SJÄLVSTÄNDIGA ARBETEN I MATEMATIK

MATEMATISKA INSTITUTIONEN, STOCKHOLMS UNIVERSITET

The second cohomology group and group extensions

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2019 - No K13

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Självständigt arbete i matematik 15 högskolepoäng, grundnivå

Handledare: Alexander Berglund

2019

Abstract

In this B.Sc. thesis we will show that for a group G and a G-module A there is a bijection between the second cohomology group $H^2(G; A)$ and the equivalence classes of extensions $\pi_0(\mathscr{E}xt(G, A)$. This will be done by showing that there is an equivalence of categories from the category of 2-cocycles $\mathcal{Z}^2(G; A)$ to the category of extensions $\mathscr{E}xt(G, A)$. We will also, by examples, illustrate and show some computational use of some of the theory developed in the proof.

Contents

1 Introduction

We will in this thesis explore ways of "putting simple groups together" to get other groups. "Putting two groups together" being interpreted as, given two groups A and B, finding groups G such that A is isomorphic to some normal subgroup N of G and that B is isomorphic to the quotient group G/N. This is part of the categorisation of groups that the Hölder Program is concerned with. The Hölder Program consists of two parts, the first is to classify all finite simple group and the second part is to find all ways of "putting simple groups together. Since extensions are such groups G, understanding of how they can be created as presented in this B.Sc. thesis helps with an understanding of the ways of combining groups. The proof in this B.Sc. thesis showing that for a group G and a G-module A there is a bijection between the second cohomology group $H^2(G;A)$ and the equivalence classes of extensions $\pi_0(\mathscr{E}xt(G,A)$ is adapted from the proof in $H^2(G;A)$ *and extensions*, [2]. This B.Sc. thesis aims to present this proof with explanation of even the more fundamental concepts and work out computations and present proofs left out as well as show examples of results and techniques used in the proof.

To appreciate this thesis fully, the reader could benefit from having taken a course in abstract algebra to be familiar with basic concepts from group theory. The book *Abstract Algebra* [1] explains the concepts needed to appreciate this thesis as well as most of the content of the first five sections.

For a deeper understanding of cohomology the book *Cohomology of groups*[3] is a good resource.

Sections 2 and 3 will present introductions to extensions and category theory respectively to the extent needed for the proof. Section 4 will introduce the semidirect product to help with the understanding of the constructions later in the B.Sc. thesis. Section 5 will present the necessary group cohomology tools needed to understand the proof and preform some computations. The proof proper starts in Section 6 where the connection between the category of 2-cocycles $\mathscr{Z}^2(G;A)$ and the category of extensions $\mathscr{E}xt(G,A)$ is introduced in form of a functor. Section 7 concludes the proof by showing that the functor from Section 6 is an equivalence of categories.

2 Extensions

In this section we will define the concept group extension from groups and exact sequences of groups, state and prove some useful theorems connected to the concept. Since however our main focus in this B.Sc. thesis will be on extensions by G-modules we will also define what these are and exemplify some extensions by them.

Definition 2.1 For groups G_i and homomorphisms $\pi_i : G_i \to G_{i+1}$, the sequence of groups with homomorphisms

$$G_1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} G_3 \cdots \xrightarrow{\pi_{n-1}} G_n$$

is said to be exact if for all i, $im(\pi_i) = ker(\pi_{i+1})$.
An exact sequence on the form

$$1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} G_3 \xrightarrow{\pi_3} G_4 \xrightarrow{\pi_4} 1$$

is called a short exact sequence and G_2 is a group extension of G_3 by G_1.

Remark By 1 in the context of groups, as in **Definition 2.1**, is taken to mean the group with one element. On occasion we may also use 0 to mean the group with one element. These two notations will be used to differentiate between the cases when the mappings to and from the trivial group are to groups usually written with additive notation where 0 will be used and groups usually written with multiplicative notation where 1 will be used. When dealing with general groups if they are known to be abelian 0 will be used as they are usually written with additive notation. When a general group is not explicitly known to be abelian we will use 1 to fit with the multiplicative notation.

Lemma 2.3 *For a short exact sequence*

$$1 \xrightarrow{\pi_1} G_2 \xrightarrow{\pi_2} G_3 \xrightarrow{\pi_3} G_4 \xrightarrow{\pi_4} 1,$$

π_2 *is injective and* π_3 *is surjective.*

Proof If $\pi_2(g_1) = \pi_2(g_2)$, we have
$\pi_2(g_1 g_2^{-1}) = \pi_2(g_1)\pi_2(g_2^{-1}) = \pi_2(g_1)\pi_2(g_2)^{-1} = \pi_2(g_1)\pi_2(g_1)^{-1} = 1$. Since the sequence is exact $ker(\pi_2) = \{1\}$ so $g_1 g_2^{-1} = 1$ if and only if $g_1 = g_2$, so π_2 is injective.
Since all of G_4 is mapped to the identity element, all of G_4 is in the image of π_3 so π_3 is surjective.$\square$

At this point the reader should recall that a group action of a group G on a set X is a function $\phi : G \times X \to X$, commonly using the notation $g \cdot x$ as $\phi(g, x)$, where the identity element of G maps any element of X to itself and where $gh \cdot x = g \cdot h \cdot x$, for $g, h \in G$ and $x \in X$.

Definition 2.4 For a group G an abelian group A, on which G acts in such a way that $g \cdot (a + a') = g \cdot a + g \cdot a'$, is a G-**module**.

Theorem 2.5 *Let* $0 \to A \xrightarrow{\pi_2} E \xrightarrow{\pi_3} G \to 1$, *be a short exact sequence and identify* A *with the* $ker(\pi_3) \subseteq E$. *There is a group action of* G *on* A *defined by*

$$g \cdot a = \bar{g} a \bar{g}^{-1}, g \in G, a \in A$$

where $\bar{g}$ *is any element such that* $\pi_3(\bar{g}) = g$.

Proof By **Lemma 2.3** π_2 is an injective homomorphism and since the sequence is exact A is mapped to $ker(\pi_3)$ so the identification is allowed. It is apparent that $g \cdot a \in A$ since

$$\pi_3(\bar{g}a\bar{g}^{-1}) = \pi_3(\bar{g})\pi_3(a)\pi_3(\bar{g}^{-1}) = g1g^{-1} = 1$$

The action defined follows the group action axioms:
identity, where the identity action is performed by the identity of the group G. For if e is the identity element in G then $\bar{e} \in ker(\pi_3) = A$ and since A is abelian we have $e \cdot a = \bar{e}a\bar{e}^{-1} = \bar{e}\bar{e}^{-1}a = a$.
compatibility, which follows from associativity in E since

$$gg' \cdot a = (gg')a(gg')^{-1} = (gg')a(g'^{-1}g^{-1}) = g(g'ag'^{-1})g^{-1} = g \cdot g' \cdot a.$$

Lastly the choice of $\bar{g}$ is irrelevant. For two choices for $\bar{g}$ labelled $\bar{g}_1$ and $\bar{g}_2$ we want to show that $\bar{g}_1a\bar{g}_1^{-1} = \bar{g}_2a\bar{g}_2^{-1}$ if and only if $\bar{g}_2^{-1}\bar{g}_1a = a\bar{g}_2^{-1}\bar{g}_1$, which holds since $\pi_3(\bar{g}_2^{-1}\bar{g}_1) = g^{-1}g = 1$ so $\bar{g}_2^{-1}\bar{g}_1 \in A$ and the expression commutes.

Definition 2.6 A commutative diagram is, for a collection of sets A_i, a collection of maps $\phi : A_n \to A_m$ such that any composition of maps beginning at a set A_j and ending in a set A_k result in the same composite map.

Theorem 2.7 *(The short five lemma) Letting α, β, γ be homomorphisms of two short exact sequences such that the following is a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, "\alpha"] & B \arrow[r, "\phi"] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 1 \\
0 \arrow[r] & A' \arrow[r, "\psi'"] & B' \arrow[r, "\phi'"] & C' \arrow[r] & 1
\end{tikzcd}$$

we have

1. *If α and γ are injective then β is injective.*
2. *If α and γ are surjective then β is surjective.*

Proof 1. It is enough to show that $Ker\beta = 0$. Assume $b \in Ker\beta$. Since the diagram is commutative and $\phi'(\beta(b)) = 0$ we know that $\gamma(\phi(b)) = 0$. Since γ is injective $\phi(b) = 0$. Then since the ABC sequence is exact there exists an $a \in A$ such that $\psi(a) = b$. By commutativity in the diagram
$\psi'(\alpha(a)) = \beta(\psi(a)) = \beta(b) = 0$. Since α and ψ' are injective $a = 0$. Finally $b = \psi(a) = \psi(0) = 0$

2. Let $b' \in B'$. Since γ is surjective there exists a $c \in C$ such that $\gamma(c) = \phi'(b')$. Since ϕ is surjective there exists a $b \in B$ such that $\phi(b) = c$. Since the diagram is commutative $\phi'(\beta(b)) = \gamma(\phi(b)) = \phi'(b')$ so $\phi'(b' * \beta(b)^{-1}) = 0$. Since A'B'C' sequence exact there exists an $a' \in A'$ such that $\psi'(a') = b' * \beta(b)^{-1}$ and since α is surjective there exists an $a \in A$ such that $\alpha(a) = a'$. Finally, by commutativity of the diagram, $\beta(\psi(a) * b) = b' * \beta(b)^{-1} * b = b'$ and $\psi(a) * b \in B$ so β is surjective.

Now we will look at some examples of short exact sequences with the G-module structure described above.

2.1 Examples

- The following is a short exact sequence with $\phi(x) = 3x, \psi(x) = x(\text{mod } 3)$.

$$0 \to \mathbb{Z}/2\mathbb{Z} \overset{\phi}{\to} \mathbb{Z}/6\mathbb{Z} \overset{\psi}{\to} \mathbb{Z}/3\mathbb{Z} \to 0.$$

  By Theorem 2.5 we can find the group action of $\mathbb{Z}/3\mathbb{Z}$ on $\mathbb{Z}/2\mathbb{Z}$ defined by $b \cdot a = \bar{b}a\bar{b}^{-1}$ for $a \in \mathbb{Z}/2\mathbb{Z}$ and $b \in \mathbb{Z}/3\mathbb{Z}$. The group action ends up being described by $b \cdot a = a$ since $\mathbb{Z}/6\mathbb{Z}$, where the actions computation takes place, is a commutative group.

- Similarly we have the short exact sequence

$$0 \to \mathbb{Z}/3\mathbb{Z} \overset{\phi}{\to} \mathbb{Z}/6\mathbb{Z} \overset{\psi}{\to} \mathbb{Z}/2\mathbb{Z} \to 0$$

  with $\phi(x) = 2x$ and $\psi(x) = x(\text{mod } 2)$. For $a \in \mathbb{Z}/2\mathbb{Z}, b \in \mathbb{Z}/3\mathbb{Z}$ the group action defined as dictated by the theorem will be $a \cdot b = b$, again because $\mathbb{Z}/6\mathbb{Z}$ is abelian.

- In fact we can generalise the previous examples to the short exact sequences

$$0 \to \mathbb{Z}/n\mathbb{Z} \overset{\phi}{\to} \mathbb{Z}/nm\mathbb{Z} \overset{\psi}{\to} \mathbb{Z}/m\mathbb{Z} \to 0$$

  with $\phi(x) = mx, \psi(x) = x(\text{mod } m)$ with the same trivial group action, for integers n, m.

- For an example when the group action works differently we look at the short exact sequence

$$1 \to A_3 \overset{\phi}{\to} S_3 \overset{\psi}{\to} \mathbb{Z}/2\mathbb{Z} \to 0$$

  where ϕ is sending a permutation in A_3 to the same permutation in S_3 and ψ sending odd permutations to $1 \in \mathbb{Z}/2\mathbb{Z}$ and even permutations to $0 \in \mathbb{Z}/2\mathbb{Z}$. For $a \in A_3$ we have that our conjugation group action works by $0 \cdot a = a$ and $1 \cdot a = a^{-1}$. That the identity element 0 acts trivially should not be surprising as it is mapped to only by elements in the abelian group A_3 therefore commutes and the conjugation cancels out. As discussed in the proof of Theorem 2.5 any choice of $\bar{g}$ will give the same result, so using cycle notation for permutations and choosing (12) as our $\bar{1}$ we get
  $1 \cdot (123) = (12)(123)(12)^{-1} = (132) = (123)^{-1}$,
  $1 \cdot (132) = (12)(132)(12)^{-1} = (123) = (132)^{-1}$
  and of course $1 \cdot (1) = (12)(1)(12)^{-1} = (1)^{-1}$, which can be expressed simpler as $1 \cdot a = a^{-1}$. Also notice that A_3 is isomorphic to $\mathbb{Z}/3\mathbb{Z}$ so we have found two short exact sequences with only the middle group and group action differing.

- For another example where where where the non-trivial group action is inversion we inspect the short exact sequence

$$1 \to \langle i \rangle \overset{\phi}{\to} Q_8 \overset{\psi}{\to} \langle -1 \rangle \to 1$$

where ϕ sends an element in $\langle i \rangle$ to the same element in Q_8 and ψ forced to send $1, i, -i, -1$ to $1 \in \langle -1 \rangle$ and the rest to $-1 \in \langle -1 \rangle$. Calculating similarly to the last example with $j \in Q_8$ as our $\overline{-1}$ we get
$-1 \cdot i = ji(-j) = -i$,
$-1 \cdot -i = j(-i)(-j) = i$,
$-1 \cdot -1 = j(-1)(-j) = -1$ and
$-1 \cdot 1 = j(-j) = 1$.
So for $a \in \langle i \rangle$ we have $1 \cdot a = a$ and $-1 \cdot a = a^{-1}$. Note here that $\langle i \rangle$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and that $\langle -1 \rangle$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.

- We will show one last thing with the dihedral group of order 8 D_8 and the short exact sequence

$$1 \to \langle r \rangle \overset{\phi}{\to} D_8 \overset{\psi}{\to} \langle r^2 \rangle \to 1$$

where ϕ sending an element in $\langle r \rangle$ to the same element in D_8 and again ψ is forced. We are using a characterisation with r as the rotation element and s as the reflection element. Checking our prescribed group action again we get that for $a \in \langle r \rangle$ we have $1 \cdot a = a$ and choosing s as our $\overline{r^2}$ we get
$r^2 \cdot r = srs^{-1} = r^3$,
$r^2 \cdot r^2 = sr^2s = r^2$,
$r^2 \cdot r^3 = sr^3s = r$ and
$r^2 \cdot 1 = ss = 1$.
This again can simply be written as $r^2 \cdot a = a^{-1}$. This time note that $\langle r \rangle$ is isomorphic to $\mathbb{Z}/4\mathbb{Z}$ and that $\langle r^2 \rangle$ is isomorphic to $\mathbb{Z}/2\mathbb{Z}$, so with this and the previous example we have found two short exact sequences with the middle group as the only difference, even using the same group action.

3 Category Theory

This section will briefly lay out the category theory that will be needed for the proof of the main result, defining categories as well as functors.

Definition 3.1 A category C consists a class of objects, $ob(C)$, and a class of morphisms, $hom(C)$, between the objects. For two objects A and B in the category we denote the class of all morphisms from A to B $hom_C(A, B)$. For any three objects A, B and D the binary composition mapping
$hom_C(A, B) \times hom_C(B, D) \to hom_C(A, D)$ takes $(f, g) \mapsto gf$. A Category must also satisfy the following axioms for any four objects A, B, D and E.

(i) $A \neq B$ or $D \neq E \Longrightarrow hom_C(A, B) \cap hom_C(D, E) = \varnothing$

(ii) associativity: $h(gf) = (hg)f$ holds for every $f \in hom_C(A, B)$, $g \in hom_C(B, D)$ and $h \in hom_C(D, E)$

(iii) identity: there exists a morphism $1_A \in hom_C(A, A)$ such that for any morphims $f \in hom_C(A, B), g \in hom_C(B, A), f1_A = f$ and $1_A g = g$.

Definition 3.2 A category which has only invertible morphisms is a groupoid.

Example:
As an example of a category we can take the class of all sets as our object class, the class of all functions from a set to a set as our class of morphisms and composition of morphisms being composition of functions. It is easy to see that this makes a category as composition of functions is associative and identity functions from a set to itself exist.

Definition 3.3 A functor from one category to another $F : C \to D$ maps all objects in C to objects in D by $A \mapsto F(A)$. A functor also maps all morphisms in C to morphisms in D , mapping a morphism $f \in hom_C(A, A')$ to a morphism $F(f) \in hom_D(F(A), F(A'))$ while preserving identity morphisms and morphism compositions.

- If $F : C \to D$ maps , for any $X, Y \in C$, $hom_c(X, Y)$ to $hom_d(F(X), F(Y))$ bijectively then F is called fully faithful.

- If for any object $Y \in D$ there is an object $X \in C$ such that there is an isomorphism from $F(X)$ to Y in D then F is said to be essentially surjective.

- If F is fully faithful and essentially surjective, F is said to be an equivalence of categories.

Definition 3.4 A path component of a category C contains only all objects in C that are connected to another object in the component by a morphism. So two objects are in the same path component iff there is some sequence of morphisms connecting them, the direction of the morphisms are not taken into account. The components of C are taken to be $\pi_0(C)$.

Theorem 3.5 *An equivalence of categories $F : C \to D$ induces a bijection on path components.*

Proof We know that F maps path components in C to path components in D since if A_m and A_n belong to the same path component in C then there is a sequence of morphisms f_i connecting them and since F is in particular fully faithful it will map the morphisms bijectively connecting $F(A_m)$ to $F(A_n)$ by the sequence of morphisms $F(f_i)$.

We also know that F maps path components surjectively since if there were a path component in D not mapped to by F then none of the objects in the path component could be mapped to by F, however all, but in particular one, objects in the component is isomorphic to an object mapped to by F, because F is essentually surjective, this introduces an object mapped to by F to the path component so the component is mapped to from a component in C by F. We know that F maps path components injectively since F could not map two path components in C to the same path component in D because by F being fully faithful any morphism connecting the image of the path components of C in D would be mapped from a morphism connecting the components in C, this includes any composition of morphisms in D using objects in D not mapped to by F that one might worry about.
Since F maps the components surjectively and injectively it does so bijectively.$\square$

3.1 Category of extensions $\mathscr{E}xt(G, A)$

Equipped with the category theory we have seen so far we can create a category of extensions which will be used in the main result.

Fixing a group G and a G-module A we define the category $\mathscr{E}xt(G, A)$. The category has as objects all group extensions E, such that

$$0 \to A \overset{\pi_2}{\to} E \overset{\pi_3}{\to} G \to 0$$

is an exact sequence with the induced G action agreeing with the group action defined in **Theorem 2.5**. The category $\mathscr{E}xt(G, A)$ has as morphisms the commutative diagrams between the objects.

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r, "\psi"] \arrow[d, equal] & E \arrow[r, "\phi"] \arrow[d, "\beta"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r, "\psi'"] & E' \arrow[r, "\phi'"] & G \arrow[r] & 0
\end{tikzcd}$$

By the short five lemma β is an isomorphism.

4 Semidirect product

As a steppingstone to the groups we will later define to bridge the gap between 2-cocycles and extensions we now define the semidirect product of a module and its group. The semidirect product is related to the direct product but uses the same group action as before in its multiplication to alter the first component of a pair, this makes it more versatile and able to create a more varied set of groups.

Definition 4.1 Having groups A and G with elements of G acting on A by automorphisms the set of all pairs (a, g), where$a \in A$ and $g \in G$, is a group with the operation $\bullet$, defined by $(a, g) \bullet (a', g') = (a(g \cdot a'), gg')$. Denote this group by $A \rtimes G$, the semidirect product of A and G.

We will now show that the semidirect product, as stated, satisfies the group axioms and is in fact a group

Group axioms

Closure: For all $a, a' \in A, g, g' \in G$, $(a(g \cdot a'), gg')$ is clearly in $A \rtimes G$ since $g \cdot a' \in A$.

Associativity:

$$\begin{aligned}
&((a,g) \bullet (a',g')) \bullet (a'',g'') = (a + g \cdot a' +, gg') \bullet (a'',g'') \\
&= (a + g \cdot a' + gg' \cdot a'', gg'g'') = (a + g \cdot (a' + g' \cdot a''), gg'g'') \\
&= (a,g) \bullet (a' + g' \cdot a'', g'g'') = (a,g) \bullet ((a',g') \bullet (a'',g'')).
\end{aligned}$$

Identity:(0,1);

$$(a,g) \bullet (0,1) = (a + g \cdot 0, g1) = (a,g) = (0 + 1 \cdot a, 1g) = (0,1)(a,g).$$

Inverse:$(g^{-1} \cdot a^{-1}, g^{-1})$;

$$\begin{aligned}
&(a,g) \bullet (g^{-1} \cdot a^{-1}, g^{-1}) = (a + g \cdot (g^{-1} \cdot a^{-1}), gg^{-1}) \\
&= (0,1) = (g^{-1} \cdot a^{-1} + g^{-1} \cdot a, gg^{-1}) = (g^{-1} \cdot a^{-1}, g^{-1})(a,g).
\end{aligned}$$

Following are some properties of the semidirect product which are useful to work with.

Properties

Isomorphisms:

There is an isomorphism from A to the group of all pairs in $A \rtimes G$ on the form $(a,1)$, $a \to (a,1)$, since $(a,1) \bullet (a',1) = (a + 1 \cdot a', 11) = (a + a', 1)$.

There is an isomorphism from G to the group of all pairs in $A \rtimes G$ on the form $(0,g)$, $g \to (0,g)$, since $(0,g) \bullet (0,g') = (0 + g \cdot 0, gg') = (0, gg')$.

Identifying A and G with these sets of pairs we can see that $A \cap G = \{(0,1)\}$ and that $gag^{-1} = g \cdot a$ since
$(0,g) \bullet (a,1) \bullet (0,g^{-1}) = (g \cdot a, g) \bullet (0, g^{-1}) = (g \cdot a + g \cdot 0, gg^{-1}) = (g \cdot a, 1)$.

4.1 Examples

Let us now by examples demonstrate some capabilities and limitations of the semidirect product, specifically we want to see if we can, from a short exact sequence

$$0 \to A \overset{\pi_2}{\to} E \overset{\pi_3}{\to} G \to 0,$$

of the sort found in the category of extensions, create a group isomorphic to the middle group E as the semidirect product of the two group on either side $A \rtimes G$.

- If we look over our list of examples of short exact sequences we might want to try both $\mathbb{Z}/6\mathbb{Z}$ and S_3 as candidates for semidirect products on the form $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$. We already know that $\mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and looking over the definition of semidirect product we realise that the direct product corresponds to the semidirect product with completely trivial group action. We recall that this was the group action we worked out in the example for the exact sequence corresponding to this semidirect product. To examine what group action might give us an isomorphism $\mathbb{Z}/3\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z} \cong S_3$ we attempt the group action defined as $0 \cdot a = a$ and $1 \cdot a = a^{-1}$ as our short exact sequence from before hints at. With some calculations we can convince ourselves that with that group action the following correspondence between pairs and permutations will be an isomorphism from S_3 to $\mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$;
$(0,0) \leftrightarrow (1), (0,1) \leftrightarrow (12), (1,1) \leftrightarrow (23), (2,1) \leftrightarrow (13), (2,0) \leftrightarrow (123),$
$(1,0) \leftrightarrow (132)$.

- Given our luck with S_3 and $\mathbb{Z}/6\mathbb{Z}$ we might want to examine our exact sequence with Q_8 however Q_8 is not isomorphic to any semidirect product $\langle i \rangle \rtimes \langle -1 \rangle$.
Clearly it is not the semidirect product with trivial the automorphism since that would be the direct product of two abelian groups, make it abelian. The remaining option is letting -1 act on $\langle i \rangle$ by the inverse: meaning $(a,-1)^2 = (aa^{-1}, 1)$,the identity element, for all a. This already gives us four elements of order 2, more than there are in Q_8, making an isomorphism impossible. Further exploration would show that this group is in fact D_8, this might not surprise us if we remember that the short exact sequence we set up for Q_8 and D_8 were very similar, even obtaining the same group action.

5 Group cohomology

Definition 5.1 For abelian groups A^i and homomorphisms $\pi^i : A^i \to A^{i+1}$, the sequence $\mathcal{A} : \cdots A^1 \xrightarrow{\pi^1} A^2 \xrightarrow{\pi^2} A^3 \cdots \xrightarrow{\pi^{n-1}} A^n \cdots$ is said to be a cochain complex if for all i $im(\pi^i) \subset ker(\pi^{i+1})$. The quotient group $ker(\pi^{n+1})/im(\pi^n)$ is the nth cohomology group of the cochain complex, denoted $H^n(\mathcal{A})$.

Remark An exact sequence is a cochain complex with $ker(\pi^{i+1}) = im(\pi^i)$ for all i so all $H^n(\mathcal{A})$ are trivial. We interpret this as the cohomology groups measuring the non-exactness of the complex.

Definition 5.3 For two cochain complexes $\mathcal{A} : \cdots A^1 \xrightarrow{\pi^1} A^2 \xrightarrow{\pi^2} A^3 \cdots \xrightarrow{\pi^{n-1}} A^n \cdots$ and $\mathcal{B} : \cdots B^1 \xrightarrow{\psi^1} B^2 \xrightarrow{\psi^2} B^3 \cdots \xrightarrow{\psi^{n-1}} B^n \cdots$ a collection of homomorphisms $\phi_i : A^i \to B^i$ such that the resulting diagram

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{\pi^0} & A^1 & \xrightarrow{\pi^1} & A^2 & \xrightarrow{\pi^2} & A^3 & \xrightarrow{\pi^3} & \cdots \\
 & & \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_3} & & \\
\cdots & \xrightarrow{\psi^0} & B^1 & \xrightarrow{\psi^1} & B^2 & \xrightarrow{\psi^2} & B^3 & \xrightarrow{\psi^3} & \cdots
\end{array}$$

is commutative is called a homomorphism of complexes.

Theorem 5.4 *A homomorphism of complexes $\phi : \mathcal{A} \to \mathcal{B}$ induces a group homomorphism from $H^n(\mathcal{A})$ to $H^n(\mathcal{B})$.*

Proof By commutativity of the diagram we have that for all $a \in Ker(\pi^n), \phi_n(a) \in Ker(\psi^n)$, thus ϕ maps the kernel to the kernel.
By commutativity of the diagram we have that for all $b \in Im(\pi^{n-1}), \phi_n(b) \in Im(\psi^{n-1})$, since for all b there is necessarily a $b' \in A^{n-1}$ such that $\pi^{n-1}(b') = b$ and $\psi^{n-1}(\phi^{n-1}(b')) \in Im(\psi^{n-1})$, thus ϕ maps the image to the image.

Definition 5.5 The sequence of complexes $0 \to \mathcal{A} \overset{\alpha}{\to} \mathcal{B} \overset{\beta}{\to} \mathcal{C} \to 0$ is a short exact sequence iff $0 \to A^n \overset{\alpha_n}{\to} B^n \overset{\beta_n}{\to} C^n \to 0$ is a short exact sequence for all n.

Definition 5.6 For a group G and a G-module A we let $G^n = G \times G \times \cdots \times G \times G$ be the direct product of G n times. We will define $C^n(G; A)$ as A for $n = 0$ and as all maps from G^n to A for $n \leq 1$. The elements of $C^n(G; A)$ are called n-cochains of G with values in A.

We want to construct a cochain complex out of $C^n(G; A)$. To do this we must know that all $C^n(G; A)$ are abelian, which is given for $C^0(G; A) = A$ and can be seen to hold for $n > 0$ if we consider that the pointwise addition of maps $f_1 + f_2(g_1, g_2, g_3 \cdots g_n) = f_1(g_1, g_2, g_3 \cdots g_n) + f_2(g_1, g_2, g_3 \cdots g_n)$ commutes. Next we will need homomorphisms.

Definition 5.7 We define the coboundary homomorphisms $\delta_n : C^n(G; A) \to C^{n+1}(G; A)$ by

$$\begin{aligned}
\delta_n(f)(g_1, g_2, g_3 \cdots g_{n+1}) = & g_1 \cdot f(g_2, g_3 \cdots g_{n+1}) \\
& + \sum_{i=1}^{n} (-1)^i f(g_1, g_2, g_3 \cdots , g_i g_{i+1}, \cdots g_{n+1}) \\
& + (-1)^{n+1} f(g_1, g_2, g_3 \cdots g_n)
\end{aligned}$$

By how addition of maps worked we can see that δ is a homomorphism. It can also be calculated that $\delta_{n+1} \circ \delta_n = 0$. Given this we have set up a cochain complex and we can calculate the cohomology groups.

Definition 5.8 The elements of $ker(\delta_n)$ are called n-cocycles.
The elements of $im(\delta_{n-1})$ are called n-coboundaries
We define $H^n(G, A)$ as the quotient $ker(\delta_n)/im(\delta_{n-1})$ with the exception of $H^0(G, A)$ which will be defined as $ker(\delta_n)/1$ for lack of a δ we could take the image of.

5.1 Category of 2-cocycles $\mathcal{Z}^2(G;A)$

Definition 5.9 The category of 2-cocycles has as objects all normalised 2-cocycles, the functions $\tau : G^2 \to A$ satisfying for all $x, y, z \in G$

$$x\tau(y,z) - \tau(xy,z) + \tau(x,yz) - \tau(x,y) = 0,$$

the cocycle condition given by the definition of δ, and $\tau(1,y) = 0 = \tau(x,1)$, being normalised. As morphisms the category has normalised 1-cochains $\beta : \tau \to \tau'$ satisfying $\delta^1\beta = \tau - \tau'$, functions $\beta : G \to A$ such that for all $x, y \in G$

$$x\beta(y) - \beta(xy) + \beta(x) = \tau(x,y) - \tau'(x,y)$$

and $\beta(1) = 0$.

Remark Composition for β is addition, and 0 acts as the identity morphism of every object τ. The category axioms are obviously satisfied. Every morphism β is invertible, with inverse $-\beta$ so $\mathcal{Z}^2(G;A)$ is a groupoid.

Definition 5.11 For a ring R with multiplicative identity 1 an abelian group A and an operation $* : R \times A \to A$ such that
$r * (a + a') = r * a + r * a'$
$(r + r') * a = r * a + r' * a$
$rr' * a = r * r' * a$ and
$1 * a = a$,
A with $*$ is a R**-module**. If R-module A has a basis, that is to say there is a subset $S \subseteq A$ such that for all $a \in A$ we can write $a = \sum_{i=0}^{n} r_i * s_i$ and the elements of S linearly independent, then A is called free.

Definition 5.12 For a R-module A, a possibly infinite exact sequence

$$\cdots E_3 \to E_2 \to E_1 \to E_0 \to A \to 0$$

with E_i being R-modules is called a resolution.

Definition 5.13 For a commutative ring R and a finite group $G = \{g_1, g_2 \cdots g_n\}$, the group ring, RG, of G with coefficients in R is defined to be the set of sums $\sum_{i=1}^{n} r_i g_i$ where r_i are elements in R, addition of two sums is defined as $\sum_{i=1}^{n} r_i g_i + \sum_{i=1}^{n} s_i g_i = \sum_{i=1}^{n} (r_i + s_i) g_i$ and with $(r_i g_i)(s_j g_j) = (r_i s_j)(g_i g_j)$ multiplication of two sums is defined as $(\sum_{i=1}^{n} r_i g_i)(\sum_{j=1}^{n} s_j g_j) = \sum_{i,j} (r_i g_i)(s_j g_j)$.

Theorem 5.14 *For group G and G-module A, Given a resolution*

$$\cdots \mathbb{Z}G \xrightarrow{f_2} \mathbb{Z}G \xrightarrow{f_1} \mathbb{Z}G \xrightarrow{f_0} \mathbb{Z}G \to \mathbb{Z} \to 0,$$

where is the group ring of G with coefficients in $\mathbb{Z}$, we have that $H^n(G;A) = ker(f_n)/im(f_{n-1})$ for the chain complex

$$0 \to A \xrightarrow{f_0} A \xrightarrow{f_1} A \xrightarrow{f_2} A \xrightarrow{f_3} A \to \cdots$$

where f_i in the second chain complex are maps induced, by maps from $\mathbb{Z}G$ to A, from the respective maps f_i in the resolution.

The theorem is constructed from a collection of results not proved in this text. The theorem will be used to easily calculate cohomology groups in our final set of examples. The reader is directed to chapter 17 of abstract algebra [1] if the reader wishes to read about the theory behind the theorem.

6 2-cocycles to extensions

There is a morphism between two objects in the groupoid of 2-cocycles $\mathcal{Z}^2(G;A)$ iif they represent the same cohomology class, so

$$H^2(G;A) = \pi_0(\mathcal{Z}^2(G;A)).$$

If we manage to prove that there is an equivalence of categories

$$F : \mathcal{Z}^2(G;A) \to \mathscr{E}xt(G,A)$$

from the category of 2-cocycles to the category of extensions then it will trivially follow that there is a bijection

$$H^2(G;A) = \pi_0(\mathcal{Z}^2(G;A)) \cong \pi_0(\mathscr{E}xt(G,A))$$

between the second cohomology group, equivalent to the group of components of the category of 2-cocycles, and the group of components of extensions.

6.1 Construction: between objects

Given $\tau : G^2 \to A$ a normalised 2-cocycle we construct the extension

$$0 \to A \xrightarrow{\pi_2} A \times_\tau G \xrightarrow{\pi_3} G \to 1,$$

where $A \times_\tau G$ is the group of the set $A \times G$ with multiplication defined by $(a,g)\bullet(b,g') = (a + gb + \tau(g,g'), gg')$, π_2 is the inclusion morphism defined by $\pi_2(a) = (a,1)$ and π_3 is the projection morphism defined by $\pi_3(a,g) = g$. Given the definitions of the morphisms it is clear that the sequence is exact. Now we show that the group axioms are satisfied.

Group axioms

Closure: For all $a, a' \in A, g, g' \in G$, $(a+gb+\tau(g,g'), gg')$ is clearly in $A \times_\tau G$ since $g \cdot a', \tau(g,g') \in A$.

Associativity:

$$
\begin{aligned}
&((a,g)\bullet(a',g'))\bullet(a'',g'')\\
&=(a+g\cdot a'+\tau(g,g'),gg')\bullet(a'',g'')\\
&=(a+g\cdot a'+\tau(g,g')+gg'\cdot a''+\tau(gg',g''),gg'g'')\\
&=(a+g\cdot a'+g\cdot\tau(g',g'')+gg'\cdot a''+\tau(g,g'g''),gg'g'')\\
&=*=(a+g\cdot a'+gg'\cdot a''+g\cdot\tau(g',g'')+\tau(g,g'g''),gg'g'')\\
&=(a,g)\bullet(a'+g'\cdot a''+\tau(g',g''),g'g'')\\
&=(a,g)\bullet((a',g')\bullet(a'',g'')),
\end{aligned}
$$

where we use that A is abelian and the cocycle condition at $*$.

Identity:(0,1); Using normalisation, $\tau(g,1)=\tau(1,g')=0$ we get:

$$
\begin{aligned}
(a,g)\bullet(0,1)&=(a+g\cdot 0+\tau(g,1),g1)=\\
&=(a,g)=(0+1\cdot a+\tau(1,g),1g)=(0,1)\bullet(a,g).
\end{aligned}
$$

Inverse:$(-g^{-1}\cdot a-g^{-1}\cdot\tau(g,g^{-1}),g^{-1})$;

$$
\begin{aligned}
&(a,g)\bullet(-g^{-1}\cdot a-g^{-1}\cdot\tau(g,g^{-1}),g^{-1})=\\
&=(a+g\cdot(-g^{-1}\cdot a-g^{-1}\cdot\tau(g,g^{-1}))+\tau(g,g^{-1}),gg^{-1})=\\
&=(a-a-\tau(g,g^{-1})+\tau(g,g^{-1}),1)=(0,1).
\end{aligned}
$$

For left multiplication we will use the cocycle condition with $x=g^{-1},y=g,z=g^{-1}$ and normalisation.

$$
\begin{aligned}
&(-g^{-1}\cdot a-g^{-1}\cdot\tau(g,g^{-1}),g^{-1})\bullet(a,g)=\\
&=(-g^{-1}\cdot a-g^{-1}\cdot\tau(g,g^{-1})+g^{-1}\cdot a+\tau(g^{-1},g),g^{-1}g)=\\
&=(-g^{-1}\cdot\tau(g,g^{-1})+\tau(g^{-1},g),1)=\\
&=(-\tau(g^{-1}g,g^{-1})+\tau(g^{-1},gg^{-1}),1)=(0,1).
\end{aligned}
$$

6.2 Morphisms of extensions and 1-cochains

Let $\beta:\tau\to\tau'$ be a morphism in the category of cocycles $\mathcal{Z}^2(G;A)$. Recall that β is a normalised 1-cochain from G to A such that $\delta^1\beta=\tau-\tau'$ or equivalently $g\cdot\beta(g')-\beta(gg')+\beta(g)=\tau(g,g')-\tau'(g,g')$ for all $g,g'\in G$. To obtain a corresponding morphism ϕ_β between extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & A\times_\tau G & \longrightarrow & G & \longrightarrow & 0\\
 & & \| & & \downarrow{\scriptstyle\phi_\beta} & & \| & & \\
0 & \longrightarrow & A & \longrightarrow & A\times_{\tau'} G & \longrightarrow & G & \longrightarrow & 0
\end{array}
$$

We define $\phi_\beta : A \times_\tau G \to A \times_{\tau'} G$ by $\phi_\beta(a, g) = (a + \beta(g), g)$.
Since $\delta^1\beta = \tau - \tau'$ we have

$$\begin{aligned}
&\phi_\beta((a,g)\bullet(a',g')) = \\
&= (a + g\cdot a' + \tau(g,g') + \beta(gg'), gg') = \\
&= (a + g\cdot a' + g\cdot\beta(g') + \beta(g) + \tau'(g,g'), gg') = \\
&= (a + g\cdot(a' + \beta(g')) + \beta(g) + \tau'(g,g'), gg') = \\
&= \phi_\beta((a,g))\bullet\phi_\beta((a',g'))
\end{aligned}$$

and therefore that ϕ_β is a group homomorphism. To see that the diagram is commutative we need only check that $\phi_\beta((a,1)) = (a,1)$, since ϕ_β only changes the fist position of the double, which can be seen by remembering that β is normalised i.e. $\beta(1) = 0$. Since composition of morphisms in $\mathscr{Z}^2(G;A)$ works by addition it is clear that if β and β' are composable then $\phi_{\beta\circ\beta'} = \phi_\beta \circ \phi_{\beta'}$. Lastly the identity morphisms exist and we can conclude that we have defined a functor $F : \mathscr{Z}^2(G;A) \to \mathscr{E}xt(G,A)$.

7 The functor F is an equivalence of categories

Now that we have defined our two categories and our functor F between them it is time to show that F is an equivalence of categories, done by showing that it is fully faithful and essentially surjective.

7.1 Fully Faithfulness

To show that F is fully faithful we need to show that there is a unique 1-cochain β for every extension

$$\begin{tikzcd}
0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \times_\tau G \arrow[r] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & A \arrow[r] & A \times_{\tau'} G \arrow[r] & G \arrow[r] & 0
\end{tikzcd}$$

such that $\phi_\beta = \phi$. For this to hold, by how we defined ϕ_β, we would need that $\phi(0,g) = (\beta(g), g)$ and this can and will be used to uniquely define β. As in the previous section, for the diagram to be commutative $\phi(a,1) = (a,1)$ and in particular this implies that $\phi(0,1) = (0,1) = (\beta(1),1)$ so $\beta(1) = 0, \beta$ is normalised. To see that for this $\beta, \phi_\beta = \phi$ note that $(a,g) = (a,1)\bullet(0,g)$ so knowing that ϕ is a group homomorphism we get

$$\begin{aligned}
&\phi(a,g) = \phi(a,1)\bullet\phi(0,g) = \\
&= (a,1)\bullet(\beta(g),g) = \\
&= (a + \beta(g), g) = \phi_\beta(a,g).
\end{aligned}$$

7.2 Essential surjectivity

To show essential surjectivity we need to show that any extension in our category of extensions is isomorphic to one our functor F gives us from the category of 2-cocycles. Given such an extension

$$0 \to A \to E \xrightarrow{\pi} G \to 0,$$

to find a corresponding 2-cycle we first find a function $f : G \to E$ such that $\pi \circ f = 1$ and $f(1) = 1$ which is always possible by the axiom of choice, specifically the formulation of the axiom of choice as "every surjective function has a right inverse", since π is surjective. Usually f will not be a homomorphism however $\tau(g, g') = f(g)f(g')f(gg')^{-1}$ can be used to tell "how" f fails at being a homomorphism. Now note that $\pi(\tau(g, g')) = 1$, since π is a group homomorphism and $\pi \circ f = 1$, meaning that τ defines a function $\tau : G^2 \to A$. Now to see that τ is a normalised 2-cocycle. Since $f(1) = 1$ we have,

$$\begin{aligned} &\tau(1, g') = f(1)f(g')f(1g')^{-1} = 0 = \\ &= f(g)f(1)f(g1)^{-1} = \tau(g, 1), \end{aligned}$$

so we know that τ is normalised. To show that τ is a 2-cocycle, that it satisfies the cocycle condition, we work in E with $A \subseteq E$ using that the G-module structure on A agrees with the one from our extension. Since $\pi \circ f = 1$ we can let $\bar{g} = f(g)$ and for all $g \in G, a \in A$, $g \cdot a = f(g)af(g)^{-1}$ so together with the definition of τ the cocycle condition becomes the lengthy equation

$$\begin{aligned} &f(g)f(g')f(g'')f(g'g'')^{-1}f(g)^{-1}(f(gg')f(g'')f(gg'g'')^{-1})^{-1}f(g)f(g'g'')f(gg'g'')^{-1} \\ &(f(g)f(g')f(gg')^{-1})^{-1} = 1 \end{aligned}$$

Now we will be simplifying and since π is a homomorphism inverting f the terms in square brackets are in the kernel of π and therefore in the abelian group A so we can use that it can commute. Given these observations we get

$$\begin{aligned} &f(g)f(g')f(g'')[f(g'g'')^{-1}f(g)^{-1}f(gg'g'')][f(g'')^{-1}f(gg')^{-1}f(g)f(g'g'')]f(gg'g'')^{-1} \\ &f(gg')f(g')^{-1}f(g)^{-1} = \\ &f(g)f(g')f(g'')f(g'')^{-1}f(gg')^{-1}f(g)f(g'g'')f(g'g'')^{-1}f(g)^{-1}f(gg'g'')f(gg'g'')^{-1} \\ &f(gg')f(g')^{-1}f(g)^{-1} = \\ &f(g)f(g')f(g'')f(g'')^{-1}f(gg')^{-1}f(gg'g'')f(gg'g'')^{-1}f(gg')f(g')^{-1}f(g)^{-1} = 1 \end{aligned}$$

Equipped with a 2-cocycle τ we create the morphism of extensions

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & A \times_\tau G \arrow[r] \arrow[d, "\phi"] & G \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & E \arrow[r, "\pi"] & G \arrow[r] & 0 \end{tikzcd}$$

where $\phi(a,g) = af(g)$.
Since $f(1) = 1$ and $\pi(af(g)) = \pi(a)\pi(f(g)) = g$ the diagram commutes. Finally ϕ is a group homomorphism, since

$$\begin{aligned}
&\phi((a,g)\bullet(a',g')) = \\
&= \phi((a + g\cdot a' + \tau(g,g'), gg')) = \\
&= (a + g\cdot a' + \tau(g,g'))f(gg') = \\
&= af(g)a'f(g)^{-1}f(g)f(g')f(gg')^{-1}f(gg') = \\
&= af(g)a'f(g') = \phi(a,g)\phi(a',g'),
\end{aligned}$$

so by the short five lemma ϕ is an isomorphism.

7.3 Examples

- Remembering we could not find a way to create a semidirect product $\langle i\rangle \rtimes \langle -1\rangle$ isomorphic to Q_8 we make a new attempt with the techniques from the proof of essential surjectivity. Since $0 \to \langle i\rangle \xrightarrow{\pi_1} Q_8 \xrightarrow{\pi_2} \langle -1\rangle \to 0$, with π_1 being the inclusion morphism and π_2 forced, is a short exact sequence we know that there is a fitting 2-cocycle making $\langle i\rangle \times_\tau \langle -1\rangle$ isomorphic to Q_8. We are in luck because $\langle -1\rangle$ only has two elements and we only have to decide on four values for τ, one if we notice that by normalisation we are given
$\tau(1,1) = \tau(-1,1) = \tau(1,-1) = 1$.
Using our formula for τ and knowledge that squares in Q_8 are either 1 or -1 we get
$\tau(-1,-1) = f(-1)f(-1)f((-1)(-1)) = f(-1)^2 = 1, -1$.
Since 1 would have no effect in the calculations of the semidirect product -1 must be chosen. However $\langle i\rangle \times_\tau \langle -1\rangle$ still depends on which group action is chosen, however the trivial group action can be ruled out because it would result in $(1,1), (-1,1), (i,-1), (-i,-1)$ all being of order two, too many for the group to be Q_8. By process of elimination we have found the characterisation of the sought after group. To tie this to the cohomology group $H^2(G;A)$ we calculate the group using Theorem 5.13 and the resolution

$$\cdots \mathbb{Z}\langle -1\rangle \xrightarrow{f_3} \mathbb{Z}\langle -1\rangle \xrightarrow{f_2} \mathbb{Z}\langle -1\rangle \xrightarrow{f_1} \mathbb{Z}\langle -1\rangle \xrightarrow{f_0} \mathbb{Z}\langle -1\rangle \to \mathbb{Z} \to 0$$

where $f_i(x) = x * (\overline{-1} + (-1)^{i+1})$, notice that
$f_{i+1} \circ f_i(x) = x * (\overline{-1} - 1)(\overline{-1} + 1) = x * (\overline{-1}^2 - 1) = x * (1 - 1) = 0$.
So $H^2(G;A) = ker(f_2)/im(f_1)$ for the chain complex

$$0 \to \langle i\rangle \xrightarrow{f_0} \langle i\rangle \xrightarrow{f_1} \langle i\rangle \xrightarrow{f_2} \langle i\rangle \xrightarrow{f_3} \langle i\rangle \to \cdots.$$

Remembering that $\overline{-1}$ acts by inversion we can see that the kernel of f_2 are the elements g of $\langle i \rangle$ for which $g^{-1}g^{-1} = \bar{1}$, so $ker(f_2) = \{\bar{1}, \overline{-1}\}$ and $im(f_1) = \{\bar{1}\}$ since the action of $(\overline{-1}+1)$ amounts to sending every element to the identity. Now noticing the size of the cohomology group is $|H^2(G;A)| = |ker(f_2)|/|im(f_1)| = 2/1 = 2$, so only two extensions exist.

- Next we explore the possibilities for $\mathbb{Z}/2\mathbb{Z} \times_\tau \mathbb{Z}/2\mathbb{Z}$ with trivial group action, we already know that the direct product results in
the Klein four-group K_4 but is there another alternative. Again $\mathbb{Z}/2\mathbb{Z}$ only has two elements so we only really need decide what value $\tau(1,1)$ should have, and for it to make a difference it better take the value 1, the element of order 2. If we assume this and compute we notice that the element $(0,1)$ has order 4, since $(0,1)(0,1)^3 = (1,0)(0,1)^2 = (1,1)(0,1) = (0,0)$. There is only one group of order 4 with elements of order 4 and it is the cyclic group of order 4. With some further work we can work out that we indeed have found a characterisation of the cyclic group of order 4.

- We might now look at products $\mathbb{Z}/8\mathbb{Z} \times_\tau \mathbb{Z}/2\mathbb{Z}$, where the action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathbb{Z}/8\mathbb{Z}$ is trivial , so if we calculate the cohomology group we can use the same resolution as in the example with Q_8. The calculation becomes $H^2(G;A) = ker(f_2)/im(f_1)$ for the chain complex

$$0 \to \mathbb{Z}/8\mathbb{Z} \xrightarrow{f_0} \mathbb{Z}/8\mathbb{Z} \xrightarrow{f_1} \mathbb{Z}/8\mathbb{Z} \xrightarrow{f_2} \mathbb{Z}/8\mathbb{Z} \xrightarrow{f_3} \mathbb{Z}/8\mathbb{Z} \to \cdots.$$

This time, because the action of $\overline{-1}$ is trivial, f_2 sends everything to the identity and f_1 sends elements to their squares so

$$|H^2(G;A)| = |ker(f_2)|/|im(f_1)| = 8/4 = 2$$

and we can only have two distinct extensions. To do things differently lets first find two short exact sequences and then check what 2-cocycle that belong to them. With a small search we might find the sequences

$$0 \to \mathbb{Z}/8\mathbb{Z} \xrightarrow{\phi_1} \mathbb{Z}/8\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \xrightarrow{\psi_1} \mathbb{Z}/2\mathbb{Z} \to 0$$

and

$$0 \to \mathbb{Z}/8\mathbb{Z} \xrightarrow{\phi_2} \mathbb{Z}/16\mathbb{Z} \xrightarrow{\psi_2} \mathbb{Z}/2\mathbb{Z} \to 0$$

where $\phi_1(a) = (a,0), \psi_1(a,g) = g, \phi_2(a) = 2a$ and $\psi_2(g) = g(\text{mod}2)$, both with trivial group action. We only need to search for $\tau(1,1)$ and using the method from Section 7.2. We choose for the first sequence $f(1) = (0,1)$, this gives us
$\tau(1,1) = (0,1)(0,1)f(1+1) = (0,1)(0,1)(0,0) = (0,0) = 0$, to no surprise the direct product is not affected by the cocycle. For the second sequence we choose $f(1) = 1$ giving us $\tau(1,1) = f(1)+f(1)+f(0) = 1+1+0 = 2$, where 2 is mapped to from $1 \in \mathbb{Z}/8\mathbb{Z}$.

8 Summary

In Section 6 we noted that finding an equivalence of categories from the category of 2-cocycles to the category of extensions to establish an isomorphism between the second cohomology group, which we identified with the components of the category of 2-cocycles, and the components of the category of extension. We continued by constructing a functor between the categories which we would continue to prove was an equivalence of categories. In Section 7 we first showed that the functor was fully faithful by proving how to find a unique 1-cochain for any morphism of extensions. We then proved essential surjectivity by, for any valid extension E, presenting a way to find the 2-cocycle that would be mapped to an extension isomorphic to E with the help of a right inverse function. We had thereby proved that the functor we defined is an equivalence of categories since it was both essentially surjective and fully faithful.

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[3] Kenneth S. Brown. *Cohomology of Groups*. Springer-Verlag New York Inc. 1982.