this operator into account, the model graphs_n^{dec} is extended, by adding integer decorations to the vertices of the graph, so that when a graph from the Lie algebra acts at a vertex decorated by k, we first act by L^k on the graph before inserting it.

The extension of the map from $H^\bullet(BG)\hat{\otimes}\mathsf{GC}_n$ into the homotopy derivations of the chains on $\mathcal{D}_n^{fr}$ that this extension results in is then used by Brun and Willwacher to compute *all* the (bi)derivations of this operad[3]. This thus results in a model for the framed little n-disks operad, together with an action which gives a model for the homotopy derivation of the operad. It is thus primarily as a piece in the solution of this problem that the new model that we have constructed is important in our context.

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[3] As in the article by Khoroshkin and Willwacher where the original model for the framed little n-disks operad is introduced, Brun and Willwacher actually work in the dual setting to ours, where instead of chains on $\mathcal{D}_n^{fr}$, one works with PA-forms. We therefore call automorphisms that are both derivations with respect to the associative algebra structure and coderivations with respect to the cooperad structure as *bi*-derivations.

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